notes

THE COURSES OF HISTORY:
IDEAS FOR DEVELOPING A HISTORY OF MATHEMATICS COURSE

Amy Shell Gellasch
Dick Jardine
Editors

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The Courses of History

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Ideas for Developing a History of Mathematics Course

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MAA

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Introduction

Teaching a history of mathematics (HoM) course is a great opportunity to expose a wide variety of students to the breadth, depth, and importance of mathematics. As Judy Grabiner wrote in her contribution to this volume, "Students need to understand that history isn't about 'who did what and when?' but about 'how did people make these choices and why?' " It is our goal in compiling this volume to provide useful examples and suggestions to enable both novice and experienced HoM instructors help students achieve that understanding.

The authors of the various chapters of this volume are well established in the HoM and experienced in its teaching. We have grouped the sections of the volume according to the type, level, and audience of the course. The first section contains courses designed with mathematics majors as the target students. The next section contains a collection of courses devoted to students who may be majoring in mathematics but intend to be teachers upon graduation. Some of us teach HoM courses developed for the general education of students, and those courses are found in the third section. More and more of us are developing courses with a significant online component, whether required to by administrators or of our own volition, and examples of those can be found in the fourth section. The last section contains descriptions of courses of a specialized nature, courses addressing specific branches of mathematics, time periods, or geographical regions. Some of the chapters could have been placed in several sections. Certainly there are similarities in the content of the courses described; after all, they are all HoM courses. However, each of the chapters is distinguished by elements that make it different from the others.

For the benefit of the reader, we challenged the contributors to structure their contributions in a standard format. To this end, each chapter has sections devoted to the course overview, course design, resources, assignments, lessons learned, references, and appendices providing necessary details, to include a schedule of how the course is implemented in a term or semester. For the most part, readers will find that format throughout, though some of the courses did not lend themselves to that structure.

Because most HoM courses are designed for mathematics majors, we lead with the section devoted to courses for those students. Judy Grabiner offers a course in which those students are challenged to wrestle with philosophical issues of mathematics from ancient Egypt to the current century as well as how to do the mathematics as it was done in various periods of history. Danny Otero has long been an advocate of the use of primary sources, so it comes as no surprise that his contribution emphasizes their use in helping students learn the HoM. Capstone courses have become an important part of the mathematics curriculum at many colleges and universities, and the last three offerings in this section were developed as capstones for the mathematics major. Like Danny, David Pengelley emphasizes the inclusion of primary sources in his capstone course. Betty Mayfield describes a capstone experience that involves students in a two-semester sequence, with the first semester designed to tie together students' undergraduate mathematics with a HoM course and a research project due the second semester. Larry D'Antonio's capstone course uses geometric construction as a unifying theme for various branches of mathematics that students have encountered as undergraduates.

The second section of our volume includes descriptions of HoM courses principally for pre-service or in-service teachers. There certainly is some overlap with the previous section, as many mathematics majors intend to become teachers and have taken the courses described previously. Similarly, many mathematics majors who do not intend to become teachers have been students in the courses in this section. Janet Barnett contributes the description of a course that provides course questions that help students develop their understanding of the rich traditions of mathematics. Charles Lindsey's course focuses on the historical background for the mathematical topics taught in high school, but emphasizes the multicultural context. Sarah Greenwald and Gregory Rhoads describe an intensive HoM experience for teachers that is completed within two weeks. Amy Shell-Gellasch describes a course in which she engages pre-service teachers in deep discussions about the material, challenging their views on mathematics and education while also requiring them to present their learning in a variety of modes. Dominic Klyve presents a course intent on developing students as scholars in the HoM as they learn the subject. The innovative in-class activities differentiate Christine Latulippe's offering from others, as do the varied assignments described in Diana White's course cross-listed for undergraduate and graduate students.

Some of us have taught and continue to teach HoM to students with little mathematical preparation beyond the topics typically taught in high school. If the course of interest to you is a general education course or a course that requires no upper-level mathematics, then the seven chapters of the third section fit that description. Janet Beery describes her activity-based elementary mathematics history course for non-science students in the first offering in this section. College algebra may be the only mathematics course a student takes to fulfill the mathematics graduation requirement at some institutions, and Amy Shell-Gellasch tells us a way to teach college algebra with its history. The theme of the perspectives course that Dorothee Blum provides is to connect topics from older mathematics with newer mathematics of the same topic throughout the course. Dorothee's course is interdisciplinary by design, as is Dick Jardine's course which not only emphasizes connections to other disciplines but includes faculty from other fields as guest lecturers for his students. While many of us welcome humanities students to our HoM course, Glen Van Brummelen describes the design of a course that targets that audience and includes potential team-teaching with a humanities faculty member. Matthew Hallock and Alex McAllister provide an example of just such a team-teaching experience in which a drama professor and a mathematics professor weave ancient Greek theater and mathematics together. To round out this section, David Pengelley entices students from other disciplines to his general education course offering, one that has had success attracting students to the mathematics major.

Because of the growth in the number of online and blended learning courses, it is important that we include examples of those in this volume. Kathy Clark contributes her experiences in teaching a graduate-level, blended-learning course for teachers. She describes a blended-learning course, one that is taught partially face-to-face and partially online, but she has taught the course both totally online and completely face-to-face. Tammy Muhs teaches a HoM course entirely online, as does her colleague Joseph Brennan. Her offering includes how she incorporates student projects and mathematical exercises in an online course, while Joseph's contribution provides many helpful suggestions on developing an online HoM course.

As mentioned earlier, the last section of the volume presents the experiences of those who have developed HoM courses on specific topics. The first offering in this section communicates how Chris Baltus uses geometry to strengthen pre-service secondary teachers' mathematical preparation, providing the historical foundations for the subjects they will be teaching. Owing to his great experience with specialized HoM courses, Stacy Langton provides three courses: a history of calculus, a history of probability, and one focused on Newton's foundational work on mathematical physics. Jill Thomley developed a history of statistics seminar, an important contribution considering the proliferation of statistics courses at the undergraduate level. David Pengelley continues his emphasis on primary sources, sharing his experiences implementing HoM courses emphasizing discrete mathematics, combinatorics, number theory, and geometry. Janet Beery provides a course integrating HoM and number theory sufficient to prepare preservice school teachers in both subjects. Clemency Montelle and John Hannah provide a wonderfully innovative and graphically appealing course employing diagrams as a theme to explore the evolution of mathematical developments through the ages. Stephen Kennedy's contribution describes a course in which students relive the skeptical acceptance of complex numbers, exploring the drama involved in the discovery through reading primary sources written by the principals involved. While complex numbers proved to not be impossible, Alex McAllister provides the description of a course on mathematical impossibilities, in which students learn the difference between open questions and impossible questions, and the mathematical foundations of those that are impossible. Joe Albree's course on the HoM in America brings the subject closer to home to our students, a second course for students that serves as an example of a course offering specialization in a specific time period and geographic region.

Joe's course is also typical of each of the contributions to this volume in that it can be replicated at the reader's institution in every detail or can be modified extensively. When considering the courses described in this volume, the reader intending to develop a HoM course can follow each author's suggestions or take them as general guidance that serve as a point of departure for adaptation according to personal pedagogical philosophy, mathematical/historical interests, or comfort zone. Between us, the editors have taught the HoM almost every year for more than the past 15 years, and every year our courses change as new and interesting material is discovered and innovative instructional ideas and resources are made available. Benefitting from reviewing all the great ideas presented by our contributors, one of us modified a HoM course taught last fall, extensively incorporating ideas presented by contributors to this volume. The purpose of compiling this volume is to serve the needs of both new and experienced instructors. We are confident that readers will find a wide variety of useful experiences and best practices that will contribute to the successful delivery of a HoM course appropriate for their students.

Contents

I

Courses for Mathematics Majors

Student Excursions Beyond the Textbook in a Survey Course

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Course Overview

Why a course in the history of mathematics? Because it explains what mathematics is. Mathematics is created or discovered by human beings from a variety of backgrounds. To appreciate this, it is necessary, though not sufficient, to get the history right. So a survey course needs a pedagogically sound and historically reliable textbook. The choice of textbook depends on the preparation of the students. The course described here requires at least a year of university-level calculus, and some of the students have far more. But the excursions beyond the textbook, described in the section "Assignments: Five Excursions Beyond the Textbook" (pp. 5–8 below), can be used for students at a wide variety of levels of preparation. The "excursions" approach can be used in courses with minimal prerequisites as well.

These "excursions" humanize my survey course. The key idea behind them is to develop a broader context for the mathematics by interrupting the forward motion through history at strategic points, by means of readings, exercises, projects, and in-class discussion. The excursions include ethnomathematics, philosophy, and material pertaining to the twentieth century and beyond. The students grapple with the philosophical questions that have surrounded mathematics for thousands of years. They get a sense of the social and cultural background of key mathematical achievements so that all students can see both compatriots and ancestors, and also people unlike themselves, all engaged in the same endeavor. Finally, they will get a sense of the variety of ways different groups of people—crucially, including the students themselves—do mathematics.

Course Design

Given the prerequisites, the student audience is mathematically prepared to be taken from ancient Egypt and Babylon through the invention of the calculus and eighteenth-century mathematics. The common readings end with the rigorization of the calculus. (Those who have not encountered deltas and epsilons, experience suggests, understand them far better when they meet them first in historical context.) More modern mathematics is covered by giving the individual students some choice of readings and topics.

In the classroom, the time is split between lectures and discussions. There are topics that, even though they're covered in the text, seem so important that presenting the instructor's take on them seems essential. Student questions are encouraged, and the instructor provides some that help provoke discussion. For example, after looking at the way the seventeenth-century predecessors of the invention of the calculus found areas, volumes, tangents, maxima and minima, and arc lengths, one can ask, "What is there left, then, for

Newton and Leibniz to do?" or "Couldn't Fermat already have passed the placement test to get into this course?"

Resources

The textbook chosen is Victor Katz's *A History of Mathematics: Brief Edition* [**9**]. It is an abridgment of Katz's longer *A History of Mathematics: An Introduction*, 3d edition, 2009 [**8**], and I "abridge" it more in assignments, but I share Katz's view that it is best to make one's own selections from something sound and detailed. Katz is a highly respected scholar, and his book is thorough, balanced in its historical judgments, and accurate. He covers the historical and social background of important mathematics. And he describes many important and influential works from the history of mathematics in enough detail so that the reader is brought close to the style and thought of the original.

Katz's book has excellent exercises to draw on. Some are problems to be done in the style of the historical epoch; some are problems from the past for students to solve using more modern tools; some require the student to work through an original argument from the past based on the text; some ask broad historical or philosophical questions; and some—for prospective teachers the most valuable—ask how the student would use a particular historical approach to teach a mathematical topic today. And the book has appendices useful both to the teacher and the student: a pronunciation guide in the index, an extensive general bibliography, a list of useful and reliable websites, a historical timeline, and eight pages on ways of using history of mathematics to teach the subject. One must be aware, though, that the book is sometimes tough going, and the information sometimes very dense from the student's point of view.

There are three other books whose purchase is required, each of which gives the students an important perspective on the nature of mathematics. The first, Plato's dialogue *Meno* [**12**], illustrates the philosophically important idea that truth can be reached by logical reasoning, as exemplified by Greek geometry—an idea that explains how mathematics has so long been central to liberal arts education [**6**]. The second, Descartes' *Discourse on Method* [**3**], embodies the new scientific approach, marks the flowering of the analytic method in philosophy, and is intimately linked with Descartes' and Fermat's independent invention of analytic geometry, the revolutionary merging of Renaissance symbolic algebra with classical geometry that made the calculus possible. The third, Hadamard's *Psychology of Invention in the Mathematical Field* [**7**], draws on the experience of creative mathematicians to address how mathematicians think about what they are doing. All three of these books are fairly short and readable, and produce lively class discussion. Details on how these discussions proceed in the context of the course are found in the section "Assignments: Five Excursions Beyond the Textbook" below.

Whatever the level of one's prospective students, I would warmly recommend that those wanting to teach any course in the history of mathematics, whatever approach or textbook they choose, read through the five hundred pages of Katz's "brief" textbook [**9**]**,** or, even better, the 900 pages of his longer version *A History of Mathematics: An Introduction* [**8**]. Working through Katz's books with care makes one quite well informed about all aspects of the subject, from ancient times to the present day.

For the instructor's further reading, one might also want to supplement Katz's book with material published more recently. The MAA's "Convergence" website [**2**] is both of scholarly value and attractive to students. Another important recommendation is to look at other proof traditions than the Euclidean in Karine Chemla, ed., *The History of Mathematical Proof in Ancient Traditions*, Cambridge University Press, Cambridge, 2012 [**1**].

Especially if the instructor plans to work with English translations of important mathematical writings, Benjamin Wardhaugh, *How to Read Historical Mathematics*, Princeton: Princeton University Press, 2010 [**15**] is highly recommended. The most valuable collections of primary sources in translation are Dirk J. Struik, A *Source Book in Mathematics, 1200–1800* (extensive selections, superbly annotated) [**14**], Harvard, 1969; John Fauvel and Jeremy Gray, eds., *The History of Mathematics: A Reader* (a wide variety of well-chosen short selections with introductory remarks), Macmillan, 1987 [**5**]; Victor J. Katz, ed., *The Mathematics of Egypt, Mesopotamia, China, India, and Islam: A Sourcebook*, Princeton University Press, Princeton, 2007 [**10**]; and Victor J. Katz, Menso Folkerts, Barnabas Hughes, Roi Wagner, and J. L. Berggren, eds., *Sourcebook in the Mathematics of Medieval Europe and North Africa,* Princeton University Press, Princeton, 2017 [**11**].

In addition, the Course Outline (Appendix A) includes a wide range of recommended readings, arranged by topic. More resources for the excursions on Ethnomathematics and Women in Mathematics are found in Appendices C and D, respectively

For more details of how the course is structured, see the sections on Assignments below, and the Course Outline and Syllabus, Appendices A and B, at the end of this article.

Assignments: Problems, Essays, Tests

Given the excellent problems and exercises in Katz's book, I let the students choose which of them to do. Letting the students choose is worthwhile even though it makes grading the assignments less routine. The publisher provides an answer key for the mathematics problems. The students are asked to do three problems a week, and one essay question. The students appreciate being able to choose their problems and essay topics. My written comments on essay questions turn the process into a dialogue between me and each student. Some students use only the textbook to do the essays, but others go and do additional research (and some of Katz's essay questions ask for such research).

