# 2024 Session A

**A1.** Determine all positive integers $n$ for which there exist positive integers $a$, $b$, and $c$ satisfying

$$2a^n + 3b^n = 4c^n.$$

**Answer:** $n = 1$ only. For $n = 1$, the equation is satisfied by $(a, b, c) = (1, 2, 2)$.

**Solution 1:** Consider $n > 1$. If $d$ is the greatest common divisor of $a, b, c$, so $a = dx$, $b = dy$, $c = dz$, then $x, y, z$ satisfy the same equation and we can assume that the greatest common divisor is 1. We see that $2 \mid 3b^n$, so $2 \mid b$. Letting $b = 2b_1$, the equation becomes

$$2a^n + 3 \cdot 2^n b_1^n = 4c^n.$$

Since $n \geq 2$, we have that $4 \mid 2a^n$, so $2 \mid a^n$ and $2 \mid a$. Setting $a = 2a_1$, we get that

$$2^n(2a_1^n + 3b_1^n) = 4c^n$$

and $2^{n-2} \mid c^n$. Since we assumed the greatest common divisor of $a, b, c$ is 1, we must have that $2 \nmid c$. Thus, we must have $n = 2$.

Then

$$2a^2 + 3b^2 = 4c^2,$$

and so $a^2 + c^2 = 3a^2 + 3b^2 - 3c^2$ is divisible by 3. Considering all possible cases for remainders of $a$ and $c$ by division by 3, we see that $a^2$ has remainder 0 or 1, and $c^2$ has remainder 0 or 1. Thus, both $a^2$ and $c^2$ must have remainder 0, so $3 \mid a$ and $3 \mid c$. Writing $a = 3a_2, c = 3c_2$ we have $b^2 = 3(4c_2^2 - 2a_2^2)$, so $3 \mid b$, contradicting the assumption that $a, b, c$ have common divisor 1.

**Solution 2:** To prove that there are no solutions for $n \geq 2$, assume to the contrary that there is such a solution. Let $d$ be the greatest common divisor of $a$, $b$, and $c$, and let $x = a/d$, $y = b/d$, and $z = c/d$. Then $2x^n + 3y^n = 4z^n$, and the greatest common divisor of $x$, $y$, and $z$ is 1. In particular, at least one of $x$, $y$, and $z$ is odd. Since $3y^n = 4z^n - 2x^n$ is even, $y$ is even. Then since $n \geq 2$, it follows that $2x^n = 4z^n - 3y^n$ is a multiple of 4, so $x$ is even too, whence $z$ is odd. If $n \geq 3$, we then have the contradiction that $2x^n + 3y^n$ is a multiple of 8, but $4z^n$ is not. If $n = 2$, we can write $2(x/2)^2 + 3(y/2)^2 = z^2$. It follows that $y/2$ is odd. Since all odd squares are congruent to 1 modulo 8, we have $2(x/2)^2 \equiv z^2 - 3(y/2)^2 \equiv 1 - 3 \equiv 6 \pmod 8$, which is impossible.

**A2.** For which real polynomials $p$ is there a real polynomial $q$ such that

$$p(p(x)) - x = (p(x) - x)^2 q(x)$$

for all real $x$?

**Answer:** Only $p(x) = \pm x + c$ for $c$ a constant.

**Solution 1:** Let $f(x) = p(x) - x$ and let $d$ denote its degree. Then the desired property is equivalent to $f(x + f(x)) + f(x) = [f(x)]^2 q(x)$. By the Taylor series expansion of $f$ at $x$,

$$f(x + f(x)) = f(x) + f'(x)f(x) + \frac{f''(x)}{2}[f(x)]^2 + \cdots + \frac{f^{(d)}(x)}{d!}[f(x)]^d.$$

Thus, the factorization exists if and only if $2f(x) + f'(x)f(x) = [f(x)]^2 r(x)$ for some polynomial $r$, which in turn is equivalent to $f(x) = 0$ or $2 + f'(x) = f(x)r(x)$. The factorization holds when $d = 0$, hence when $p(x) = x + c$. If $d > 0$, then $2 + f'(x)$ has degree $d - 1$, a contradiction, unless $2 + f'(x) = 0$, which is equivalent to $p(x) = -x + c$.

**Solution 2:** Let $r(x) = p(x) - x$, then $p(x) = x + r(x)$ and the equation becomes

$$r(r(x) + x) + r(x) = r(x)^2 q(x)$$

Let $r(x) = c_n x^n + \cdots + c_1 x + c_0$ be its expansion in monomials. Then

$$\begin{aligned} r(r(x)+x) &= \sum_{k=0}^{n} c_k (r(x)+x)^k = \sum_{k=0}^{n} c_k \sum_{i=0}^{k} \binom{k}{i} x^i (r(x))^{k-i} \\ &= r(x)^2 \left( \underbrace{\sum_{k=2}^{n} c_k \sum_{i=0}^{k-2} \binom{k}{i} x^i r(x)^{k-i-2}}_{q_1(x)} \right) + r(x) \left( \sum_{k=1}^{n} c_k k x^{k-1} \right) + \sum_{k=0}^{n} c_k x^k \\ &= r(x)^2 q_1(x) + r(x) r'(x) + r(x). \end{aligned}$$

Thus, the original equation is equivalent to

$$r(x)^2 q_1(x) + r(x) r'(x) + 2r(x) = r^2(x) q(x).$$

Thus, either $r(x)$ is identically 0 (so $p(x) = x$) or

$$r'(x) + 2 = r(x)(q(x) - q_1(x)).$$

If the degree of $r(x)$ is not 0, then $\deg(r'(x) + 2) = \deg(r(x)) - 1 < \deg(r(x)(q(x) - q_1(x)))$, unless $q(x) - q_1(x) = 0$. Thus, either $\deg(r(x)) = 0$, so $p(x) = x + c$, or $q(x) = q_1(x)$ and $r'(x) = -2$, so $r(x) = -2x + c$, so $p(x) = -x + c$.

