The 85th William Lowell Putnam Mathematical Competition 2024

A1 Determine all positive integers n for which there exist positive integers a, b, and c satisfying

$$2a^n + 3b^n = 4c^n.$$

A2 For which real polynomials p is there a real polynomial q such that

$$p(p(x)) - x = (p(x) - x)^2 q(x)$$

for all real x?

A3 Let S be the set of bijections

$$T\colon \{1,2,3\} \times \{1,2,\dots,2024\} \to \{1,2,\dots,6072\}$$

such that $T(1,j) < T(2,j) < T(3,j)$ for all $j \in \{1,2,\dots,2024\}$ and $T(i,j) < T(i,j+1)$ for all $i \in \{1,2,3\}$ and $j \in \{1,2,\dots,2023\}$. Do there exist a and c in $\{1,2,3\}$ and b and d in $\{1,2,\dots,2024\}$ such that the fraction of elements T in S for which $T(a,b) < T(c,d)$ is at least $1/3$ and at most $2/3$?

A4 Find all primes $p > 5$ for which there exists an integer a and an integer r satisfying $1 \le r \le p-1$ with the following property: the sequence $1, a, a^2, \dots, a^{p-5}$ can be rearranged to form a sequence $b_0, b_1, b_2, \dots, b_{p-5}$ such that $b_n - b_{n-1} - r$ is divisible by p for $1 \le n \le p-5$.

A5 Consider a circle Ω with radius 9 and center at the origin $(0,0)$, and a disk Δ with radius 1 and center at $(r,0)$, where $0 \le r \le 8$. Two points P and Q are chosen independently and uniformly at random on Ω. Which value(s) of r minimize the probability that the chord $\overline{PQ}$ intersects Δ?

A6 Let $c_0, c_1, c_2, \dots$ be the sequence defined so that

$$\frac{1 - 3x - \sqrt{1 - 14x + 9x^2}}{4} = \sum_{k=0}^{\infty} c_k x^k$$

for sufficiently small x. For a positive integer n, let A be the n-by-n matrix with i,j-entry c_{i+j-1} for i and j in $\{1,\dots,n\}$. Find the determinant of A.