Besides the problems and exercises, every student hands in, in writing, "two enlightening sentences" for every reading assignment, at the beginning of the class devoted to that reading. By "enlightening sentences" is meant, not saying something like "Archimedes was from Syracuse and did a lot of cool geometry," but "When Archimedes uses the law of the lever to find areas he seems less a mathematician and more an engineer who cuts out an area from a metal sheet and weighs it" would qualify. Students catch on as the course proceeds and say striking things. Comments are provided to each student, and especially instructive examples (without naming the student) are shared with the entire class. These practices both give each student another chance for a private dialogue with me, and help expand the range of ideas that the class gets to see.

Except for the most motivated, the students do not read every word and go through every mathematical argument in Katz's book, but this very brief writing assignment gets them to grapple with some important things from every historical period and culture, and to think about the material before class discussion.

The final examination for the course includes short-answer questions of the form "Identify and say why it's important." Students also answer two essay questions that require synthesis. Students are given study questions before the exam, and are also given some choice of essay topics on the exam itself. This type of final gets them to study and pull the material together, and to review the important contributions of individual mathematicians and the key ideas. Still, more than half the course grade is based on the assigned mathematics problems, discussion questions from the text, enlightening sentences, and contributions to class discussions, because no three-hour examination can capture the variety of ideas and events covered in the course.

What I'm selling in this article, though, is what I do beyond the relatively conventional survey course described so far. I find that these excursions work well with a wide variety of students, and am happy to share them with others. This particular approach, and these assignments, can be used no matter what one's textbook, level of prerequisites, or general approach to the subject may be.

Assignments: Five Excursions Beyond the Textbook

At strategic points in the course, the students are invited to consider five important issues in the history of mathematics in depth. Here's how it works.

(a) Ancient Greek mathematics is distinguished from other ancient traditions by its justifying even obvious results by means of logical proof from self-evident assumptions, and using proof by contradiction in mathematics. Often the only time high school students today learn logic is in their geometry class, and it is there, if at all, that they learn the powerful intellectual tool of counterfactual reasoning. The idea that truth could be achieved by logical reasoning—exemplified in geometry—influenced many philosophers, and is historically why mathematics has so long been a part of liberal education [**6**].

Students appreciate these facts best not by being told what happened, but by becoming immersed in the process themselves. So we read Plato's dialogue *Meno* [**12**]. The *Meno* is short, it's fun, it's available cheaply (and also on-line), it addresses "hypothetical reasoning" and proof by contradiction, and it also addresses broad philosophical questions, like the nature of virtue and whether one can teach people to behave ethically. (It also has a famous example of teaching geometry by the Socratic method.) Starting class discussion with the question "What is this dialogue about?" and making a list of student answers quickly gets the major issues up on the blackboard. A key pedagogical point is that one can teach the Pythagorean theorem for isosceles right triangles by having Socrates ask questions about the diagram so that the young man in the dialogue discovers the answer by himself. Class discussions are wide-ranging and lively, and the students come to understand the relationship between Greek mathematics and Greek philosophy.

(b) Once Katz's chapters on mathematics in India, China, the Islamic world, and medieval Europe have been covered, a reading is added on Ethnomathematics: "Mathematics around the World," Chapter 11 of Katz's longer book of 2009 [**8,** 364–381]. This chapter serves as an introduction, with important examples, of how different cultures, including some traditional societies without writing, have developed interesting mathematics to solve problems that really matter in these cultures.

Each student then does a very short in-class report on a piece of mathematics from a culture other than the Greek or the modern European. Students can choose their topics, and are given a range of bibliographic suggestions (see Appendix C). For this assignment, if more than one student chooses a particular topic, that's fine; it suggests great student interest in that topic. Students can emphasize either the cultural or the mathematical aspects in their reports, as long as both are included. Of course many more sources are available, but I warn them to be careful and encourage them to be critical if they choose to use Web resources.

For the in-class presentation, the amount of time per presentation is determined—and rigorously enforced with a kitchen timer—by dividing the length of two class sessions, minus time for discussion, by the number of students. Even a four-minute report can be remarkably instructive when it is one of a large set of such reports. Then the class can have a highly informed discussion about how mathematics and culture relate. At the end of each class session of reports, the students are asked to write briefly (2–3 sentences total) about which report by a classmate they found most illuminating, and to write their own general conclusions about what we have heard on that day. They get these responses back the next day with my comments. The most common generalization I get is some version of "I never understood how universal mathematics is in all kinds of societies," with "It's amazing how symmetry (or geometry or combinatorics) is used for different purposes in different cultural groups" being a close second.

(c) When the course reaches the Renaissance and the seventeenth century in Europe, the class sees that the birth of "modern" mathematics involves both the invention of general symbolic algebra and its application to solving geometric problems. Once we have looked at the problem-solving power of Descartes' and Fermat's independent invention of analytic geometry, and the power of general symbolic notation to reveal the structure of algebraic solutions, the students are ready to see how this mathematical revolution interacted with philosophy and science. For this, we read Descartes' *Discourse on Method* of 1637 [**3**].

The *Discourse* is short and easily available in many English editions, and is a foundational work in modern philosophy. We find in it, among other things, Descartes' analytic method, his well-known proof of his own existence, his discussion of the right way to do science, and, crucially for this course, his claim that following the method of mathematics could reveal "all truths knowable to man." Again, the discussion is started with the instructor's question, "What's this about?" Once we've identified a range of topics on the board, the students especially enjoy trying to shoot down Descartes' arguments, which is a good way to find out both how the arguments work and what evidence there might be in the seventeenth century to support those arguments.

(d) When the class reaches the end of the common mathematical readings in the course, the students are ready to look at the question of mathematical creation. The reading for this is Jacques Hadamard, *The Psychology of Invention in the Mathematical Field* [**7**]. I begin the discussion by assigning an in-class problem for them all to do. In my experience, the following problem from Polya's *How to Solve It* [**13**] is ideal, since almost everyone solves it before five minutes have elapsed: "Given a regular hexagon and an arbitrary point in the same plane, to draw a line through the point that divides the hexagon into two parts having equal area." The purpose of asking them all to do it is not, needless to say, to find the solution; it's to see how everybody found it.

Once virtually everyone has solved the problem, the discussion begins with my asking the students how they did it, first with leading questions: Did you draw a physical diagram? If so, raise your hand. (Those who didn't can't imagine how others need to, and vice versa.) Did you imagine anything moving? (Those who did thought it was natural to rotate the hexagon or to rotate the line around the point; those who didn't are amazed by that approach.) Did you use analytic methods in any way? (Those who didn't are in the overwhelming majority, and say "how in the world could you use analytic methods on this problem?" But some say they did.) Did you explicitly think about symmetry? Did you imagine folding the plane? (Chemistry majors are more likely to have done that.) Were you motivated by the analogous problem with the circle? (That's the way Polya suggests solving the problem, but in my experience most students don't do it that way.) How many of you consciously used words when you thought about the problem? How many didn't? (Again, the students marvel at those whose choice about words was different from their own.). Then I ask whether anybody had some other method I didn't ask about, and sometimes they do. (Once, notably, a student said, "Aristotle thought that the center of the universe was a special point so I asked myself whether there is a special point in the hexagon." This student went on to become a historian specializing in Greek mathematics.) Finally, I ask, "How many of you were sure your answer was right once you had solved the problem?" Although occasionally students try to prove, or to generalize, their solution, for most of them it was enough to have the conviction, "Aha! That's the right answer."

At this point, the ideas in Hadamard's book come alive. The students are astonished by the diversity of approaches to mathematical discovery among their classmates, even on an elementary problem among students with more or less the same mathematical education. Also more meaningful after this exercise is Poincaré's description, quoted by Hadamard, of mathematical discovery as a period of conscious work followed by sudden illumination. This discussion broadens into talking about how the students themselves actually work on mathematics when they're doing non-routine homework. Hadamard's account, the historical examples from the course, and their own experiences all support the conclusion that the highest levels of mathematical creation are punctuated by errors and false starts and fruitless thrashing around before the "Aha!" This lets the students see themselves as engaged in the same enterprise as creative mathematicians.

(e) Near the end of the course, we look at the history of women in mathematics. Again, each student does a very short report, typically limited to two minutes. As resources, they are given a list of women mathematicians (including those on the MAA poster "Women of Mathematics" [**4**]) and a few further topics, including interviewing women faculty members here in Claremont and the statistics about women in mathematics over time; these resources include particular websites and books (See Appendix D). The students can choose their own topics, with the restriction that there be no overlaps. The biographical reports are supposed to focus on out how the woman got interested in mathematics, how she got her mathematical education, and what her principal achievements were. The interviews and the statistics help round out the historical picture that the student reports create. After all of the reports, there is a discussion.

Typical questions I use in this discussion are to ask what general trends the students see from most of the biographies, whether there were one or a few individuals who made a difference in the students' own interest in mathematics, and what the students thought could have helped them or their classmates in the light of the experiences we've learned about.

Lessons Learned

What works and what could be improved?

All of these "excursions" intend to link mathematics to the rest of human activity, and to help the students see themselves in the world of mathematics. Students participate in some key interactions between the development of mathematics and the rest of human life. The students seem to learn a lot, and to enjoy these assignments. And my classes continually amaze me with questions I have not thought about. This is good for us all.

There are things that don't work as well as I would like. The standard historical march from Egypt and Babylonia to modern mathematics, especially in the detail Katz provides, is illuminating for many students, but those with marginal background find it difficult. A few students who enroll because they are looking for an easy mathematics course find the work too demanding and try to minimize the time they put in. As teachers in every subject in the curriculum have discovered, the resources available on the Web allow students to answer take-home questions by Googling the key words and copying—another reason to have an actual final in-class examination—and some students have taken this easy way out on the exercises in Katz's book. I've decided that it's better to identify and criticize such lazy responses than to change what are educationally good assignments that work for most students.

Another difficulty, which every teacher of mathematics faces, is that a few students who have completed the formal prerequisite don't know the material. Even though my class requires a year of university-level calculus, I find myself having to review precalculus and logic on occasion.

Also, sometimes students with strong science and mathematics backgrounds can be historically and philosophically naive. Students need to understand that history isn't about "who did what and when?" but about "how did people make these choices and why?" One must hammer away to convince students that people in the past were just as smart as we are, and that they didn't think the way we do about symbolism, limits, curves, areas, and tangents because they didn't have the benefit of the thousands of years of experience distilled into modern introductory courses. And one must also point out explicitly that, although we have become accustomed to working with irrational numbers and studying lines that (unlike the physical ones we draw) have only one dimension, this doesn't mean that there are no philosophical difficulties about a subject whose subject-matter is not part of the so-called real world.

All these difficulties are inherent in the task of teaching the history of mathematics. Being open to in-class questions, sharing student insights from the "enlightening sentences," having students share their research with the class, and having a back-and-forth written dialogue with individual students through their assignments, are essential in dealing with these difficulties. But grappling with the difficulties is eminently worth it. Studying the history of mathematics reveals that mathematics is made by human beings, and shows how, where, when—and why—it has developed as it has. Knowing all of this helps us learn mathematics, teach mathematics, understand mathematics, and contribute to its future.

References

1. Karine Chemla, ed., *The History of Mathematical* Proof *in Ancient Traditions*, Cambridge University Press, Cambridge, 2012.

2. “Convergence” (www.maa.org/press/periodicals/convergence), the Mathematical Association of America’s online magazine on the history of mathematics and its uses in the classroom.
3. René Descartes, *Discourse on Method*, 2d edition, Liberal Arts Press, New York, 1956.
4. Florence Fasanelli, Joe Gallian, Judith Grabiner, Susan Landau, Carl Pomerance, Amy Shell-Gellasch, and Jim Tattersall, “Women of Mathematics,” Poster. Mathematical Association of America, Washington, DC, 2008.
5. John Fauvel and Jeremy Gray, eds., *The History of Mathematics: A Reader*, Macmillan, Basingstroke and London, 1987.
6. Judith V. Grabiner, The Role of Mathematics in Liberal Arts Education, in M. Matthews, ed., *International Handbook of Research in History, Philosophy and Science Teaching,* Springer, New York, 2014, vol. I, pp. 793–836.
7. Jacques Hadamard, *The Psychology of Invention in the Mathematical Field*, Princeton University Press, Princeton, 1945; Reprinted by Dover, 1954.
8. Victor J. Katz, *A History of Mathematics: An Introduction,* 3d edition, Boston, Pearson, 2009.
9. ———, *A History of Mathematics: Brief Edition*, Boston, Pearson, 2004.
10. ———, ed., *The Mathematics of Egypt, Mesopotamia, China, India, and Islam: A Sourcebook*, Princeton University Press, Princeton, 2007.
11. Victor J. Katz, Menso Folkerts, Barnabas Hughes, Roi Wagner, and J. L. Berggren, eds., *Sourcebook in the Mathematics of Medieval Europe and North Africa,* Princeton University Press, Princeton, 2017.
12. *Plato’s Meno*. Translated with annotations by George Anastaplo and Laurence Berns, Focus Publishing, Newburyport, MA, 2004.
13. George Polya, *How to Solve It*: *A New Aspect of Mathematical Method*, with a Foreword by John H. Conway, Princeton University Press, Princeton, NJ, 2014.
14. Dirk J. Struik, A *Source Book in Mathematics, 1200–1800*, Harvard, Cambridge, MA, 1969.
15. Benjamin Wardhaugh, *How to Read Historical Mathematics*, Princeton University Press, Princeton, 2010.

Appendix A
Course Outline: (taken from the course syllabus)

Dates Topic and Assignments

(My "Recommended" books are for your personal further education)

Week 1: Introduction to the course.

READ THE SYLLABUS (it's a contract between us), look at pp. 521–543 of Katz.

Mathematics in ancient Egypt

Read Katz, 1–9.

Babylonian mathematics

Katz, 10–25, 29–36.

Recommended: H. Frankfort, *Before Philosophy*; O. Neugebauer, *Exact Sciences in Antiquity*; M. Ascher, *Ethnomathematics*; A. Aaboe, *Episodes from Early Mathematics*.

Super-highly Recommended: Victor J. Katz, ed., *The Mathematics of Egypt, Mesopotamia, China, India, and Islam: A Sourcebook* (Princeton, 2007), Chapter I, Annette Imhausen, Egyptian Mathematics; Chapter II, Eleanor Robson, Mesopotamian Mathematics.

J. V. Grabiner, The Role of Mathematics in Liberal Arts Education, in M. Matthews, ed., *International Handbook of Research in History, Philosophy and Science Teaching,* New York, Springer, 2014, vol. I, pp. 793–836.