Plugging the two possibilities in the original equation we see the following. For $p(x) = x + c$, we have $2c = c^2 q(x)$, so all real values for $c$ give a solution with a constant polynomial $q(x)$. For $p(x) = -x + c$ we have $-(-x + c) + c - x = (-2x + c)^2 q(x)$, so $q(x) = 0$ gives a solution for all $c$.

The solutions are thus $p(x) = -x + c$ for any real $c$ or $p(x) = x + c$ for any real $c$.

**A3.** Let $S$ be the set of bijections

$$T\colon \{1,2,3\} \times \{1,2,\ldots,2024\} \to \{1,2,\ldots,6072\}$$

such that $T(1,j) < T(2,j) < T(3,j)$ for all $j \in \{1,2,\ldots,2024\}$ and $T(i,j) < T(i,j+1)$ for all $i \in \{1,2,3\}$ and $j \in \{1,2,\ldots,2023\}$. Do there exist $a$ and $c$ in $\{1,2,3\}$ and $b$ and $d$ in $\{1,2,\ldots,2024\}$ such that the fraction of elements $T$ in $S$ for which $T(a,b) < T(c,d)$ is at least $1/3$ and at most $2/3$?

**Answer:** Yes.

**Solution 1:** We consider the more general situation where the set of bijections $S$ is $T:\{1,\ldots,m\} \times \{1,2,\ldots,n\} \to \{1,2,\ldots,mn\}$ satisfying the given inequalities, where

$$m,n \geq 2$$

. In the problem we have $m=3, n=2024$. To simplify the notation, we switch to the probabilistic formulation: we are choosing elements $T$ uniformly at random from $S$ and considering the probability that $T(a,b) < T(c,d)$, which is equal to the proportion of such bijections $T$. By symmetry, if $m$ and $n$ are exchanged, then $Pr[T(a,b) < T(c,d)]$ becomes $Pr[T(b,a) < T(d,c)]$.

Consider $Pr[T(2,1) < T(1,2)]$. If $Pr[T(2,1) < T(1,2)] \in [\frac{1}{3}, \frac{2}{3}]$, then let $(a,b) = (2,1)$ and $(c,d) = (1,2)$, and we are done. If not, then without loss of generality we can assume $Pr[T(2,1) < T(1,2)] < \frac{1}{3}$ (if instead $Pr[T(2,1) < T(1,2)] > \frac{2}{3}$, then we can exchange $m$ and $n$ to get $Pr[T(1,2) < T(2,1)] > \frac{2}{3}$, in which case $Pr[T(2,1) < T(1,2)] = 1 - Pr[T(1,2) < T(2,1)] < \frac{1}{3}$). Our goal now is to show that $Pr[T(2,1) < T(1,j)] \in [\frac{1}{3}, \frac{2}{3}]$ for some $j > 2$.

Let $S_i = \{T \in S : T(2,1) = i\}$ and let $q_i = |S_i|/|S|$, i.e. the probability that the bijection $T$ has $T(2,1) = i$. If $T(2,1) = i$, we must have $T(1,j) = j < T(2,1)$ for $j \leq i-1$ and $T(1,j) > i = T(2,1)$ for $j \geq i$. So $T(2,1) < T(1,j)$ is equivalent to $j \geq T(2,1)$, and summing over all possibilities for the value $T(2,1) = 2,\ldots,j$ we have

$$Pr[T(2,1) < T(1,j)] = q_2 + q_3 + \cdots q_j.$$

In particular, $q_2 = Pr[T(2,1) < T(1,2)] < \frac{1}{3}$. Note also that $q_2 + \cdots + q_{n+1} = 1$ since these are the only possibilities for $T(2,1)$.

**Claim.** *We have that* $q_2 \geq q_3 \geq \cdots \geq q_{n+1}$.

*Proof.* We see that there is an injection $\phi : S_{i+1} \to S_i$, given by $\phi(T)(1,i) = i+1$, $\phi(T)(2,1) = i$, and $\phi(T)(r,s) = T(r,s)$ for all other $(r,s)$. □

Finally, let $k = \max\{j : q_2 + \cdots + q_j < \frac{1}{3}\}$, which exists by the above bounds. Further, since $q_{n+1} \leq q_2 < \frac{1}{3}$, we have $q_2 + \cdots + q_n = 1 - q_{n+1} > \frac{2}{3}$, so $k < n$. By maximality we must have that $\frac{1}{3} \leq q_2 + \cdots + q_{k+1}$, and since $q_{k+1} \leq q_2 < \frac{1}{3}$, we must also have $q_2 + \cdots + q_{k+1} = (q_2 + \cdots + q_k) + q_{k+1} < \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$. Then $(a,b) = (1,k+1)$ and $(c,d) = (2,1)$ are the desired pairs.