Week 2: Early Greek mathematics from Pythagoras to Euclid; Greek mathematics and philosophy; Euclid

Katz, 36–62. Plato, *Meno* (all) for Wednesday. Wednesday's class will be a discussion

Recommended: David Fowler, *The Mathematics of Plato's Academy*; Wilbur Knorr, *The Evolution of the Euclidean Elements*.

The *Elements* of Euclid are on line and well worth looking at (try his proof of the Pythagorean Theorem, Book I, Proposition 47). Full table of contents of Euclid on line, with links to the theorems, definitions, postulates, and diagrams: aleph0.clarku.edu/~djoyce/java/elements/toc.html

Week 3: Archimedes, Apollonius, Ptolemy

Review Katz, 36–62. Katz, 67–75.

Recommended: Same books as last week, and also T. L. Heath, ed., *The Works of Archimedes*; E. J. Dijksterhuis, *Archimedes* (2d ed. is better because it has an updated bibliography); Reviel Netz and William Noel, *The Archimedes Codex* (on the recent re-discovery of Archimedes' "Method").

Week 4: Hellenistic Mathematics, Greek Astronomy, Diophantus, Pappus, and Hypatia

Katz, 76–99, 104–113.

Recommended: Thomas S. Kuhn, *The Copernican Revolution*; T. L. Heath, ed., *Diophantus of Alexandria*; Michael Deakin, *Hypatia of Alexandria*: *Mathematician and Martyr*; Jacob Klein, *Greek Mathematical Thought and the Origin of Algebra*; Glen van Brummelen, *The Mathematics of the Heavens and the Earth*: *The Early History of Trigonometry.*

Week 5: China, India, and the Islamic World

Katz, 116–134, 138–156, Katz, 161–188.

Very highly recommended: Victor J. Katz, ed., *The Mathematics of Egypt, Mesopotamia, China, India, and Islam*: *A Sourcebook,* Chapter III, Joseph Dauben, Chinese Mathematics; Chapter IV, Kim Pfloker, Mathematics in India.

Recommended: George Gheverghese Joseph, *The Crest of the Peacock*: *Non-European Roots of Mathematics*; Frank J. Swetz and T. Kao, *Was Pythagoras Chinese? Right Triangle Theory in China*; Kim Pfloker, *Mathematics in India*; Jean-Claude Martzloff, *A History of Chinese Mathematics*; J. L. Berggren, *Episodes in the Mathematics of Medieval Islam*; N. L. Rabinovitz, *Probability and Statistical Inference in Ancient and Medieval Jewish Literature*; Victor J. Katz, ed., *The Mathematics of Egypt, Mesopotamia, China, India, and Islam*: *A Sourcebook,* Chapter 5, Mathematics in Medieval Islam; G. Van Brummelen, *The Mathematics of the Heavens and the Earth*: *The Early History of Trigonometry*; Karine Chemla, ed., *The History of Mathematical Proof in Ancient Traditions*.

Week 6: Mathematics in the Latin Middle Ages, Mathematics Around the World

Katz, 192–210. Also Katz, "Mathematics Round the World," to be handed out.

Recommended: M. Mahoney, Mathematics, and J. Murdoch and Edith Sylla, The Science of Motion, in D. Lindberg, ed., *Science in the Middle Ages*; Edward Grant, ed., *A Source Book in Medieval Science.*

Friday: In-class reports on Mathematics and Culture (the assignment will have been handed out in Week 5).

Week 7: The Renaissance; Mathematics in the Scientific Revolution

Wednesday: Katz, 213–242; 242–251.

Recommended: O. Ore, *Cardano*: *The Gambling Scholar*; J. Klein, *Greek Mathematical Thought and the Origin of Algebra;* Judith Grabiner, Mathematics, in P. F. Grendler, ed., *Encyclopedia of the Renaissance* (6 vols., 1999), vol. 4, 66–72; Victor Katz and Karen Hunger Parshall, *Taming the Unknown*: *A History of Algebra from Antiquity to the Early Twentieth Century*; A**.** Rupert Hall, *The Scientific Revolution*; I. Bernard Cohen, *The Newtonian Revolution.*

Week 8: Analytic Geometry: Fermat and Descartes; Descartes' Philosophy

Katz, 257–278.

Read Descartes, *Discourse on Method*, all, for class discussion on Friday

Recommended: M. Mahoney, *The Mathematical Career of Pierre de Fermat*; Carl Boyer, *History of Analytic Geometry*; H. J. M. Bos, *Redefining Mathematical Exactness*: *Descartes' Transformation of the Early Modern Concept of Construction*; Stephen Gaukroger*, Descartes: An Intellectual Biography.*

Week 9: 17th-century Mathematics before the Calculus; Newton and Leibniz

Katz, 282–303. Katz, 303–323.

Recommended: I. Hacking, *The Emergence of Probability*; Edith Sylla, ed., *The Ars Conjectandi of Jacob Bernoulli*; Carl Boyer, *History of the Calculus* (covers antiquity to the nineteenth century); Margaret Baron, *Origins of the Infinitesimal Calculus* (focuses on the seventeenth century); Robin Wilson and John Watkins, eds., *Combinatorics*: *Ancient and Modern* (2013); D. J. Struik, *A Source Book in Mathematics*, especially for excerpts from Leibniz's papers on calculus; D. T. Whiteside, ed., *The Mathematical Works of Isaac Newton*, vol. I, which contains Newton's published work on calculus; A. R. Hall, *Philosophers at War*: *The Newton-Leibniz Controversy.*

Week 10: Eighteenth-Century Analysis, Rigorization of the Calculus

Katz, 329–359. Katz, 430–459. Read the handouts: translations from Cauchy.

Recommended: T. L. Hankins, *Science and the Enlightenment*; William Dunham, *The Calculus Gallery: Masterpieces from Newton to Lebesgue*; William Dunham, *Euler:The Master of Us All*; Judith V. Grabiner, *The Origins of Cauchy's Rigorous Calculus* (Dover, 2005); *Cauchy's Cours d'analyse*, translated by Rob Bradley and Ed Sandifer (Springer, 2009).

Week 11: Mathematical Creation

For class discussion on Monday: Read Jacques Hadamard, *Psychology of Invention in the Mathematical Field.* Class will not meet on Wednesday and Friday (Thanksgiving). Use the time to prepare for next week's discussion.

Recommended: Peter Borwein, Peter Liljedahl, and Helen Zhai, eds., *Mathematicians on Creativity* (MAA, 2014).

Week 12: Women in Mathematics: Reports and Class Discussion, Monday and Wednesday

Handout and brief assignment, 10% of your total grade; more details forthcoming. (See Appendix D.) Your instructor was on the committee of the Mathematical Association of America that published a "Women in Mathematics" poster and readings, from which part of the assignment will be taken. It's in the hall on the second floor connecting the Fletcher and Scott buildings.

Recommended: G. J. Alexanderson and D. Albers, *Mathematical People; More Mathematical People*; Claudia Henrion, *Women in Mathematics* (her bibliography is especially worth consulting); C. C. Gillispie, ed., *Dictionary of Scientific Biography*, 16 vols., *the* place to start for reliable biographies of any deceased scientific figure; Ann H. Koblitz, *Sofia Kovalevskaya: Scientist, Writer, Revolutionary*; Constance Reid, *Julia Robinson*; "Biographies of Women Mathematicians" online: http://www.agnesscott.edu/Lriddle/women/women.html

Friday: **Topics in Eighteenth-Century Mathematics: Lecture/Discussion**

Choose *one* of these chapters in Katz's book:

13: Probability and Statistics in the eighteenth century
14: Algebra and Number Theory in the eighteenth century
15: Geometry in the eighteenth century

Recommended: Jacqueline Stedall, *From Cardano's Great Art to Lagrange's Reflections: Filling a Gap in the History of Algebra.* Also see Katz's reading suggestions in each of the listed chapters.

Week 13: Nineteenth-Century Mathematics

Monday: Choose *one* of these chapters in Katz's book:

16: Nineteenth-century Algebra
17: Nineteenth-century Statistics
18: Nineteenth-century Geometry

Recommended: Theodore Porter, *The Rise of Statistical Thinking*; Jeremy Gray, *Ideas of Space: Euclidean, Non-Euclidean, and Relativistic*. Also, see Katz's suggestions in each chapter.

Wednesday: Twentieth-Century Mathematics and Some Philosophical Questions

Skim Katz, chapter 20.

Recommended: T. Hawkins, *Lebesgue's Theory of Integration*; Joseph Dauben, *Georg Cantor*; J. L. Richards, *Mathematical Visions: Non-Euclidean Geometry in Britain*; G. Moore, *Zermelo's Axiom of Choice*; Constance Reid, *Hilbert*; Sylvia Nasar, *A Beautiful Mind* (life of John Nash); John Dawson, *Logical Dilemmas: The Life and Work of Kurt Gödel*; J. R. Newman and E. Nagel, *Gödel's Proof.*

Friday: Review and Wrapup

Week 14: Final Examination

Appendix B

Material from Course Syllabus

What's on this syllabus?

1: The four required books

2: Written assignments and grading algorithm

The Four Required Books

1. Victor J. Katz, *A History of Mathematics*: *Brief Edition* (Pearson/Addison-Wesley), 2004.
A "brief"—528 pages!—revision and updating of Katz's longer book. This "brief" edition makes a better course textbook than the longer version, *A History of Mathematics: An Introduction,* whose 3d edition was published by Addison-Wesley in 2009. The longer version, though, is the best one-volume history in existence—you can buy the longer version for yourself and use it if you prefer since the problems and questions are the same. Either edition is very much worth keeping permanently.

Note especially in the required text:

The **pronunciation guide** in the index at the end of Katz's book; this is amazingly empowering.

The g**eneral bibliography** on p. 542.

What I call the "**Guide to Decent Websites**" on p. 543. Add to this the website "Convergence" (www.maa.org/press/periodicals/convergence), the Mathematical Association of America's online magazine on the history of mathematics and its uses in the classroom.

The **historical timeline** on pp. 529–535 linking the history of mathematics to general history.

The material on pages 521–528 with **suggestions on using history of mathematics in the teaching of mathematics**.

2. Plato, *Meno* (the bookstore has the Bobbs-Merrill paperback but any edition will do).

3. René Descartes, *Discourse on Method* (Bobbs-Merrill paperback or any other edition).

4. Jacques Hadamard, *Psychology of Invention in the Mathematical Field* (Dover paperback).

Lectures will focus on specific topics, and will reflect the instructor's view of what is most important; Katz will provide an excellent overview of each topic and a wealth of examples.

There will be **discussions on the books numbered 2, 3, and 4**, and **two in-class reports**, one on mathematics and culture, and one on women in mathematics. Of course questions in lecture are welcome, as consistent with the size of the class.

Written assignments

(i) **"Enlightening sentences" Assignment**: Read the week's reading **before** each Monday's class. Hand in two enlightening sentences on the reading for the coming week, covering **different parts of the reading**. **These will be handed in at the START of class.** I'll briefly review them and hand them back with comments on the following Wednesday.

Sample good sentences and not-so-good ones (two of each; you should be able to tell which is which):

1. Leibniz, inventor of the calculus, lived in Germany.
2. Leibniz's major contribution to calculus was really his mathematically suggestive notation, since conceptually Newton's calculus would have been good enough.
3. Euclid was an important predecessor of Archimedes, who drew on the work of Euclid who lived earlier.
4. Although both Euclid and Archimedes gave logical proofs in geometry, Archimedes was willing to use less rigorous methods to discover the non-obvious results he later proved.

These "sentences" will constitute **10% of your course grade**, and doing them will help you be an intelligent discussant and lecture listener. Try to enlighten your reader. NOTE: **This is a pass/fail assignment**.

(ii) **Problems and Essays**: Each week, due at the **START** of class on **Friday** (except the first week), **40% of the course grade.** For both (a) and (b), GIVE THE PAGE AND QUESTION NUMBER!

(a) Every **Friday** at the **START** of class (except for the first week and the eleventh and twelfth weeks), hand in:

the solution of **three problems** from the current week's reading from Katz (your choice; it's most valuable if you pick the hardest ones you can reasonably do. *Do not "share" this assignment with another student; work on your own*): **2/3 of this 40% component**.

(b) **Every Friday, at the START of class**:

your answer to **one** of the **discussion questions** in the **previous TWO weeks' reading** from Katz (again, your choice; pick something that interests you. *Do not "share" this assignment with another student; work on your own*). "Discussion questions" are at the end of the problems; it should be clear which is which. **One to two double-spaced pages** usually is appropriate. **1/3 of this 40% component**. **These questions are like what you'll find on the final examination.**

You must type (b). As for (a), if you don't know how to use TeX, at least make (a) as legible as possible. I'll give it back to you if I can't read it.

(iii) **Mathematics and Culture short project and report**. This will happen in Weeks 5 and 6. Guidelines will be handed out.

(iv) **Women in Mathematics brief assignment**. This will happen in Week 12. A written assignment will be handed out.

(v) **Final Examination.** The final will combine short answers and essays. There will be some choice. Study questions will be provided ahead of time.

Grading algorithm: Problems and essays 40%, Mathematics and Culture 10%, Women in Mathematics 10%, Enlightening sentences 10%, Final 30%.

Appendix C

Some Resources for Mathematics in Many Cultures

Marcia Ascher, *Ethnomathematics*: *A Multicultural View of Mathematical Ideas* (a seminal work; this brought the subject to the English-speaking academic world).

Marcia Ascher, *Mathematics Elsewhere* (a sequel to her *Ethnomathematic*s, with a different set of examples).

J. L. Berggren, *Episodes in the Mathematics of Medieval Islam.*

Karine Chemla, ed., *The History of Mathematical Proof in Ancient Traditions*.

Ubiratan d'Ambrosio, Ethnomathematics and Its Place in the History and Pedagogy of Mathematics, *For the Learning of Mathematics*, Vol. 5, No. 1 (Feb., 1985), pp. 44–48 (this essay really started the field of Ethnomathematics).

George Gheverghese Joseph, *The Crest of the Peacock*: *Non-European Roots of Mathematics*. 3d edition.

Victor J. Katz, ed., *The Mathematics of Egypt, Mesopotamia, China, India, and Islam*: *A Sourcebook* (immensely valuable; has original writings with excellent commentary).

Victor J. Katz, Menso Folkerts, Barnabas Hughes, Roi Wagner, and J. L. Berggren, eds., *Sourcebook in the Mathematics of Medieval Europe and North Africa.*

Jean-Claude Martzloff, *A History of Chinese Mathematics*.

Kim Pfloker, *Mathematics in India*.

N. L. Rabinovitz, *Probability and Statistical Inference in Ancient and Medieval Jewish Literature*.

Amy Schell-Gellasch and Elizabeth Holmes Clark, Ethnomathematics, Poster available from the Mathematical Association of America, 2008.