**Solution 2:** The answer is yes, in particular for $a=2$, $b=2024$, $c=3$, $d=2023$. Think of the domain of $T$ as a grid with 3 rows and 2024 columns, with rows and columns numbered as for a matrix. The greatest value of $T$, namely 6072 must be in the bottom row and the rightmost column, so $T(3,2024)=6072$. Similarly, the value 6071 can only be immediately above or immediately to the left of $(3,2024)$, so either $T(2,2024)=6071$ or $T(3,2023)=6071$. Thus, the number $n_1$ of $T$ for which $T(2,2024)<T(3,2023)$ is the number of $T$ for which $T(3,2023)=6071$, and the number $n_2$ of $T$ for which $T(2,2024)>T(3,2023)$ is the number of $T$ for which $T(2,2024)=6071$. To prove that $n_1/(n_1+n_2)$ is between $1/3$ and $2/3$, it suffices to prove that $n_1/n_2$ is between $1/2$ and 2.

For integers $k\geq \ell\geq m\geq 0$, let $D_{k,\ell,m}=(\{1\}\times\{1,2,\dots,k\})\cup(\{2\}\times\{1,2,\dots,\ell\})\cup(\{3\}\times\{1,2,\dots,m\})$. Notice that $D_{2024,2024,2024}$ is the domain of $T$ in the problem statement. More generally, think of $D_{k,\ell,m}$ as a left-justified grid with $k$ elements in the first row, $\ell$ elements in the second row, and $m$ elements in the third row. Let $N(k,\ell,m)$ be the number of bijections $T: D_{k,\ell,m}\to\{1,2,\dots,k+\ell+m\}$ for which $T(i,j)<T(i+1,j)$ and $T(i,j)<T(i,j+1)$ whenever both sides of the inequality are defined. Notice that $N(2024,2024,2024)$ is the number of elements in $S$, and so is $N(2024,2024,2023)$, since $T(3,2024)$ is determined by the conditions on $T$. Furthermore, $n_1=N(2024,2024,2022)$ and $n_2=N(2024,2023,2023)$.

The bulk of the remainder of the solution is to derive a formula for $N(k,\ell,m)$. The greatest value of $T$, namely $k+\ell+m$, must occur at the right end of a row and only if this row extends beyond the bottom row, so either $T(1,k)=k+\ell+m$ or $T(2,\ell)=k+\ell+m$ or $T(3,m)=k+\ell+m$. If $k>\ell>m>0$, then all three cases are possible, and if $k=\ell$ then the largest element cannot be at $(1,k)$ etc. Each case can be reduced to a domain with one fewer element, resulting in the identity

$$N(k,\ell,m)=N(k-1,\ell,m)+N(k,\ell-1,m)+N(k,\ell,m-1), \qquad (*)$$

where we assume that the term is 0 if the arguments are not in weakly decreasing order. Furthermore, considering the conventional values $N(0,0,0)=1$ and $N(k,\ell,m)=0$ if $k,\ell$ or $\ell<m$ as "boundary conditions", (*) recursively determines $N(k,\ell,m)$ for all $k\geq\ell\geq m\geq 0$ with $k+\ell+m>0$.

We will express $N(k,\ell,m)$ as a linear combination of trinomial coefficients, which satisfy a similar recursion. Let

$$F(p,q,r)=\frac{(p+q+r)!}{p!q!r!}$$

for nonnegative integers $p,q,r$, and extend $F$ to the integers by defining $F(p,q,r)=0$ if $p<0$ or $q<0$ or $r<0$.

**Lemma.** *For $(p,q,r)\neq(0,0,0)$,*

$$F(p,q,r)=F(p-1,q,r)+F(p,q-1,r)+F(p,q,r-1).$$

*Proof.* For nonnegative $p,q,r$ with $p+q+r>0$, the desired equality is equivalent to

$$\frac{(p+q+r-1)!}{p!q!r!}(p+q+r)=\frac{(p+q+r-1)!}{p!q!r!}p+\frac{(p+q+r-1)!}{p!q!r!}q+\frac{(p+q+r-1)!}{p!q!r!}r.$$

If $p<0$ or $q<0$ or $r<0$, then the desired equality is equivalent to $0=0$. □

**Claim.** *For $k\geq\ell\geq m\geq 0$ or $k+1=\ell\geq m\geq 0$ or $k\geq\ell+1=m\geq 0$ or $k\geq\ell\geq m+1=0$,*

$$\begin{aligned}N(k,\ell,m)=\;&F(k,\ell,m)+F(k+2,\ell-1,m-1)+F(k+1,\ell+1,m-2)\\&-F(k+1,\ell-1,m)-F(k,\ell+1,m-1)-F(k+2,\ell,m-2).\end{aligned}$$

*Proof.* The claimed expression for $N(k, \ell, m)$ satisfies the recursion (*) for $k \geq \ell \geq m \geq 0$ and $k + \ell + m > 0$ as an immediate consequence of the Lemma, since $k + \ell + m > 0$ ensures that none of the triples on which $F$ is being evaluated are $(0, 0, 0)$. It remains to verify the "boundary conditions".

If $k+1 = \ell \geq m \geq 0$, then substituting $\ell = k+1$ into the claimed expression and using the fact that $F(p, q, r) = F(q, p, r)$ makes all the terms cancel out, yielding the required boundary value 0 in this case. Similarly, if $k \geq \ell + 1 = m \geq 0$, then substituting $m = \ell + 1$ and using the fact that $F(p, q, r) = F(p, r, q)$ makes all the terms cancel out. If $k \geq \ell \geq m + 1 = 0$, then all terms in the claimed expression are 0. Finally, if $k = \ell = m = 0$, first term is 1 and all other terms are 0, yielding the required value 1 in this case. $\square$

For $k \geq \ell \geq m \geq 0$, it follows that

$$\begin{aligned}
\frac{(k+2)!(\ell+1)!m!}{(k+\ell+m)!} N(k, \ell, m) &= (k+2)(k+1)(\ell+1) + (\ell+1)\ell m + (k+2)m(m-1) \\
&\quad - (k+2)(\ell+1)\ell - (k+2)(k+1)m - (\ell+1)m(m-1) \\
&= (k+1-\ell)\left((k+2)(\ell+1) - (k+\ell+2)m + m(m-1)\right) \\
&= (k+1-\ell)(\ell+1-m)(k+2-m).
\end{aligned}$$

Then for $k \geq 2$,

$$\frac{N(k, k, k-2)}{N(k, k-1, k-1)} = \frac{(k+2)!k!(k-1)!}{(k+2)!(k+1)!(k-2)!} \cdot \frac{1 \cdot 3 \cdot 4}{2 \cdot 1 \cdot 3} = 2\,\frac{(k-1)}{(k+1)}.$$

This fraction is between $1/2$ and 2 for all $k \geq 2$. In particular, with $k = 2024$, we get that $n_1/n_2$ is between $1/2$ and 2, which completes the solution.

**Remark.** This problem is a special case of the 1/3–2/3 conjecture (`https://en.wikipedia.org/wiki/1/3-2/3_conjecture`). Solution 1 is based on an argument in [S. H. Chan, I. Pak, G. Panova, "Sorting Probability for Large Young Diagrams", Discrete Analysis 24 (2021), `https://doi.org/10.19086/da.30071`]. The final formula derived in Solution 2 for $N(k, \ell, m)$ is a special case of the "hook length formula", written in the following form:

`https://en.wikipedia.org/wiki/Hook_length_formula#Related_formulas`

**A4.** Find all primes $p > 5$ for which there exists an integer $a$ and an integer $r$ satisfying $1 \le r \le p-1$ with the following property: the sequence $1, a, a^2, ..., a^{p-5}$ can be rearranged to form a sequence $b_0, b_1, b_2, ..., b_{p-5}$ such that $b_n - b_{n-1} - r$ is divisible by $p$ for $1 \le n \le p-5$.

**Answer:** Only $p = 7$.

**Solution:** For $p = 7$, $a = 3$ yields the sequence $1, 3, 9$, which can be reordered as $1, 9, 3$.

For $p \ge 11$, we work modulo $p$. Suppose $1, a, a^2, \ldots, a^{p-5}$ can be rearranged with differences between consecutive terms congruent to $r \not\equiv 0 \pmod{p}$. If two of these terms were the same modulo $p$, then $jr \equiv 0 \pmod{p}$ where $j$ is the distance between their indices in the arithmetic progression. Since $j < p$, we must have $j = 0$, and so the terms are all distinct modulo $p$. Because $p-5 > (p-1)/2$, we conclude that $a$ has multiplicative order $p-1$ modulo $p$, and so $0, 1, a, a^2, \ldots, a^{p-2}$ are distinct modulo $p$. Therefore, $1, a, a^2, \ldots, a^{p-5}$ must be congruent to a "segment" of the "cyclic" modulo-$p$ arithmetic progression $0, r, 2r, \ldots, (p-1)r, 0, \ldots$. Then $0, a^{p-4}, a^{p-3}, a^{p-2}$ must be congruent to the remaining segment that completes the cycle. Since $a^{p-1} \equiv 1 \pmod{p}$, these four terms are congruent to $0, c, c^2, c^3$, where $c$ is the residue class of $a^{p-2}$ modulo $p$. Because none of $c, c^2, c^3$ are $-1$ times another, 0 must be an end of the arithmetic progression, which we may assume is the beginning. Furthermore, if we multiply by $c^{-4}$, we obtain another arithmetic progression using the same rearrangement of the terms $0, c^{-3}, c^{-2}, c^{-1}$ as for $0, c, c^2, c^3$. Thus, with either $d = c$ or $d = c^{-1}$, we need only consider the three orderings of $0, d, d^2, d^3$ that begin with 0 and where $d$ precedes $d^3$.

The progression is $0, d, d^2, d^3$. Then $d^2 \equiv 2d \pmod{p}$ and $d^3 \equiv 3d \pmod{p}$, so $d \equiv 2 \pmod{p}$ and $8 \equiv 6 \pmod{p}$, a contradiction.

The progression is $0, d, d^3, d^2$. Then $d^3 \equiv 2d \pmod{p}$ and $d^2 \equiv 3d \pmod{p}$. Thus, $d \equiv 3 \pmod{p}$ and $27 \equiv 6 \pmod{p}$, a contradiction.

The progression is $0, d^2, d, d^3$. Then $d \equiv 2d^2 \pmod{p}$ and $d^3 \equiv 3d^2 \pmod{p}$. Thus, $d \equiv 3 \pmod{p}$ and $3 \equiv 18 \pmod{p}$, a contradiction.

**A5.** Consider a circle $\Omega$ with radius 9 and center at the origin $(0,0)$, and a disk $\Delta$ with radius 1 and center at $(r,0)$, where $0 \le r \le 8$. Two points $P$ and $Q$ are chosen independently and uniformly at random on $\Omega$. Which value(s) of $r$ minimize the probability that the chord $\overline{PQ}$ intersects $\Delta$?