Frank J. Swetz and T. Kao, *Was Pythagoras Chinese? Right Triangle Theory in China*.

Glen Van Brummelen, *The Mathematics of the Heavens and the Earth*: *The Early History of Trigonometry*.

Claudia Zaslavsky, *Africa Counts*: *Number and Pattern in African Cultures.*

Appendix D

Some Resources, and 32 Topics on Women in Mathematics

A good place to begin research (and find lots more women mathematicians listed) is the website "Biographies of Women Mathematicians": www.agnesscott.edu/lriddle/women/alpha.htm

To choose topics, I'll start with volunteers, and hand out the other topics to those who don't volunteer.

Here's the list of names; these can be viewed, with very brief biographies to get you started, on the "Women of Math" poster published by the MAA.

1. Hypatia of Alexandria (ca. 355–415)
2. Gabrielle Émilie Le Tonnelier de Breteuil, Marquise du Châtelet (1706–1749)
3. Maria Agnesi (1718–1799)
4. Caroline Herschel (1750–1848)
5. Marie-Sophie Germain 1776–1831)
6. Ada Lovelace (1815–1852)
7. Florence Nightingale (1820–1910)
8. Christine Ladd-Franklin (1847–1930)
9. Sofia Kovalevskaia (1850–1890)
10. Charlotte Angas Scott (1858–1931)
11. Grace Chisolm Young (1868–1944)
12. Emmy Noether (1882–1935)
13. Ann Johnson Pell Wheeler (1883–1966)
14. Dame Mary Cartwright (1900–1998)
15. Mina Rees (1902–1997)
16. Ruth Moufang (1905–1977)
17. Olga Taussky-Todd (1906–1995) (see #32, by the way)
18. Grace Hopper (1906–1992))
19. Emma Lehmer (1906–2007)
20. Cora Ratto de Sadosky (1912–1981) (Argentina)
21. Hanna Neumann (1914–1971) (Australia)
22. Julia Bowman Robinson (1919–1985)
23. Olga Ladyzhenskaya (1922–2004)
24. Olga Arsen'envа Oleinik (1925–2001)
25. Etta Zubner Falconer (1933–2002) (Spelman College)

Include how and why she became a mathematician, where and when she got her PhD (if she lives/lived in the modern world), any difficulties she had, including but not limited to those because of her gender, and the nature and importance of her mathematical achievements

Additional topics

26. Current job market in mathematics for women. A good source is the *Notices of the American Mathematical Society*. Find the most recent reports, usually in February; available in the library or on the Web: www.ams.org/publications/Notices/.

    For the Annual Survey of the Mathematical Sciences, try www.ams.org/careers-edu.

The key points here are the percentage of women, how those percentages change over time (which means you need to look at the *Notices* over time), and also the national backgrounds of Ph. D. mathematicians. **Even if you don't do this topic**, anybody interested in a career in mathematics should know about the *Notices* and its attention to news of the mathematical profession, and should also know that the *Notices* publishes reports on the job market every year.

27. "Four Women from Taiwan": four leading contemporary mathematicians who graduated from the Taiwan Normal University together. Read the poster and then check them out on line: Sun-Yung Alice Chang, Fan Chung, Wen-Ching Winnie Li, and Jang-Mei Wu.
28. Information on the Association for Women in Mathematics. When founded, who's in it, what it does, and the various issues involving women in mathematics with which it deals. This should be done by two people. Starting points: AWM in the 1990s, by Jean Taylor and Sylvia Wiegand, *Notices of the AMS* January 1999 (vol. 46, #1, or on line), pp. 27–38, and the latest *Newsletter* of the AWM (link on AWM's website www.awm-math.org).

Women Mathematicians here in Claremont (three possibilities):

29. Find out how many there are and at which of the Claremont colleges, and when they came (information in the college catalogues and on line). What fraction of all Claremont mathematicians are they now?
30. Interview or read up on *one* (who does not have to be from your own college) about her education, career, research, difficulties, experiences as a woman in mathematics.
31. Research the late Barbara Beechler, founding Professor of Mathematics at Pitzer, through Margaret Murray's book, *Women Becoming Mathematicians*. Includes generalizations about her entire generation of women mathematicians as well.
32. Don't have a topic yet? Pick a woman from the sources listed below on this sheet.

Suggestions for further reading, or, Sources beyond the Web. Biographies:

G. L. Alexanderson and D. Albers, eds., *Mathematical People*. Cool interviews with:
Mina Rees (applied mathematician)
Olga Taussky-Todd (Caltech mathematician, #17 on the poster)

G. L. Alexanderson and D. Albers, eds., *More Mathematical People*. More cool interviews, with:
Cathleen Morawetz (worked in PDEs)
Julia Robinson (key work on solving Hilbert's Tenth Problem)
Mary Ellen Rudin (topologist)

Claudia Henrion, *Women in Mathematics*: *The Addition of Difference*. The biographies in this book are presented as refutations of "myths" about mathematical research: for instance, "Myth: Only white males do mathematics" is followed by biographies of two African-American women mathematicians, Vivienne Malone-Mayes and Fern Hunt.

Karen Uhlenbeck (has been a MacArthur Fellow)
Miriam Pour-El (neither Pour-El nor Uhlenbeck "works in isolation")
Joan Birman (to show math is not "a young man's game")
Leonore Blum (her career combines math and gender politics)
Judy Roitman (her career combines math and gender politics)
Vivienne Malone-Mayes (African American)
Fern Hunt (African American).

Note that *Women in Mathematics* by Henrion and the Alexanderson-Albers books *Mathematical People* and *More Mathematical People* are lots of fun to read—the latter two on both male and female mathematicians. I highly recommend these books, and not just for this assignment.

Dictionary of Scientific Biography, many volumes, ed. C. C. Gillispie. This is the major biographical source for people in all of the sciences. It's in any decent library. *If you want to find out about ANY mathematician, after looking in Katz you should look here*. However, it only covers people who are no longer living.

Louise S. Grinstein and Paul Campbell, eds., *Women of Mathematics*: *A Biobibliographical Sourcebook,* Greenwood Press, 1987. There are copies in the Reference section in the library. The Library of Congress call number is QA 28 W66 1987.

One Should Study the Masters: Teaching with Primary Sources

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Course Overview

My own undergraduate education in the liberal arts convinced me early on that if the goal of education is understanding, then there are few approaches to your subject that foster deeper understanding than to follow in the footsteps of the great minds and experts in your field. That is, one should "study the masters." The course described here is one the author designed with this motivating principle in mind.

What will be described is a one-semester course in the history of mathematics offered at Xavier University (Cincinnati) as an elective to students who have completed the calculus sequence and a transition-to-proofs course. Its audience consists predominantly of math majors who are also pre-service secondary school teachers. It is offered at the junior-senior level, and expects that students have completed three semesters of calculus, an introduction to proof course, and a semester of linear algebra.

Course Design

For simplicity's sake, and because it is the author's conviction that building a narrative context is a high priority in the study of history, the course takes a chronological approach to history. However, this choice demands a heavy price, since no one-semester course can do justice to the immense scope of the history of mathematics. So it is only possible in a single semester to give attention to the history of mathematics before 1700. (See Appendix A for a full course outline.) While there is much of importance to undergraduate mathematicians in the story of our discipline in the last three centuries—the rigorous formalism of the nineteenth century, set theory's placement at the foundation of mathematics, and nearly all of the most important applications of mathematics to solving problems in the modern world having occurred after 1750—it is still true that the majority of the mathematics that appears in primary and secondary school curricula was set in place before Euler was born. So students will still derive great benefits from becoming familiar with the mathematics of the ancient world (especially of the Babylonians and Greeks), the mathematics of the non-Western world (especially of India, China, and the Islamic Middle East), and the mathematics that set in motion modernity and the Scientific Revolution.

Finally, in the last week of the semester, students are taken on a field trip across town to the Rare Book Library at the University of Cincinnati, which houses a wonderful collection of old and important mathematics books, including rare editions of some of the works they have read from over the course of the semester. This trip is the highlight of the semester, as many of the works whose contents were such significant factors in the story of the development of mathematical ideas that they had been studying in their textbook now were coming alive in their hands. These students are very lucky to have such a rich collection of beautiful volumes within a few miles of where they go to school.

Resources and Assignments

The course makes good use of the first half of Victor Katz's *A History of Mathematcs, Brief Edition* [9] as a textbook. Reading from this text is the primary preparation for much of the classroom discussion. It is supported with 1–4 page outlines prepared for students for the era under consideration. (See Appendix B for a sample era outline.) These outlines help students focus their attention on the themes and main historical milestones encountered in their reading, and later the outlines assist them in studying for exams. Their reading is paired with the assignment of a handful of exercises from the textbook. Some class days are given over to a discussion of students' struggles with these exercises. While the exercises Katz supplies are typically quite challenging (and notoriously so) since they combine elements of mathematical problem solving with an exposure of the mathematical culture and styles of the given historical era, the author has discovered that with appropriately detailed hints, students can handle them successfully. More importantly, they really get a sense of what mathematics was being done in centuries past by working through these problems.

However, even more important than these readings from a standard textbook is the experience that students gain by becoming intimate with some of the greatest mathematicians of the ancient world through the careful reading of their writing. A hallmark of this course involves the assignment of readings from a number of primary historical sources, albeit in English translation. These lessons enliven the course immensely, as students can drill deep into the perspectives of some of the greatest mathematical minds.

Six times over the course of the semester, timed to coincide with the closing of classroom discussion in a certain period of history, the reading of a substantial excerpt of two to ten pages from a text that was authored in that same period of history is assigned. The following list of six readings is the most recent lineup (seven readings turns out to be too ambitious a number to attempt in a single semester):

- Plimpton 322 [12]: this famous Old Babylonian text (c. 19th c., BCE) tabulates the whole number lengths of one side and the hypotenuse of over a dozen right triangles, a testament to the fact the Pythagorean Theorem was known at least this early in history, over 1000 years before Pythagoras.
- The Definitions, Axioms, Common Notions, the first six, and last two Propositions, of Book I of Euclid's *Elements* (3rd c., BCE) [8]: these selections are taken from easily the single most influential piece of mathematical writing of all time, and a quintessential example of the deductive axiomatic method of presenting mathematical results, employing a formal approach and style of writing that Western mathematics has preserved since Euclid's day.
- The main results from Archimedes' *Quadrature of the Parabola* (3rd c., BCE) [7]: this is an example of Archimedes' geometrical genius and an application of Eudoxus's method of exhaustion to handle the summation of a geometric series and a nontrivial quadrature of a curved plane region.
- Selections from Abū Rayḥān al-Bīrūnī's *On Shadows* (11th c., CE) [4]: this treatise is devoted to the mathematical and astronomical science of timekeeping, representing an early work in which the trigonometrical ratios known today as tangent and cotangent, secant and cosecant were identified.
- A few chapters from Girolamo Cardano's *The Great Art* (1545) [5]: here, the general algebraic formulas for solving cubic and quartic equations were first published.
- Book I of René Descartes' *Geometry* (1637) [6]: in this extremely influential work, Descartes demonstrates the ability of algebra to solve geometrical problems and of geometrical methods to shed light on algebraic problems, the origins of the subject we today call analytic geometry.

- Gottfried Leibniz's *A new method for maxima and minima*… (1684), in [11]: in this amazing three-page paper, the co-inventor of differential calculus presents many of its central results.

Katz's textbook blends well with the use of these primary source readings since he often focuses attention in his narrative on the same source from which the students will read.

The ideal approach to this kind of textual study for undergraduates involves a three-step process. Students are first asked to read the text on their own, together with a summary of its content from a secondary source (often simply a reference to where Katz discusses it in their textbook); this orients them to appreciate the general meaning of the text. Then, a single class period is given over to a seminar-style re-reading of the text, or perhaps only its most difficult or interesting passages, guided by the instructor. This helps the students to zero in on the salient mathematical ideas, as well as to make important connections with any historical or cultural features of the text that might be obscure to modern readers.

Finally, students are left to work on a writing assignment that helps them to process their reading of the text in the light of the classroom discussion. They are given a handful of essay questions that explore the more critical elements of the given reading. For instance, for the reading from Euclid's *Elements*, Book I, questions like the following are posed:

- Consider Euclid's first five Postulates: how do the first three differ from the others? What is the need for the fourth Postulate, that "all right angles equal one another"?
- The famous Fifth Postulate is not used by Euclid until his proof of Proposition I.29. Examine this Proposition and the one that comes before it. Discuss their content, and their relation to the Fifth Postulate. Does this provide you any insight into why the Fifth Postulate is so much different than the first four?
- Examine the proof of Proposition I.1. Which postulate ensures that the two circles intersect at the point *C*?
- Given that we do not have access to Euclid's original manuscript, how can we be sure that the diagrams that accompany these propositions are "correct"? Examine this question in the context of Proposition I.6 and the proper placement of the point *D* in Euclid's argument.

After they read selections from al-Bīrūnī's *On Shadows*, they are asked to respond to questions like:

- One of the Five Pillars of Islam is called *ṣalāt*; look up what this Arabic word means. What connection would this have with the need al-Bīrūnī and his contemporaries express for accurate determination of the elevation of the Sun in the sky at various times of day?
- Give a modern mathematical interpretation of:

  > … if the altitude is assumed known and the shadow of the gnomon is wanted for that time: the ratio of the sine of the altitude to its cosine, is as the ratio of the gnomon to its shadow, and from this the gnomon is multiplied by the cosine of the altitude and the result is divided by the sine of the altitude, and the shadow results.

- Al-Bīrūnī writes about the process of interpolating values of direct and reversed shadows (tangents and cotangents) for arcs that might lie between those recorded in his *zīj*. To this end, discuss what he means when he says that

  > … The greater the difference in the tabular differences, the less exact it is and the more in error, because the variation in the dependent variable due to fractions [of the argument] depends on the variation in it due to the integer parts. The shadow [functions] behave like this.

Students who take seriously these issues come away from their reading of the text with a deep understanding of the underlying mathematics, and they gain practice forging connections between ideas in the readings and notions they bring from their prior mathematical study. Taken together, this three-step approach reinforces the impact of the historical and cultural context in which the text was written as the student constructs mathematical meaning from their reading.

In addition, students write two other papers during the semester, the first being a mathematical biography of an important figure of their choice from the history of mathematics. A long list of suggested personages are given them, including Pythagoras of Samos and his school, Omar Khayyam, Niccolò Tartaglia, Qin Jiushao, Albrecht Dürer, Johannes Kepler, Blaise Pascal, and Leonhard Euler. The second paper they write describes a detailed reading of another primary historical source different from any of the ones considered in the classroom sessions detailed above. As for their mathematical biographies, students are provided a list of suggestions for texts they may choose for their primary source reading, but they are also encouraged to search out others that pique their interests. Here are some of the texts that students have chosen to write on recently:

- Problem 36 from the Rhind Papyrus
- Zeno's Paradoxes, as presented in Aristotle's *Physics*
- Chapters 1.10–11 of Ptolemy's *Almagest*
- Verses 261–272 of Bhāskarāchārya's *Līlāvati*
- Selected chapters from Leonardo of Pisa's *Book of Squares*
- The first five chapters of Henry Briggs's *Arithmetica Logarithmica*
- Selections from Isaac Newton's *Principia mathematica.*

The primary historical text readings that the students have already done (described above) prepare them for work on this second paper, due near the end of the term, and many find the work they do on this assignment their most important (and fulfilling!) experience of the course.

As these two papers represent a significant fraction of the student's work in the course, and especially since most students find it off-putting that they are expected to produce a good deal of writing as part of a mathematics course (!), we discuss in more detail the guidelines set forth for these assignments. The rubric for the evaluation of the two papers includes four components:

- Format [15%]: The mathematical biography is to be 5–8 page, word-processed, double-spaced in a standard 10 or 12 point font with one inch margins, with separate title page and final blank page (for my comments and evaluation). The primary source reading paper has the same guidelines, except that there are no specified page limits; instead, students are required to attach a copy of the text they have read along with their commentary, adding line numbers to the primary source text to help them make references to the text in their writing.
- Style [25%]: The papers should be presented in a clear and coherent writing style, using correct spelling, proper punctuation, and good grammar.
- Research [35%]: The mathematical biography should present a comprehensively researched discussion of the life history of the chosen individual; this person's significant mathematical accomplishments should be thoroughly discussed. The primary source reading paper should represent a critical reading of the text, describing in detail exactly what the author of the text is trying to do and what mathematical methods are being used to accomplish this.
- References [25%]: The body of the paper must be followed by a list of sources and references containing at least three sources, in order to avoid overuse of potentially unreliable internet sources, at least three sources must have been published in print or have undergone an obvious peer-review process. It should make appropriate (i.e., not too frequent) use of direct quotations, and should include in-text citations (as either footnotes or endnotes, following some standard reference style).

Next, some words about the class "field trip" to the University of Cincinnati Archives & Rare Books Library. In recent visits, students had the opportunity to turn the pages, with gloved fingers, of a dozen or so lovely old books. They had the opportunity to read material that was quoted in their textbook in their original volumes and to study the diagrams in these important works. Here are some of the volumes in that collection:

- Euclid's *Elements*: the 1537 Basel edition of Campanus's Latin translation of Theon's version, "in XV Books," printed in Paris in 1546.
- Diophantus' *Arithmetica*: Bachet's 1621 edition of Xylander's Latin translation (in Fermat's edition of this book, he wrote marginal notes, including the statement of his Last Theorem).
- Niccoló Tartaglia's *Quesiti et inventioni diverse*, published in Venice in 1546: including the author's writings on the motion of ballistics (prior to Galileo's investigations on the subject, wherein he correctly identifies the parabolic path of a projectile) and his rules for solving cubic equations.
- François Viète's *Opera mathematica*, edited together with writings of Francis van Schooten and published by Elzevier in Leiden in 1646.
- Adrian Vlacq's *Tables*: German edition, published in Amsterdam in 1673 of Vlacq's tables of sines, tangents, and secants, and their logarithms, along with Henry Briggs's common logarithms of numbers from 1 to 10000.
- René Descartes' *Geometria*: Florimond de Beaune's 1659 second Latin edition of Descartes' seminal work, with commentaries by van Schooten and de Beaune.
- John Wallis's *A treatise of algebra*, published in English in London in 1685: an early history and comprehensive account of algebra and its methods.
- Isaac Newton's *The method of fluxions and infinite series*, John Colson's English translation of the Latin original (London, 1736).
- Johann Bernoulli's *Ars conjectandi* (Basel, 1713): the first comprehensive modern work in combinatorics and its applications to probability theory.
- Maria Agnesi's *Instituzioni analitiche*, (Milan, 1748): the first textbook treating both differential and integral calculus, written to aid in the instruction of her younger siblings.
- Leonhard Euler's *Introductio in analysin infinitorum* (Lausanne, 1748): the first great treatise in precalculus, expounding on the nature of functions and series.
- Augustin-Louis Cauchy's *Cours d'analyse* (Paris, 1821): the first work to introduce the concept of limit as the foundation for calculus.

It should be noted, however, that most of the titles listed above are currently available through digitized online resources (and access to these and other texts in English translation continue to become more readily available every year). So if the reader does not enjoy the benefit of a quality rare books library nearby, there are ways to approximate the experience via online access to these or similar primary sources.

Lessons Learned

One of the consequences of promoting the reading of primary sources in the history of mathematics course is that an important message is clearly sent to students that mathematics is not a collection of oracles that were delivered from heaven, ready-packaged in four-page units with boldfaced definitions and highlighted theorem statements, as in many modern textbooks. Rather, mathematics has been very much an endeavor of human beings, advanced by many generations of thinkers and writers, and dating back to the earliest written artifacts. These thinkers and writers brought a wide variety of approaches to bear to struggle with problems that were pressing in their times and places. Reading their works impresses upon students that their mathematical ideas were natural and relevant for their times, and full of insight. Only later have these ideas been highly refined and abstracted into the forms more commonly presented today in textbooks and references.

Nonetheless, it is often hard work to make sense of a primary source, at least as much of a challenge as the work required to decode new mathematical ideas from a modern textbook. Luckily, the mathematical ideas in the writings my students are asked to study are typically not new to them; instead, they invest their energy in making sense of the unusual context in which the ideas are being presented. (It recalls to mind the now-proverbial motto, "The past is a foreign country: they do things differently there.") Historical texts often contain little symbolism; rather, one finds a verbal language rich in mathematical ideas, ideas redolent with context. Geometric diagrams also turn out to play a large role, as the legacy of Greek geometry looms large throughout the history of Western mathematics. The struggles with these texts reward students with deeper connections to the mathematics they encounter and with the worlds that the mathematics was meant to support.

Needless to say, a fair amount of instructor preparation is required to effectively use a primary source in the classroom. These are not "off the shelf" materials. Very few of these texts were written for students—and none for 21st century classrooms! Instructors will have to interpret the texts they wish to use for themselves before they are ready to place them in the hands of their students. Besides, there are often historical and mathematical side issues that might be explored within a given text, issues that are secondary, if not orthogonal, to the course or instructor goals. Managing the focus of your students on the instructor's goals for the tasks at hand, or at least channeling their energies toward these purposes, demands prior planning. Fortunately, the number of resources for instructors who wish to use primary sources in the classroom has been growing quickly in recent years. There is a wider collection of wonderful sourcebooks and anthologies of texts from the history of mathematics now available (e.g., [10], [13]). Also, consider the recent work of National Science Foundation grant projects that have been directed to developing course materials using primary sources in the undergraduate classroom, two projects out of New Mexico State University focused on discrete mathematics and computer science [3, 1], and their daughter project [2], which is currently (at this writing) supporting the entire college mathematics curriculum.

Finally, the great repository of mathematical ideas has been preserved between the pages of actual books, and the ability to see and handle some of these books in libraries, especially first or rare editions, helps to bring a concrete excitement to the study of the history of mathematics. Instructors are encouraged to investigate their local areas for library collections containing rare books in mathematics and science to see whether such volumes could be displayed for students to touch and examine. In the absence of the opportunity for physical contact with these artifacts of history, an increasing corpus of important primary sources in mathematics is now available electronically in facsimile (as images and pdfs); even encounters with historical texts this form can have a powerful effect on students.

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Appendix A
Course Outline

Week	Katz: [Ch.] Reading/Exercises	Primary Source Readings	Papers/Exams
1	[1] Ancient Egypt & Babylonia	Plimpton 322	
2	[2] Classical Greece		
3	[2] Euclid; [3] Archimedes	from *Elements*, Bk I	
4	[3] Hellenistic Greece	from *Quad. of the Parabola*	
5	[4] Greece Under the Romans		
6	[5] Ancient & Medieval China		
7	[6] Ancient & Medieval India		Math. Biography
8	—		Midterm Exam
9	[7] Islamic Mathematics	from *On Shadows*	
10	[8] Medieval Europe		
11	[9] Renaissance Algebra		
12	[9] Renaissance Analysis	from *The Great Art*	
13	[10] 17th Century Precalculus	from *Geometry*	
14	[11] The Origins of Calculus	from *A New Method*	Prim. Source Paper
15	—	(Rare Book Room visit)	Final Exam

Appendix B
Sample Era Outline

Mathematics in Ancient India

- Indus valley civilizations, 25th to 15th c. BCE
  - Archaeological evidence of ancient well-populated cities of Harrapa and Mohenjo-daro along the Indus River (present-day Pakistan)
- Aryan invasions, intermittent from 15th to 5th c. BCE
  - Migrations of Aryan peoples from the Central Asian steppes southward
  - Caste system instituted
  - Hinduism established
  - Religious protests against Brahmin practices (ca. 650 BCE) gave rise to Buddhism and Jainism
  - Vedas composed: long epic poems, originally transmitted orally, and later written down
  - *Bhagavad-gita*, part of the *Mahabharata*, written between 100 BCE and 100 CE, tells the story of Lord Krishna and Prince Arjuna
- Anonymous: *Sulvasūtras* (6th c. BCE)
  - These four most famous Vedic texts are composed by Baudhayana, Manava, Apastamba and Kātyāyana, about whom little is known
  - Dealt with religious issues like construction of altars and other practical problems of geometry
  - Outlined techniques for computing surds
- Persian and Greek invasions, 6th to 4th c. BCE
  - Persians led by Darius I, who expanded his empire eastward to the Indus
  - Succeeded Alexander the Great, after conquest of the Persians
  - Ruled by Seleucid Greeks from Babylon
- Mauryan Dynasty, 4th–2nd c. BCE
  - Chandragupta Maurya united northern India around 300 BCE, repelling the advancing Greeks
  - Set up a capital at Pataliputra (modern-day Patna)
  - Ashoka, Chandragupta's grandson, expanded the empire to most of the subcontinent (except for the extreme south) and converted to Buddhism
  - First Hindu numerals, found on pillar inscriptions, date to this period
- Kushan invasion, 4th–2nd c. BCE
  - Demetrius, a king of Bactria (modern-day Afghanistan), swept in over the weakened Mauryans
  - Kushans established trade with the West
  - Greek works found in India
- Sunga dynasty, 2nd c. BCE–1st c. CE
  - Controlled the northeastern subcontinent from Pataliputra
  - Expanded Buddhism across India
- Gupta dynasty, 3rd–7th c.
  - Ruled over the "golden age" of Indian culture and science
  - Opened universities

- Anonymous: *Surya Siddhanta* (*System of the Heavens*), 3rd c.
  - Astronomical text describing motion of the planets and computation of dates of solar eclipses
  - Contains the roots of modern trigonometry, using sine (*jya*, short for *jya ardha* = half chord), cosine (*kojya*, "perpendicular sine") and inverse sine (*otkram jya*), and the earliest use of tangent and secant (as ratios of sines and cosines) when discussing shadows cast by a gnomon
- Āryabhaṭa: *Āryabhaṭīya* (6th c.)
  - 121 verses in Sanskrit devoted to astronomical calculations (units of time and calendars, solving equations, shadow measurements, sine table, 5-decimal approximation of π, rising of constellations, length of daylight, etc.)
  - Shows evidence that the older multiplicative system of decimal numeration is replaced by a positional one (roots of Hindu numerals)
- Harsha Dynasty, 7th c.
  - Established diplomatic relations with China
- Brahmagupta: *Brāhmasputasiddhānta* (*Correct Astronomical System of Brahma*)—Gave rules for arithmetic with negative numbers
  - Provided a quadratic formula, solved systems of congruences, and gave a complete treatment of Pell's equation ($Dx^2 \pm b = y^2$)
  - Derived Heron's formula from a similar result for the area of a cyclic quadrilateral
  - Developed a sophisticated interpolation formula for intermediate sine values from tables
- Bhaskāra I: *Aryabhatiyabhashya* (*Commentary on the Aryabhatiya*)
  - Used Hindu numerals consistently (including a circle for zero)
  - Used *kuṭṭaka* ("pulverizer") method for indeterminate Diophantine equations $ax + by = c$
  - Gave a general method for solving Pell's equation
- Muslim incursions, 8th–14th c.
  - Influence of Muslims felt mostly in the northwest (Sindh, Punjab), occasionally further south
  - Mohammed Ghori established the Delhi Sultanate (13th–14th c.)
- Bhaskāra II (Bhāskarāchārya), 12th c.
  - In *Līlāvati* (named for his daughter) and *Bijaganita*, solved equations to the fourth power and indeterminate equations of linear and quadratic form, including Pell's equation in a general form by the *chakravala* ("cyclic") method, developed combinatorial results (counting permutations), plane and spherical trigonometry, and detailed analysis of sine and cosine values
- Jyeṣṭhadeva, 16th c.
  - Member of the Kerala school (14th–16th c.), noted for detailed study of trigonometry
  - In *Yuktibhāṣā* (written in Malayalam and in prose), he derived many rules

Capstone Mathematics from Primary Historical Sources

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Course Overview

The senior-level course *Great Theorems: The Art of Mathematics* implements the philosophy of learning mathematics directly from primary historical sources [1, 2, 3, 7, 9]. The central focus is always great mathematics, but since the approach is entirely through interconnected historical sources, students learn a great deal of history at the same time. Each offering of the course studies one or more sequences of (translated) primary sources at the upper undergraduate level, each sequence following a great theme in mathematics over centuries or millennia, leading to one or more great results in modern mathematics. The aim is for students to experience a sequence of discovery close to firsthand.

Adopting the view of mathematics as art, we examine the creation of some mathematical masterpieces from antiquity to the modern era. A careful analysis of these theorems and their original proofs illustrates the aesthetic spirit which pervades mathematics; a comparison with modern theorems and methods through demanding exercises demonstrates the progress mathematics achieves through abstraction. And a mathematical term paper on an approved topic chosen through the student's initiative rounds out a capstone experience.

We also place the theorems in a larger historical context, since mathematics is an inseparable part of human history, effecting profound changes in people's lives, and itself responding to new social and political needs. Finally, as these theorems were discovered by real flesh-and-blood people, it is only appropriate to look at how their lives and work were influenced by the intellectual and social environment of the day.