**Answer:** $r = 0$. More generally, if the larger circle has radius $\rho > 1$, then the minimum probability for $0 \le r \le \rho - 1$ occurs at (and only at) $r = 0$.

**Solution 1:** Consider more generally the case that $\Delta$ has center $(r\cos\theta, r\sin\theta)$. The probability $p(r)$ that $\overline{PQ}$ intersects $\Delta$ is independent of $\theta$, so we can compute $p(r)$ by considering $\theta$ to be a random variable chosen uniformly on $[-\pi,\pi]$, independently of $P$ and $Q$.

Next, let $O$ be the origin, and let $\Pi$ be the set of lines through $O$. Let $L$ be the line in $\Pi$ that bisects angle $POQ$. As the angle ray $\overrightarrow{OQ}$ makes with ray $\overrightarrow{OP}$ increases from 0 to $2\pi$, the angle $L$ makes with $\overrightarrow{OP}$ increases half as fast from 0 to $\pi$ (this sweeps through all the lines in $\Pi$). Thus, $L$ is uniformly distributed on $\Pi$ for each fixed $P$. Since $P$ is uniformly distributed on $\Omega$, the ordered pair $(P,L)$ is uniformly distributed on $\Omega\times\Pi$. Since $P$ and $L$ determine $Q$ (specifically, $Q$ is the reflection of $P$ through $L$), we can compute $p(r)$ with respect to the independent uniform random variables $P$, $L$, and $\theta$ (instead of with respect to $P$, $Q$, and $\theta$).

Because of the uniform distribution of $\theta$, the probability that $\overline{PQ}$ intersects $\Delta$ is independent of $L$. Thus, we can fix $L$ to be vertical and compute $p(r)$ with respect to $P$ and $\theta$; then $\overline{PQ}$ is the horizontal line through $P$. By left-right symmetry, we can compute $p(r)$ using the uniform distribution for $P$ on the half of $\Omega$ to the right of $L$. Thus, let $P = (\rho\cos\varphi, \rho\sin\varphi)$ where $\varphi$ is uniformly distributed on $[-\pi/2, \pi/2]$. For fixed $\theta$, the probability that $\overline{PQ}$ intersects $\Delta$ is then the probability that $\rho\sin\varphi$ lies between $r\sin\theta - 1$ and $r\sin\theta + 1$, which is

$$\frac{1}{\pi}\left(\arcsin\left(\frac{r\sin\theta+1}{\rho}\right) - \arcsin\left(\frac{r\sin\theta-1}{\rho}\right)\right).$$

Thus,

$$p(r) = \frac{1}{2\pi^2}\int_{-\pi}^{\pi}\left(\arcsin\left(\frac{r\sin\theta+1}{\rho}\right) - \arcsin\left(\frac{r\sin\theta-1}{\rho}\right)\right)d\theta.$$

It follows that

$$\begin{aligned}
p'(r) &= \frac{1}{2\pi^2}\int_{-\pi}^{\pi}\frac{\sin\theta}{\rho}\left(\frac{1}{\sqrt{1-(r\sin\theta+1)^2/\rho^2}} - \frac{1}{\sqrt{1-(r\sin\theta-1)^2/\rho^2}}\right)d\theta \\
&= \frac{1}{2\pi^2}\int_{-\pi}^{\pi}\sin\theta\left(\frac{1}{\sqrt{\rho^2-(r\sin\theta+1)^2}} - \frac{1}{\sqrt{\rho^2-(r\sin\theta-1)^2}}\right)d\theta.
\end{aligned}$$

The integrand is positive when $0 < r < \rho - 1$ and $\sin\theta > 0$, because then $(r\sin\theta - 1)^2 < (r\sin\theta+1)^2 < \rho^2$. Notice that the integrand is also an even function of $\theta$, since it is the product of two odd functions. Thus, $p'(r) > 0$ for $0 < r < \rho - 1$, and therefore $p(r)$ is minimized at $r = 0$ only.

**Solution 2:** Let $P = (\rho\cos\theta, \rho\sin\theta)$, where $\theta$ is uniformly distributed on $[0, 2\pi)$. Let $B$ be a point on $\Delta$ for which $\overline{PB}$ is tangent to $\Delta$, and let $C = (r,0)$ be the center of $\Delta$. Then $PBC$ is a right triangle, and since length $BC = 1$, we have

$$\sin\angle BCP = \frac{BC}{PC} = \frac{1}{\sqrt{(\rho\cos\theta - r)^2 + (\rho\sin\theta)^2}} = \frac{1}{\sqrt{\rho^2 + r^2 - 2\rho r\cos\theta}}.$$

Let A be the arc between the two tangent rays from $P$ to $\Delta$. For fixed $P$, the conditional probability that $\overline{PQ}$ intersects $\Delta$ is the probability that $Q$ lies in A, which is the angle measure $\alpha$ of A divided by $2\pi$. Notice that $\alpha$ is twice the angle between the tangent rays from $P$ to $\Delta$, and hence $\alpha = 4\angle BCP$. Thus, the conditional probability that $\overline{PQ}$ intersects $\Delta$ is $(2/\pi)\arcsin(1/\sqrt{\rho^2+r^2-2\rho r\cos\theta})$. It follows that the overall probability $p(r)$ that $\overline{PQ}$ intersects $\Delta$ is given by

$$\begin{aligned} p(r) &= \frac{1}{2\pi}\int_0^{2\pi} \frac{2}{\pi}\arcsin\left(\frac{1}{\sqrt{\rho^2+r^2-2\rho r\cos\theta}}\right) d\theta \\ &= \frac{2}{\pi^2}\int_0^{\pi} \arcsin\left(\frac{1}{\sqrt{\rho^2+r^2-2\rho r\cos\theta}}\right) d\theta. \end{aligned}$$

Notice that $p(0) = (2/\pi)\arcsin(1/\rho)$.

Since arcsin is a convex function on the interval $[0,1]$, Jensen's inequality implies that

$$p(r) \geq \frac{2}{\pi}\arcsin\left(\frac{1}{\pi}\int_0^{\pi} \frac{1}{\sqrt{\rho^2+r^2-2\rho r\cos\theta}}\, d\theta\right).$$

The proof that $p(r) > p(0)$ for $0 < r \leq \rho - 1$ will be complete after we prove the following claim for such $r$:

$$\frac{1}{\pi}\int_0^{\pi} \frac{1}{\sqrt{\rho^2+r^2-2\rho r\cos\theta}}\, d\theta > \frac{1}{\rho}.$$

This claim turns out to be true for $0 < r < \rho$, in fact. Let $x = r/\rho \in (0,1)$; multiplying the inequality above by $\rho\pi$ yields the equivalent claimed inequality

$$\int_0^{\pi} \frac{1}{\sqrt{1+x^2-2x\cos\theta}}\, d\theta > \pi.$$

Let $t = (\sqrt{1+x^2-2x\cos\theta}-1)/x$, so that $1+x^2-2x\cos\theta = (1+xt)^2$, and notice that $t$ goes from $-1$ to $1$ as $\theta$ goes from $0$ to $\pi$. To change variables from $\theta$ to $t$ in the integral above, we compute $2x\sin\theta\, d\theta = 2x(1+xt)dt$, and

$$\begin{aligned} 2x\sin\theta = 2x\sqrt{1-\cos^2\theta} &= 2x\sqrt{1-\left(\frac{1+x^2-(1+xt)^2}{2x}\right)^2} \\ &= \sqrt{(2x+1+x^2-(1+xt)^2)(2x-1-x^2+(1+xt)^2)} \\ &= \sqrt{(1+x+(1+xt))(1+x-(1+xt))(1+xt+(1-x))(1+xt-(1-x))} \\ &= \sqrt{(2+x+xt)x(1-t)(2-x+xt)x(1+t)} = x\sqrt{1-t^2}\sqrt{(2+xt)^2-x^2}. \end{aligned}$$

Thus,

$$\begin{aligned} \int_0^{\pi} \frac{1}{\sqrt{1+x^2-2x\cos\theta}}\, d\theta &= \int_{-1}^{1} \frac{1}{1+xt}\cdot\frac{2(1+xt)}{\sqrt{1-t^2}\sqrt{(2+xt)^2-x^2}}\, dt \\ &= \int_{-1}^{1} \frac{2}{\sqrt{1-t^2}\sqrt{(2+xt)^2-x^2}}\, dt. \end{aligned}$$

Let $f_x(t) = (2+xt)^2 - x^2$, so that the integrand above can be written $2f_x(t)^{-1/2}/\sqrt{1-t^2}$. Since the function $y \mapsto y^{-1/2}$ on the positive real numbers is convex,

$$\frac{f_x(t)^{-1/2} + f_x(-t)^{-1/2}}{2} \geq \left(\frac{f_x(t) + f_x(-t)}{2}\right)^{-1/2} = (4 + x^2t^2 - x^2)^{-1/2} > \frac{1}{2}$$

for $0 < x < 1$ and $-1 < t < 1$. Thus,

$$\int_{-1}^{1} \frac{2f_x(t)^{-1/2}}{\sqrt{1-t^2}}\,dt = \int_0^1 \frac{2(f_x(t)^{-1/2} + f_x(-t)^{-1/2})}{\sqrt{1-t^2}}\,dt > \int_0^1 \frac{2}{\sqrt{1-t^2}}\,dt = \pi$$

as claimed.

**A6.** Let $c_0, c_1, c_2, \ldots$ be the sequence defined so that

$$\frac{1 - 3x - \sqrt{1 - 14x + 9x^2}}{4} = \sum_{k=0}^{\infty} c_k x^k$$

for sufficiently small $x$. For a positive integer $n$, let $A$ be the $n$-by-$n$ matrix with $i, j$-entry $c_{i+j-1}$ for $i$ and $j$ in $\{1, \ldots, n\}$. Find the determinant of $A$.

**Answer:** $10^{(n^2-n)/2}$.

**Solution 1:** More generally, let

$$F(x) = \frac{1 - \alpha x - \sqrt{(1 - \alpha x)^2 - 4\beta x}}{2\beta} = \sum_{k=0}^{\infty} c_k x^k.$$

We show that the determinant of the $n \times n$ matrix defined as in the problem statement is $(\beta(\alpha + \beta))^{(n^2-n)/2}$. When $\alpha = 3, \beta = 2$, we get the problem statement.