For most students, the course is a huge breath of fresh air and an eye-opening capstone experience. For mathematics majors, the exposure to deep mathematics through its historical development is a great way to round out and enrich their major and make connections between their prior courses. For the mathematically inclined engineering and science majors who qualify for this course, it is a refreshing and stimulating alternative to the often heavily cookbook-oriented mathematics courses required by their major, and it leaves them with a much happier and more positive intellectual view of mathematics.

Great Theorems: The Art of Mathematics began in 1988 as an honors mathematics course attracting students majoring in mathematics, engineering, computer science, or the physical sciences at New Mexico State University, and has been offered every year since then [5]. The prerequisite is a year of calculus and some upper division mathematics course, all with good grades. Students are expected to contribute actively to class discussion in a course with enrollment restricted to twenty.

The course qualifies as a senior elective for either the major or minor in mathematics, the major in secondary mathematics education, and as an upper-division general education breadth elective for engineering majors. Usually about half the students are mathematics majors, with the rest from secondary mathematics education, engineering, computer science or the physical sciences.

Course Design

Our primary resource is a collection of historical texts, ideally with generous annotation, commentary, and exercises. The goal is to study the original proofs of the theorems in these texts, in the words of the discoverers, in order to understand the most authentic possible picture of the evolution of major branches of mathematics during a span of over two thousand years. It is exciting and illuminating to read original works in which great mathematical ideas were first revealed, and we aim for lively discussion involving everyone.

At home and in class we read, discuss, and interpret the theorems and their proofs, with students writing their thoughts and questions about these works, and we discuss how the various sources tie together in the development of major mathematical results. Regular written assignments based on the primary sources consist of proving related results, filling in missing parts of proofs, and related historical questions. One of our main aims is to have students actually do mathematics themselves by creating some ideas and devising proofs on their own.

Classroom pedagogy varies by instructor. I have a three-part non-lecture assignment approach for each day of my teaching, explained in a handout for students (Appendix C), and in more detail in [8]. Two of these parts are pre-class, and one is post-class. For each day, I expect students first to read new material well before class, and to write questions about their mathematical reading to give to me to read before class. Second, I expect students to prepare pre-assigned mathematical problems to bring to class based on their reading. These two pre-class assignments are graded based only on preparation, with a quick plus, check, or minus.

In class we begin by discussing their reading questions. Class discussions are often challenging, because primary sources provide fabulous grist for deep and wide-ranging considerations. Today they frequently raise as many questions as they answer, a fabulous pedagogical tool. Then most of class time is spent with students working together on the previously assigned problems, along with whole-class discussions or student presentations suggested by me when interesting questions and approaches arise.

Finally, the third part of each day's assignment occurs post-class, consisting of final homework on the topic, a very few challenging exercises not tackled in class. Students are encouraged to discuss their ideas with others, but are then expected to finish and write up their polished post-class homework entirely on their own, in their own words, to hand in for me to read and mark carefully. I often request rewriting for improvement. This post-class homework part ultimately receives a single letter grade for quality, one for each class day.

The course grade is based on a final holistic evaluation of student work: roughly one half on daily assignments (i.e., student writings on the primary sources, and related mathematical assignments); roughly one quarter on class participation; and the remaining quarter on a term paper and a brief oral presentation on it, as described below. There are no exams. The course overview handout for students is in Appendix B.

The course has a very flexible timetable (hence there is no course outline appended), often influenced by what explorations happen in the classroom based on student response and activities, according to the classroom methods described above. Whether we engage one topic chapter or two from the textbook during a term varies.

On the first day I introduce and discuss the nature and expectations of the course, we dive into some mathematics, and I ask students to skim the entire four topic chapter materials of the textbook as homework, to provide feedback, as explained earlier, at the beginning of the second class period. At the beginning of the second day I select a topic on the spot, based on their feedback, and we begin right away with primary source material from that topic. Succeeding days always have reading/writing in advance, preparatory mathematical work on exercises, in class group and whole class work, and final homework exercises, as described earlier. By mid-term students begin work on individual term paper research, in addition to our regular classroom work, as described earlier. The very end of term is spent on short term paper presentations and discussion.

Resources

When I first co-created and taught the course in 1988, we had little more than a handful of chosen primary sources on a few topics, some of which we had to translate ourselves, often with no annotation, context, or exercises. We assigned some essay readings to supplement the sources, created assignments as we went, and started writing annotation to tie it all together. And indeed this is how anyone can still develop their own materials; I enthusiastically recommend it. Guidance on the pedagogical principles, and on design of materials, can be found in [1,2,3,7,9]. Today primary sources are much more easily available, and in translation as well, than when we started. The reader may be pleasantly surprised that finding promising and appropriate primary sources for teaching on a given topic is not as hard as may be feared. The bibliography [10] provides a window to many historical sources for teaching. The recent sizeable source book [12] would be a good place to find many good sources on a variety of topics (see the review [11]).

As the course evolved, with new instructors developing their own sequences of primary sources on new topics, we coalesced on sequences of primary sources for four great themes in the evolution of mathematics leading to important modern results, added extensive annotation, contextual, historical, and mathematical commentary as a guide and overview of each big story, and numerous mathematical exercises for students. We also included copious references to the literature for deeper understanding by both teachers and students. These four sequence themes became the four chapters of the course textbook *Mathematical Masterpieces: Further Chronicles by the Explorers* [4]. Each chapter has an extensive introduction, which tells the story of a large theme from the beginning, both mathematically and historically. As it proceeds, the chapter introduction points the reader to the subsequent chapter sections, which focus on sources by specific authors. Therefore we have students read the introduction in tandem with work on individual chapter sections, going back and forth between the main story and the featured primary source sections.

Each one-semester offering of the course typically covers only one or two of the four independent chapter themes described below. The chapters are available individually online from the publisher.

The chapter themes and the authors of the primary sources they include are:

- *The Bridge Between Continuous and Discrete.* Primary sources follow two millennia of the theme of sums of numerical powers. Archimedes sums squares to find the area inside a spiral, Fermat and Pascal sum powers using figurate numbers and binomial expansions, Jakob Bernoulli discovers the pattern in sums of powers formulas, and Euler develops his summation formula for making astonishing approximations by summing divergent series, and solves the Basel problem that the sum of the reciprocal squares is $\pi^2/6$.
- *Solving Equations Numerically: Finding Our Roots.* Qin solves a fourth degree equation, Newton develops a proportional method, Simpson explains a fluxional method ("Newton's Method"), and in 1981 Smale proves that probabilistically the fluxional method almost always converges.
- *Curvature and the Notion of Space.* In primary sources Huygens discovers the isochrone, Newton derives the radius of curvature, Euler studies the curvature of surfaces, Gauss defines an independent notion of curvature, and Riemann explores higher dimensional space.
- *Patterns in Prime Numbers: The Quadratic Reciprocity Law.* Euler discovers patterns in prime divisors of quadratic forms, Lagrange develops a theory of quadratic forms and divisors, Legendre asserts the quadratic reciprocity law, Gauss proves it, Eisenstein creates a geometric proof, and Gauss composes quadratic forms, foreshadowing the class group.

As illustrations, Appendix A introduces small excerpts from selected primary source material for each theme, along with connected sample exercises for students. The website [6] provides sample sections from each chapter.

Assignments

Regular homework and related classroom work are the heart of the course. Assignments are largely mathematical in nature, based directly on the primary sources, since the course is first and foremost mathematics, set authentically in its history. Exercises often strengthen students' understanding of a primary source, and are sometimes open-ended. To give a diversity of flavors, Appendix A provides sample exercises from our four general themes, each exercise preceded by a little context and a small excerpt from the relevant primary source.

Regarding the term paper and brief oral presentation (see the handout for students in Appendix D), the choice of topic is up to the student, subject to my approval, but should include a meaningful mathematical component (it should not be mostly biographical) that they can genuinely understand and explain to others in a presentation. I do not suggest topics, so students must keep their eyes open for something along the way of interest, since this is an opportunity to delve into something personally exciting or innovative. I consciously place this responsibility on each student, to try to encourage them to take initiative. This results in a great variety of quality in topics, and while not all are inspiring, many are truly fascinating

Students are expected to pick a term paper topic by mid-semester, and I help students in refining ideas for a topic. I first ask each student to come up with two ideas for a topic, to do a preliminary library search to see that adequate research materials are to be found there (required usually to be books, not just internet sources), and to write a paragraph describing each topic to me, along with references to what was found in the library. After possible further refinement, I approve each topic. I then sometimes require that students show me their writing progress along the way, to help them complete an acceptable paper on time. We use the mandatory final exam period, as well as the last few class days as needed, for term paper presentations, with papers due several days before presentations.

Lessons Learned

This is an engaging capstone course for seniors who have studied a good amount of upper-level mathematics. It greatly broadens their horizons, ties together other course material, and simply excites most of them no end. The material is thrilling, and most of them love learning from primary sources.

The class discussions and work are often exhilarating for students and instructor alike; this course is probably my favorite, most beloved, repeating course of all time to teach.

However, it is important, as with any senior-level mathematics course, that students have sufficient strength entering the course to succeed. Since the only specific content prerequisite is a year of calculus, insisting on the requirement of some upper-division mathematics with good grades is essential to assure that students are all at the necessary level and can succeed.

Even with good mathematical preparation, students will vary in their ability to tackle primary sources and contribute to class discussions and presentations. The instructor may need to spend extra time assisting and encouraging some, draw some out in discussion and presentation, and pair them up in class with other students who can be helpful in a teaching capacity.

A final caveat: It is highly rewarding to find and incorporate one's own primary sources, but they vary tremendously in their pedagogical value and appropriateness. Some are gems, whereas some I have aimed for have turned out to be too opaque for any kind of classroom use. And the amount of each source to use, and how, requires much thought. Some of these issues are addressed further in [3].

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Appendix A
Sample primary source materials and exercises

Here I describe a small primary source excerpt and related exercise for each major theme we have used in teaching the course.

The Bridge Between Continuous and Discrete

As part of never-ending discoveries of the interplay between the continuous and the discrete, Euler found the Euler-Maclaurin summation formula, which among other things enables incredibly accurate approximations for the sums of very slowly converging series.

> “The general expression that we found in the previous chapter for the summative term of a series, whose general term corresponding to the index x is z, namely
>
> $$Sz = \int z dx + (1/2)z + (\mathfrak{A} dz)/(1 \cdot 2 dx) - (\mathfrak{B} d^3 z)/(1 \cdot 2 \cdot 3 \cdot 4 dx^3) + (\mathfrak{C} d^5 z)/(1 \cdot 2 \cdots 6 dx^5) - etc.,$$
>
> actually serves to determine the sums of series whose general terms are integral rational functions of the index x, because in these cases one eventually arrives at vanishing differentials. On the other hand, if z is not such a function of x, then the differentials continue without end, and there results an infinite series that expresses the sum of the given series up to and including the term whose index $= x$. The sum of the series, continuing without end, is thus given by taking $x = \infty$, and one finds in this way another infinite series equal to the original.”

Exercise: Follow in Euler’s footsteps by using his summation formula to approximate $\sum_{i=1}^{\infty} 1/i^3$ to seventeen decimal places. Euler obtained 1.20205690315959428. Study whether it appears to be a simple rational multiple of π^3, as Euler hoped it might be.

Solving Equations Numerically: Finding our Roots

What we now call “Newton’s Method” for approximating roots is actually due to Simpson, who developed the method, with examples, for not just one, but two unknowns.

> “When there are two Equations given, and as many Quantities (x and y) to be determined.
> Take the Fluxions of both the Equations, considering x and y as variable, and in the former collect all the Terms, affected with $\dot{x}$, under their proper Signs, and having divided by $\dot{x}$, put the Quotient = A; and let the remaining Terms, divided by $\dot{y}$, be represented by B: In like manner, having divided the Terms in the latter, affected with $\dot{x}$, by $\dot{x}$, let the Quotient be put = a, and the rest, divided by $\dot{y}$, = b. Assume the Values of x and y pretty near the Truth, and substitute in both the Equations, marking the Error in each, and let these Errors, whether positive or negative, be signified by R and r respectively: Substitute likewise in the Values of A, B, a, b, and let $\frac{Br-bR}{Ab-aB}$ and $\frac{aR-Ar}{Ab-aB}$ be converted into Numbers, and respectively added to the former Values of x and y; and thereby new Values of those Quantities will be obtained; from whence, by repeating the Operations, the true Values may be approximated ad libitum.”

Exercise: Formulate Simpson’s fluxional method geometrically for three functions each in three variables. Tangent planes now become tangent “hyerplanes.” Derive his formulas or their modern counterparts.

Curvature and the Notion of Space

Gauss, building on work of Euler on principal curvatures, developed a general theory of curvature of surfaces.

> "The solution of the problem, to find the measure of curvature at any point of a curved surface, appears in different forms according to the manner in which the nature of the curved surface is given. When the points in space, in general, are distinguished by three rectangular coordinates, the simplest method is to express one coordinate as a function of the other two. In this way we obtain the simplest expression for the measure of curvature. But, at the same time, there arises a remarkable relation between this measure of curvature and the curvature of the curves formed by the intersections of the curved surface with planes normal to it. Euler, as is well known, first showed that two of these cutting planes which intersect each other at right angles have this property, that in one is found the greatest and in the other the smallest radius of curvature; or, more correctly, that in them the two extreme curvatures are found. It will follow then from the above mentioned expression for the measure of curvature that this will be equal to a fraction whose numerator is unity and whose denominator is the product of the extreme radii of curvature. The expression for the measure of curvature will be less simple, if the nature of the curved surface is determined by an equation in x, y, z. And it will become still more complex, if the nature of the curved surface is given so that x, y, z are expressed in the form of functions of two new variables p, q."

Exercise: Look up Gauss's formula for the curvature of a surface given in terms of two parameters, p, q [87, p. 18], and use this to compute the curvature of the torus in Exercise 3.19 at an arbitrary point.

Patterns in Prime Numbers: The Quadratic Reciprocity Law

After many decades of struggle, in 1772 Euler gave a clear statement of his final vision, still not proven, of the role reversal [reciprocity] between quadratic residues [residues] and moduli [divisors].