By the quadratic formula, $F(x)$ is a root of

$$\beta F(x)^2 + (\alpha x - 1)F(x) + x = 0.$$

From its definition, observe that $c_0 = F(0) = 0$. Examining the coefficient of $x^n$ in the functional equation, we find

$$c_n = \begin{cases} 1, & \text{if } n = 1, \\ \alpha + \beta, & \text{if } n = 2, \\ (\alpha + 2\beta)c_{n-1} + \beta \sum_{k=2}^{n-2} c_k c_{n-k}, & \text{if } n > 2. \end{cases}$$

(We use the convention that a sum with strictly decreasing limits of summation is 0.)

Thus, the 1-by-1 matrix has determinant 1 and the 2-by-2 matrix has determinant

$$(\alpha + 2\beta)(\alpha + \beta) \cdot 1 - (\alpha + \beta)^2 = \beta(\alpha + \beta).$$

We proceed by induction; assume the claim for some $n \geq 2$ and consider the $(n+1)$-by-$(n+1)$ matrix.

From row $n + 1$, subtract $\alpha + 2\beta$ times row $n$ and $\beta c_{n+1-k}$ times row $k$ for rows $k = 2, \ldots, n - 1$. The entry in row $n + 1$, column $j$ is now

$$\begin{aligned} c_{n+j} - (\alpha + 2\beta)c_{n+j-1} - \beta \sum_{k=2}^{n-1} c_{n+1-k}c_{j+k-1} &= c_{n+j} - (\alpha + 2\beta)c_{n+j-1} - \beta \sum_{k=2}^{n-1} c_k c_{n+j-k} \\ = \beta \sum_{k=n}^{n+j-2} c_k c_{n+j-k} &= \beta \sum_{k=1}^{j-1} c_{k+n-1}c_{1+j-k}. \end{aligned}$$

Next, reduce rows $n, n - 1, \ldots, 3$ similarly. Finally, subtract $\alpha + \beta$ times row 1 from row 2, so the entry in the $j$th column of row 2 is now

$$\beta \sum_{k=1}^{j-1} c_k c_{1+j-k}.$$

At this point, column $j$ of rows 2 through $n+1$ is the column vector

$$\beta \sum_{k=1}^{j-1} \begin{bmatrix} c_k c_{1+j-k} \\ c_{k+1} c_{1+j-k} \\ \vdots \\ c_{k+n-2} c_{1+j-k} \\ c_{k+n-1} c_{1+j-k} \end{bmatrix} = \beta \sum_{k=1}^{j-1} c_{1+j-k} \begin{bmatrix} c_k \\ c_{k+1} \\ \vdots \\ c_{k+n-2} \\ c_{k+n-1} \end{bmatrix}.$$

Therefore, the entire first column is now the standard basis vector $\mathbf{e}_1$. The determinant of the reduced matrix (which is the same as the determinant of the original matrix) is then the determinant of its lower right $n$-by-$n$ submatrix. Pull the factor $\beta^n$ out of the determinant of the submatrix. Noting that, for $j \geq 3$,

$$\sum_{k=1}^{j-2} c_{1+j-k} \begin{bmatrix} c_k \\ c_{k+1} \\ \vdots \\ c_{k+n-2} \\ c_{k+n-1} \end{bmatrix}$$

is in the span of the columns $2, \ldots, j-1$ of this submatrix, we may reduce its columns from left to right, yielding $c_2 = (\alpha + \beta)$ times the original $n$-by-$n$ matrix. Therefore, the determinant for $(n+1)$-by-$(n+1)$ matrix is $\beta^n(\alpha+\beta)^n$ times that for the $n$-by-$n$ matrix, completing the induction.

**Remark.** Letting $A_n$ denote the $n$-by-$n$ matrix, the row and column reductions above can be summarized as follows:

$$
\begin{bmatrix}
1 & & & & & & & \\
-(\alpha+\beta) & 1 & & & & \mathbf{0} & & \\
0 & -(\alpha+2\beta) & 1 & & & & & \\
0 & -\beta c_2 & -(\alpha+2\beta) & 1 & & & & \\
 & & & & & \ddots & & \\
0 & -\beta c_{n-2} & -\beta c_{n-3} & \dots & -\beta c_2 & -(\alpha+2\beta) & 1 & \\
0 & -\beta c_{n-1} & -\beta c_{n-2} & \dots & -\beta c_3 & -\beta c_2 & -(\alpha+2\beta) & 1
\end{bmatrix}
$$

$$
\cdot A_{n+1} \cdot
\begin{bmatrix}
1 & 0 & 0 & & & 0 \\
 & 1 & -c_3/c_2 & -c_4/c_2 \ \dots & -c_n/c_2 & -c_{n+1}/c_2 \\
 & & 1 & -c_3/c_2 \ \dots & -c_{n-1}/c_2 & -c_n/c_2 \\
 & & & \ddots & & \\
 & & \mathbf{0} & & & \\
 & & & & 1 & -c_3/c_2 \\
 & & & & & 1
\end{bmatrix}
$$

$$
= \left[\begin{array}{c|ccc}
c_1 & c_2 & \dots & c_{n+1} \\
\hline
0 & & & \\
\vdots & & \beta(\alpha+\beta)A_n & \\
0 & & &
\end{array}\right].
$$

**Solution 2:** We will show that $A = LDL^t$ where $[L_{i,j}]_{i,j=0}^{n-1}$ is a lower triangular matrix with 1's on the diagonal and $[D_{i,j}]_{i,j=0}^{n-1}$ is a diagonal matrix with $D_{k,k} = 10^k$. (Here we start the indexing at 0.) Then $\det A = \det(L)^2 \det(D) = 1 \cdot 10^{0+1+\cdots+(n-1)} = 10^{\binom{n}{2}}$.