> "Conclusion. These … theorems, of which the demonstration from now on is desired, can be nicely formulated as follows:
>
> Let s be some prime number, let only the odd squares 1, 9, 25, 49, etc. be divided by the divisor $4s$, and let the residues be noted, which will all be of the form $4q+1$, of which any may be denoted by the letter α, and the other numbers of the form $4q+1$, which do not appear among the residues, be denoted by some letter $\mathfrak{U}$, then we shall have

divisor a prime number [P] of the form	then [modulo P]
$4ns+\alpha$	*$+s$ is a residue, and $-s$ is a residue;*
$4ns-\alpha$	*$+s$ is a residue, and $-s$ is a nonresidue;*
$4ns+\mathfrak{U}$	*$+s$ is a nonresidue, and $-s$ is a nonresidue;*
$4ns-\mathfrak{U}$	*$+s$ is a nonresidue, and $-s$ is a residue."*

Exercise: Use Euler's claims in his Conclusion of 1772 about the quadratic character of $-s$ to find the linear forms of nontrivial prime divisors of numbers of the quadratic form x^2+5y^2.

Appendix B

Handout for students on course overview

GREAT THEOREMS: THE ART OF MATHEMATICS

Required Text: *Mathematical Masterpieces: Further Chronicles by the Explorers,* available at the University Bookstore.

Prerequisites: Grades of B or better in second semester calculus, and any upper division MATH/STAT course, with overall GPA of 3.2 or better, or consent of instructor. Please speak with me if you do not have the prerequisites.

Course grade:

Written homework assignments: 50%
Term Paper: 25%
Class Participation: 25%

The ancient Greeks counted arithmetic and geometry among the Seven Liberal Arts. Adopting the view of mathematics as art, we will examine the creation of some mathematical masterpieces from antiquity to the modern era. A careful analysis of these theorems and their original proofs will illustrate the aesthetic spirit which pervades mathematics; a comparison with modern theorems and methods will demonstrate the progress mathematics achieves through abstraction.

We will also place the theorems in a larger historical context, since mathematics is an inseparable part of human history, effecting profound changes in people's lives, and itself responding to new social and political needs. Finally, as these theorems were discovered by real flesh-and-blood people, it is only appropriate to look at how their lives and work were influenced by the intellectual and social environment of the day.

The text contain the primary sources and theorems we will study. We have endeavored to provide the original proofs of these theorems, in order to present the most authentic possible picture of the evolution of mathematics during a long span of time. In class we will discuss and interpret these theorems and their proofs, and we hope for lively discussion involving everyone. As we examine the development of mathematical ideas, we will also discuss their historical context and the biographies of their creators. Regular written assignments based on the original mathematical sources will consist of proving related results, filling in missing parts of proofs, and related historical questions. One of our main aims is to have you actually do mathematics yourself by creating some ideas and devising proofs on your own.

In addition to the regular assignments there will be a term paper based on library research into a mathematical topic you will choose with instructor advice and approval. As the semester goes along, be looking for a topic for your paper, perhaps related to what we do in class. By midsemester you should have chosen the paper's topic. Near the end of the semester, after your written paper is finished, each person will give a brief class presentation to tell everyone else what they explored and discovered.

Appendix C

Handout for students on homework assignment guidelines

Keep this sheet

Guidelines for all regular homework assignments

Please put your name (and any nickname your prefer) on the first page, *staple* your pages together, and *do not* fold them. Use both sides of the paper if you wish, to save paper. Please *do not* write in light pencil. Please write clearly. Thank you.

Parts A, B, C of each homework are equally important.

Part A: Advance preparation. Hand this in at the beginning of class, one class period before our class discussion and work on new reading. Reading responses (a), questions (b), reflection (c), and time spent (d):

You do *not* need a new page for each part (a), (b), (c), (d).

a) Read assigned material. Reread as needed for complete understanding. Then write clear *responses* to assigned questions about the reading.

b) Write down some of your own explicit *questions* about your reading, ready to bring up in class. This may involve new or old concepts that are confusing to you, and connections to other ideas. You should also consider writing down what was well explained and interesting, what was confusing, what you had to reread but eventually understood.

c) Reflection: Write two or three sentences *reflecting* on the process of your work; this should only take a few minutes. Write about how things went with any assignment or reading done for class, and other course work. This should reflect both your ongoing personal feelings about the course as a whole and your interaction with the material at hand.

d) Write how much *time* you worked on part A.

Part B: Warmup exercise preparation to present in class. This is due during class when we begin to discuss new material. Work individually, and then with others in your group outside class time, on a few assigned easy warmup exercises on the new material we will discuss, based on your advance reading in Part A. Write up the solutions to these individually, to hand in in class. I will ask individuals and groups to present some of these to the class, to get us started discussing new material. Be sure to hand these in before leaving class.

Also always write how much time you worked on part B, and with whom.

Part C: Main exercises. These will be assigned after class discussion and work on new material. They will normally be due next period. Work individually and with others in your group on these. Also come to see me during office hours or at other appointment times about these. I am happy to help you. Then go home and write up your final solutions completely by yourself, without comparing with other people. The paper you hand in should be entirely your own writing, not the same as anyone else's.

Appendix D

Handout for students on term paper guidelines

Guidelines for term paper and presentation

One quarter of your work for the course is a well written term paper on a mathematical topic of your choice, along with a brief class presentation.

- You should take the initiative in finding a term paper topic. First come up with at least two ideas for topics, do a preliminary library check that adequate research materials are likely to be found there, and write a paragraph describing each topic, along with what you found in the library. Borrow the library books you might need now, before someone else does. Use your imagination and interests in selecting topics! Hand this in by the date assigned in class.
- The principal requirement for a topic is that it should be about mathematics. The other main requirement is that you should be able to discuss the mathematics in your paper with some genuine understanding of it. Writing a paper that lists mathematical results you have no understanding of is not fruitful. The paper should be written in a style understandable by your fellow students.
- After you hand in your topic paragraphs, you will receive feedback on your tentative topics, and may be asked to seek further source material in the library to make sure a topic is appropriate. Your selected topic should then receive final approval.
    - Your paper should be well written in your own words. It should include a complete list of detailed references, and frequent citations to your references, indicating page numbers from your references. I expect you will use at least 2 or 3 references, and that at least some will come from the library. If you wish to quote directly from a source somewhere in your paper, instead of writing something in your own words, you should indicate clearly that it is a quotation. References from the internet should also be cited with full information on the exact internet location of the information, and a copy of an internet printout attached. If you copy directly from a source without attribution this is plagiarism, and you will fail the course. Any consistent format for the paper and references is acceptable.
    - Your paper should be 6–12 pages long.
    - You may handwrite or type your paper clearly. Don't waste time trying to type mathematics or drawings. If you type it, please use 1½ or double spacing.
    - You are strongly urged to consult about your work in progress, and to submit a rough draft before the final version of your paper. These steps will greatly enhance the likely quality of your paper.
    - When you hand in your paper, by the date assigned in class, keep a copy to prepare for your presentation to the class. Please **do not** put your paper in a plastic cover. Just staple it! Thanks.

History as Capstone
A Course for Senior Mathematics Majors

Betty Mayfield
Hood College

Course Overview

This paper seeks to answer the twin questions

- If you think your senior mathematics majors should have a capstone experience, what should it be?
- If you want to include a history of mathematics course in your curriculum, where does it belong?

Our senior seminar in the history of mathematics has evolved as the rather unlikely confluence of answers to those two questions.

The MAA's Committee on the Undergraduate Program in Mathematics has long recommended the inclusion of a capstone experience in the major, leading to a written and oral report [9, p. 48], [10, p. 13]. Departments have many options for ways in which to fulfill those recommendations; we believe that a history of mathematics course at the end of a student's undergraduate career can offer a broad view of mathematics and an opportunity for exploration, collaboration, research, and exposition.

Similarly, there are many ways in which to introduce a history of mathematics course into the curriculum, from a first-year quantitative literacy course to a standard survey course, for majors or not; the prerequisites can range from intermediate algebra to abstract algebra. Our dean tried to convince us to create a course with very few mathematics prerequisites, so that it could be included as a history course in our Core Curriculum—and that was very tempting. But, in the end, we decided that we wanted our seniors to have an experience that tied together all the loose ends of their mathematics education and helped them see the big picture, and we think that a history of mathematics course is the perfect vehicle for doing that.

Our capstone "course" is actually a two-semester sequence: a seminar in the history of mathematics in the fall, followed by a research project in the spring. In the fall seminar, students read about the history of mathematics—in books, in journal articles, in primary sources, in print, and online—and present it to their peers. Each student writes a book review of a book of his or her choosing, and the class creates a time line for the department as a group. In the spring, each student chooses a research topic and submits a paper, a poster, and an annotated bibliography. At the end of the semester, we hold a celebratory poster session for the whole building and invite students, faculty, and administrators to admire the seniors' work.

Course Design

The goals for the two-semester sequence of courses are of several types:

1. **General goals for the mathematics major** that are addressed in this seminar (and in other courses):
    - Students will learn about the history of mathematics and communicate it to others.

- Students will learn about the contributions of women and non-Western scholars to the development of mathematics.
- Students will write about mathematics with clarity and precision. In particular, students will write about a topic in mathematics that they have researched independently.
- Students will speak about mathematics with clarity and precision.

2. **Content goals specific to this seminar:** Students will be able to…
    - Identify and discuss important people, events, and topics in the history of mathematics.
    - Understand the "big picture" of the development of mathematics—to see major trends, and not just isolated discoveries.
    - Have an appreciation for the development of mathematics in different cultures, at different times.
    - Read and understand primary sources in the history of mathematics.
    - Place events and trends in mathematical history in the larger context of cultural history.
3. **Process goals:** Students will be able to …
    - Identify and find well-known resources in the history of mathematics: books, journals, websites, films.
    - Read and review a popular book about the history of mathematics.
    - Use appropriate technology (MathSciNet, LaTeX, Beamer) to conduct research and present its results.
    - Conduct a thorough literature search and prepare an annotated bibliography.
    - Complete a research paper on an approved topic in the history of mathematics.
    - Present their research to their peers and others.

And, because it is these students' senior seminar, we have some more seminar-related goals for them: to bond as a group of senior mathematics majors, to become familiar with professional societies in mathematics, to attend a mathematics conference, to find where the mathematics journals are in the library (print and online), and to learn about graduate study and careers in mathematics.

For many years this course was offered every spring semester as a 3-credit senior seminar [27], so it really was the very last course mathematics students took in the major. Bowing to external pressures (secondary education majors now complete their internship in the spring semester), we have recently split the course into two distinct parts: a 2-credit history course in the fall and a 1-credit research course in the spring. While we grumbled about this new arrangement at the time, we now realize that it is ideal. Students spend an entire semester learning about the history of mathematics and about the many print and non-print resources at their disposal; then they spend another semester researching a topic of their choice and presenting it in several different ways.

The fall history seminar

This class meets for two hours each week. It is a seminar in the sense that students make almost all of the class presentations and lead the discussions. The instructor provides the administrative structure for the course—which topic is discussed on which day, which students are in charge, which homework problems are due, and so forth.

If you walked into this class on a random day, you might see...

- A student making a presentation, using the Beamer LaTeX class, of a chapter from one of the texts. She asks other students to answer questions or to come to the board to solve a problem.
- Students reading journal articles in groups and then describing and discussing them with other groups.

- Students as Pythagoreans, learning to play a musical scale on a monochord.
- Students constructing a giant time line, to be posted in the main mathematics department hallway and added to each week as students make their presentations.

The grade for the course is based on class leadership and participation, written solutions to problems, and a review of a book about the history of mathematics.

The spring research seminar

This class meets for one hour each week. The main goal is for each student to choose, research, and present —as a paper, as a poster, and as a conference talk—a topic in the history of mathematics.

During the one-hour class meetings, students gain more practice in using LaTeX and Beamer, especially in creating a professional-quality poster. They learn about conducting research and writing about historical topics. We visit the college library, and a research librarian gives a guest lecture on the many resources available there. We generally schedule a visit to another nearby library or museum, where students may get their first glimpse of old and rare books and other artifacts. Major university libraries often offer a wealth of resources; those who do not live close enough to a library or museum may still find resources online, as in the Object Groups of the Smithsonian's National Museum of American History [28].

Deadlines for various parts of the project (topic approval, annotated bibliography, outline, first and second drafts, final paper) are spread throughout the semester; see the course outline in the Appendix.

Resources

Fall history course

Texts

Students purchase these three texts:

- Berlinghoff and Gouvêa, *Math through the Ages* [4]. The organizing text for the course.
- Dunham, *Journey through Genius* [14]. Focus on several famous theorems and their history.
- Wardhaugh, *How to Read Historical Mathematics* [50]. An introduction to historiography.

We use them virtually every week in class. The course outline provided in the Appendix indicates how we use these and other resources during the course of the semester.

In addition, the following sourcebooks of primary sources are placed on reserve in the College library for the students in this course: Fauvel and Gray, *The History of Mathematics: A Reader* [16]; Katz, *The Mathematics of Egypt, Mesopotamia, China, India, and Islam: A Sourcebook* [24]; Struik, *A Source Book in Mathematics, 1200–1800* [45]; Smith, *A Source Book in Mathematics* [41]; and Stedall, *Mathematics Emerging: A Sourcebook 1540–1900* [43].

Journals and journal articles

In the fall semester, as students become familiar with mathematics journals, they are assigned specific articles to read. Some examples: On the first day of class, we read and discuss "Gauss's Day of Reckoning" [19] by Brian Hayes, an important lesson about the nature of history and the way it is passed down to us. After we have read about Mesopotamian mathematics, students locate two articles about Plimpton 322: "Sherlock Holmes in Babylon" by R. Creighton Buck [8], and "Words and Pictures: New Light on Plimpton 322" by Eleanor Robson[1] [38]. The class is divided into two groups (men vs. women is sort of fun), and each is

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[1] Of course the article by Robson that is really tempting to use is "Neither Sherlock Holmes nor Babylon: A Reassessment of Plimpton 322" [37], but that sort of gives it away.

assigned an article to read and discuss—and then to describe to the other group. In this exercise they read scholarly journal articles, get to know the *Monthly* a little bit, discuss a historical reading with their peers, and learn about the tools mathematicians and historians bring to understanding artifacts. One of the exercises in Wardhaugh's book [50, p. 19]mentions *The Ladies' Diary*. For homework, the students find and read Ivars Peterson's *Mathematical Tourist* blog entry on that topic [35] and write a page or so about what they have learned.

And then there is *Convergence* [11]. We spend one entire class in the computer lab, looking at this online journal and its many parts—articles, mathematical treasures, book reviews, quotations, and problems from another time. Especially for students who are going to teach mathematics, this will be a valuable resource.