Let $F(x) = \frac{1-3x-\sqrt{1-14x+9x^2}}{4} = x + 5x^2 + O(x^3)$. We have that $(4F(x)+3x-1)^2 = 1 - 14x + 9x^2$, so

$$2F(x)^2 + (3x-1)F(x) + x = 0.$$

Let $f(u) := F(u)/u = 1 + 5u + O(u^2)$ and $g(u) := \frac{F(u)-u}{5u} = u + O(u^2)$, and define $L_{i,j}$ for $i, j = 0, 1, \dots$ as the coefficients of the expansion of

$$\ell(u,v) = \sum_{i,j\geq 0} L_{i,j} u^i v^j = \frac{f(u)}{1 - vg(u)} = f(u)\left(1 + vg(u) + v^2 g(u)^2 + \cdots\right).$$

We see that $L_{i,j} = 0$ for $i < j$ and $L_{i,i} = 1$. Consider $B = LDL^t$ where $D_{k,k} = d^k$ for $k \geq 0$ and all other entries of $D$ are zero. Then the entries of $B$ are

$$B_{i,j} = \sum_{k\geq 0} L_{i,k} D_{k,k} L_{j,k}.$$

(Only a finite number of terms in the sum above are nonzero, because $L$ is lower triangular.) Denote by $[z^k]H(z)$ the coefficient of $z^k$ in the expansion of $H(z)$. Set

$$\begin{aligned} b(u,v) &= \sum_{i,j\geq 0} B_{i,j}u^i v^j = \sum_{k\geq 0}[w^k t^k] \sum_{i,j,r,s\geq 0} L_{i,r}u^i w^r d^r L_{j,s} v^j t^s \\ &= \sum_{k\geq 0}[w^k t^k]\ell(u,dw)\ell(v,t) = f(u)f(v)\sum_{k\geq 0}[w^k t^k]\sum_{r,s\geq 0} d^r w^r g(u)^r t^s g(v)^s \\ &= f(u)f(v)\sum_{k\geq 0} d^k g(u)^k g(v)^k = \frac{f(u)f(v)}{1-dg(u)g(v)}. \end{aligned}$$

The lower-triangular property $L_{i,j}=0$ for $i<j$ implies that $B_{i,j}$ depends only on values of $L_{i,k}$, $D_{k,k}$, and $L_{j,k}$ with $k\leq i$ and $k\leq j$. Thus, the equation $B = LDL^t$ holds also for the finite matrices $[B_{i,j}]_{i,j=0}^{n-1}$, $[L_{i,j}]_{i,j=0}^{n-1}$, and $[D_{i,j}]_{i,j=0}^{n-1}$. The following claim proves the desired decomposition for $A$, and finishes the solution.

**Claim.** *For $d=10$, we have $A_{i+1,j+1}=B_{i,j}$.*

*Proof.* Define $A_{i,j}=c_{i+j-1}$ for all $i,j=1,2,\ldots$, and let

$$\begin{aligned} a(u,v) &= \sum_{i,j\geq 0} A_{i+1,j+1}u^i v^j = \sum_{i,j\geq 0} c_{i+j+1}u^i v^j \\ &= \sum_{r\geq 0} c_{r+1}(u^r + u^{r-1}v + \cdots + v^r) = \sum_{r\geq 0} c_{r+1}\frac{u^{r+1}-v^{r+1}}{u-v} = \frac{F(u)-F(v)}{u-v}, \end{aligned}$$

since $c_0=0$.

Next, consider

$$\begin{aligned} a(u,v)-b(u,v) &= \frac{F(u)-F(v)}{u-v} - \frac{f(u)f(v)}{1-10g(u)g(v)} \\ &= \frac{F(u)(1-10g(u)g(v)) - uf(u)f(v) - F(v)(1-10g(u)g(v)) + vf(u)f(v))}{(u-v)(1-10g(u)g(v))}. \end{aligned}$$

We have that $F(u)=\frac{F(u)-u}{2F(u)+3u}$, and so the first half of the numerator above is

$$\begin{aligned} F(u)(1-10g(u)g(v)) - uf(u)f(v) &= F(u)(1-10g(u)g(v)-f(v)) \\ &= \frac{(F(u)-u)}{2F(u)+3u}\left(1-2\frac{(F(u)-u)(F(v)-v)}{5uv} - \frac{F(v)}{v}\right) \\ &= \frac{(F(u)-u)}{(2F(u)+3u)(5uv)}(5uv - 2F(u)F(v) + 2vF(u) + 2uF(v) - 2uv - 5uF(v)) \\ &= \frac{(F(u)-u)}{(2F(u)+3u)(5uv)}(3uv - 2F(u)F(v) + 2vF(u) - 3uF(v)) \\ &= \frac{(F(u)-u)(F(v)-v)(-2F(u)-3u)}{(2F(u)+3u)(5uv)} \\ &= -\frac{(F(u)-u)(F(v)-v)}{5uv}. \end{aligned}$$

The second half of the numerator is the negative of the first half, with $u$ and $v$ interchanged, so performing the same manipulations on the second half verifies that it cancels with the first half. Thus, $a(u,v)=b(u,v)$ and the claim is proved. □

**Remark.** This problem was inspired by the determinants of Hankel matrices used to count tilings of the Aztec diamond (`https://en.wikipedia.org/wiki/Aztec_diamond`).