Internet resources

In a homework assignment, students search for interesting, helpful websites in the history of mathematics and share them via an online class discussion board. They usually find the standard ones, including the MacTutor History of Mathematics archive [30], Donald Allen's history of mathematics course from Texas A&M [1], the website of The British Society for the History of Mathematics [7], the Teaching with Original Historical Sources in Mathematics project [25], David Joyce's site at Clark University [23], and an interesting side-by-side time line of mathematical and cultural history, from William Richardson at Wichita State University [36].

Wardhaugh recommends several excellent sites, including The Newton Project [29] and the website of the French Bibliothèque Numérique [5], which has digitized many primary sources.

A couple of favorites are the TimeMaps History Atlas: History of the World site [46], a collection of maps of the world, with important places and events marked, from 3500 BCE to the present; and the Cuneiform Digital Library Initiative [13], which presents "… the form and content of cuneiform tablets dating from the beginning of writing… until the end of the pre-Christian era." You can even download a CDLI app for your iPad or iPhone!

Students also learn to evaluate websites for accuracy, possible bias, and reputation of the author. The tutorial [48] and checklist [49] provided by the librarians at the University of California are very helpful.

Films

When I first started teaching this course, I amassed a great collection of 16-mm films. That collection gradually became one of VHS tapes, then of DVDs. My students have taught me that pretty much anything we need is now available on the internet, and that seems to be true. There is an amazing variety of video resources there, such as "The Man Who Loved Numbers," the NOVA program about Ramanujan (originally produced as "Letters from an Indian Clerk") [26]; the BBC documentary about the proof of Fermat's Last Theorem [17]; and the NOVA episode "Infinite Secrets," about the Archimedes Palimpsest [47]. Even "Donald in Mathmagic Land" [2] is posted by many users. We may be able to squeeze enough time out of the schedule to watch one or two films in a semester, and we always enjoy and learn from them.

Women in mathematics

Students may encounter a name or two of women mathematicians as they read their texts, but they are few and far between. To make sure students are aware of the contributions of women, and of the obstacles they have faced, we spend at least a week devoted entirely to women in mathematics. Our main tool is the department's collection of Teri Perl's *Math Equals: Biographies of Women Mathematicians + Related Activities* [33]. Although this book is written for younger readers, it is still a valuable resource. As its name implies, it includes not only a biography of each of nine women, from Hypatia to Emmy Noether, it also includes some mathematical activities related to the work of each mathematician. For a book that includes more twentieth-century women, a more diverse group of women, or women scientists more broadly, we also use Perl's *Women and Numbers: Lives of Women Mathematicians plus Discovery Activities* [34]; and *Celebrating Women in Mathematics and Science* from NCTM [12].

For a deeper discussion of *why* there have been relatively few women mathematicians, we read Lynn Osen's (admittedly somewhat dated) *Women in Mathematics* [31]. A quirky book that we use to introduce Pythagoras as well as a consideration of gender is Australian author Margaret Wertheim's *Pythagoras' Trousers: God, Physics, and the Gender Wars* [51].

We hold an annual Sonia Kovalevsky Math Day for high school girls at our college, and students from the seminar serve as assistants. At lunchtime, two seminar students give a "Who was Sonia Kovalevsky?" presentation, and we hand out t-shirts with Kovalevsky's name, and sometimes her picture, on them.

Non-Western mathematics

For a deeper appreciation of the contribution of non-European scholars to the development of mathematics, we also spend a week reading G. G. Joseph's *Crest of the Peacock* [22]. The author supplies several alternative trajectories for the development and transmission of mathematical knowledge, involving cultures in Egypt, Mesopotamia, India, China, the Mayan Empire, and the Arab world. Most current histories include at least a brief discussion of the mathematics in those cultures; the publication of the sourcebook [24] by Katz et al. has been a welcome addition to the literature.

Another very nice text that offers examples from cultures other than the usual ones is *Mathematics Across Cultures: The History of Non-Western Mathematics* [40]. Editors Helaine Selin and Ubiritan D'Ambrosio have collected works by scholars on ethnomathematics in general, and on mathematics in Pacific, Hebrew, Inca, Sioux, Aboriginal Australian, and other fascinating cultures.

Finally, a new text from NCTM provides instruction and classroom activities in "culturally situated" mathematics. In *Math is a Verb: Activities and Lessons from Cultures Around the World* [3], authors Jim Barta, Ron Eglash, and Cathy Barkley describe the trajectory of the use of culture in teaching mathematics—from traditional, to multicultural, to "culturally responsive mathematics instruction" (p. 2). In this text, they explore the ways in which mathematics and culture influence each other, both now and in the past. There are chapters on the mathematics of cornrow hair braiding, Navajo beading and weaving patterns, and fabric patterns from the Asante people of Ghana. Best of all, the book is accompanied by an interactive website [15], where students use Java applets to simulate the artifacts of other cultures and create their own designs. There are carefully-thought-out lesson plans for teachers and links to relevant scholarly publications. We spent one class watching a video of Ron Eglash's 2007 TED Talk (on the website) about African fractals and then worked through the online activity on fractals. But we could have explored skateboarding, or the symmetry of pre-Columbian pyramids. While these activities—both in the book and on the website—are designed for K–12 students, many of them are interesting and appropriate for college students.

The research seminar

Texts

The two basic texts[2] we use in this class are

- Booth, Colomb, and Williams, *The Craft of Research* [6]
- William Kelleher Storey, *Writing History* [44].

These texts introduce students to the two themes of this course: conducting research and writing about it. Both are written as common-sense guides for students as they navigate the process of choosing a topic, researching it, writing, revising, and repeating until they have a final product they can be proud of.

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[2] Many thanks to Amy Ackerberg-Hastings for introducing me to these resources.

Journal articles and internet resources

In this part of the seminar, students find their own resources. (See Assignments.) It would be unusual for a student to rely on a website for information at this stage of the course—except perhaps to find more scholarly resources.

Assignments

… in the fall seminar

Homework problems

In addition to making class presentations, students submit weekly homework assignments. Both *Math through the Ages* and *How to Read Historical Mathematics* have ample exercises for students, about mathematics and about its history. Problems to accompany *Journey through Genius* are posted on the *Convergence* website.

Making a time line

Early in the fall semester, after we have finished reading and discussing the "Nutshell" portion of *Math through the Ages*, the students start constructing a physical time line. They start with a long roll of white paper, markers, rulers, scissors, and adhesive, and spread it all out in the hallway of the science building. Students agree on a scale (one year they chose a logarithmic one!) and then organize the time line; they print out pictures and glue them to the paper; and they write on it. As the semester progresses, students add to the time line. It is a way to keep a visual image of what happened when and help to keep events straight. Everyone in the building looks forward to seeing the time line each year and watching it grow.

Attending outside events

Students must attend at least three mathematically-related outside events during the semester and write a page or so about what they have learned. They may be department seminars, lectures on campus or elsewhere, an MAA Section meeting, or volunteering at our Sonia Kovalevsky day for high school girls.

A book review

At the end of *Math through the Ages*, there is an intriguing list of "Fifteen Historical Books You Ought To Read… a short list of history books that we think are both readable and worth reading" [4, p. 248]. It includes books like *Poetry of the Universe* [32] (about geometry), *Longitude* [42] (about sailing and the longitude problem), and *The Lady Tasting Tea* [39] (about statistics)—all enormously instructive and entertaining. Each student chooses one of these books—if you have more students, you can add to this list—and reads it and then writes a review of it. We spend some time in class talking about the difference between a book *report* and a book *review*, and we read some reviews—in *Convergence* and other journals, in MathSciNet, and in *Aestimatio: Critical Reviews in the History of Science* [21].

… and in the spring research course

Finding resources

Students complete a series of assignments in which they learn to search the college library's catalog for print and electronic resources; find out how to use interlibrary loan; and learn to use several databases (MathSciNet, JSTOR, Academic Search Complete/EBSCO) to find scholarly journal articles. We walk over to the library together and tour the reference section (where they are introduced to the 16-volume *Dictionary* *of Scientific Biography* [18] and similar resources) as well as the stacks where the math history books are shelved.

Annotated bibliography

As students find resources, they add them to their annotated bibliography[3]; they must evaluate each article or book and explain how it will be helpful to them. This process helps them to organize their thoughts and refine their topic, and it is a quick indication of how much work they are doing and whether they are finding appropriate resources.

A poster

Now that posters are more common at mathematics conferences, especially for students, students need to know how to make a professional-looking poster. We use the poster-making capabilities of Beamer, which is already familiar to them. Sometimes we make a preliminary poster as a class ("Here are some great math history books that you should read!") and display it on a department bulletin board, so that students get practice in layout and design, including importing lots of graphics. Then each student produces his or her own poster. We have a Senior Poster Session at the end of the year and invite everyone in our building, plus the dean, to see the final results.

The final paper

Students know from the first day of the fall semester that this is where they're headed—writing a substantial paper on a topic in the history of mathematics, due at the end of the spring semester. Over the years I have shortened this assignment and required shorter and shorter papers, mostly for my own sanity. The current requirement is a paper of about twelve pages, 12-point font, double-spaced, not counting references. Students have a checklist of important questions (Do I have good primary sources? Is there mathematics in my paper? Is my paper different from what has already been written about this subject? How can I use all of my resources to say something interesting and new?), and I have a conference with each of them to discuss their choice of topic and availability of resources. Some recent topics: Women and Mathematics at the University of Göttingen, The Major Mathematical Contributors to Population Genetics (from a double mathematics/biology major), The History of the Mathematics Major at Hood College.

Final presentation

Each student is required to make either a poster presentation or give a talk at our local student research day, at our MAA Section meeting, or at another conference, like the Smoky Mountain Undergraduate Conference in the History of Mathematics in North Carolina. This final step in the research process makes students feel like professionals.

Lessons learned

First of all, as anyone who has tried to teach a history of mathematics course will tell you, it is impossible to do everything you want to do. You cannot cover every mathematician, every theorem, every event, every culture that you think is important. The sooner you make peace with that fact, the happier you will be. Focus on a few things—maybe different things each time you teach the course—and do them well. Know that your students are leaving your course knowing much more about the history of mathematics than they did when they came into it, and that they now have the tools to learn more on their own.

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[3] Many thanks to Sloan Despeaux for sharing her course materials on this topic.

When I first started teaching this course, I imagined that my students would arrive bearing the fruits of the first three years of a liberal arts education. They would be excellent readers and writers; they would bring to our class discussions a diverse array of knowledge about history and world cultures; they would have a firm grasp of many areas of mathematics. After all, we have a core curriculum, one that requires students to take courses in history, philosophy, literature, and Western and non-Western cultures. In a given semester, we would have students who had taken courses in ancient history, in medieval studies, in Middle Eastern culture, in Chinese philosophy. We would all share our knowledge and learn from each other. I chose Luke Hodgkin's wonderful text *A History of Mathematics: From Mesopotamia to Modernity* [20] and tried to lead discussions on Marxist approaches to historiography and paradigm shifts. I'm not sure what students I thought I was teaching. I had gotten to know them in mathematics classes for three years. I had acted as their academic advisor and watched the courses they chose to satisfy college requirements. I should have known that every single one of them chose History of the U.S. Since 1865 as the one history course they took in college. And apparently students no longer write term papers in freshman English. So I rediscovered the most basic rule of teaching: teach the students you have. Appreciate their strengths, and help them develop as students, as writers, and as scholars.

When I started using the *Math through the Ages* text, I tried to require each student to write a Sketch like the ones in that book for a final project: choose a topic in mathematics and trace its history, from its earliest days to the present. It didn't take me long to discover that the authors of the book had already done that. They found all the good topics and wrote about them. Not a good plan for a student project. In the very early days of the course, I also accepted a biography of a mathematician or a book review as a final project. Now I incorporate those things into other course assignments and do not allow them as final topics. In its current incarnation, the project must address something new. Ideally, it would state a question or a hypothesis and then an argument for it. (Gerbert could not have studied in Muslim Spain; Are there paradigm shifts in mathematics?) But I also accept expository papers on topics that have at least a little twist of something new, as in the examples cited earlier in this paper. I also no longer expect a 20-page single-spaced thesis. My expectations have become more realistic and, I think, more helpful to students.

Finally, teaching this course has given me a real appreciation for librarians. There are the ones I've never met—the ones who create online tutorials on how to write a book review or how to evaluate a website. There are the ones at the Library of Congress who lead research orientation sessions and set aside materials for our students. And there are the ones at our own college library who become genuinely excited about and involved in the seniors' research projects and want to know what they discover. Their knowledge and their helpfulness have made life much easier for me.

I learn something every time I teach this course. And I enjoy watching the students learn. I am convinced that a course in the history of mathematics can serve as a perfect capstone to an undergraduate degree in mathematics.

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Apendix A
Course Outlines

Fall Seminar in the History of Mathematics

Week	**Topic**	***Math through the Ages***	***Journey through Genius***	**Wardhaugh, other resources**	**Research and presentation skills**
1	Overview	History of mathematics in a large nutshell		*Mathematical Universe*, “Origins”	LaTeX mathematics mark-up language
2	Numerals, symbols	Nutshell, continued Sketches 1,2	Preface	Wardhaugh, Chap. 1. What does it say?	Beamer presentation software
3	Zero, fractions Negative numbers, metric measurement	Sketches 3-6	1. Hippocrates	Wardhaugh 2: How was it written?	Department rubric for oral presentations
4	Reading about ancient mathematics Primary sources			Two articles about Plimpton 322 Sourcebooks on reserve	Finding and evaluating web sites in the history of mathematics
5	Greek mathematics	Sketches 4, 12	2-3. Euclid	Wardhaugh 3: Paper and ink Monochord “lab”	*Convergence*, the online journal of the history of math
6	Greek mathematics, continued		4. Archimedes 5. Heron	Wardhaugh 4: Readers	How to write a book review
7	Non-Western mathematics			Excerpts from *Crest of the Peacock*	
8	The beginnings of algebra	Sketches 8-11	6. Cardano	Wardhaugh 5: What to read, and why	The Mathematical Genealogy Project
9	The discovery of calculus	Sketch 13	7. Newton 8. The Bernoullis		
10	Calculus, continued Oral reports of book reviews		9-10. Euler		
11	Geometry & trigonometry Complex numbers Logic and set theory	Sketches 14-16, 19-20, 25	11-12. Cantor		

12	Women and mathematics			Teri Perl book	
13	Women and mathematics			Teri Perl book	
14	Mathematics at Hood College			Hood archives, catalogs, yearbooks	

Spring Research Project

Week	**Class activities**	***The Craft of Research***	***Writing History***	**Due this week**
1	Introduction	I. Research, researchers, and readers	1. Getting started	
2	Guest instructor: Hood College research librarian			
3	Using the library's databases to find journal articles	II. Asking questions, finding answers	2. Interpreting source materials	Topic: first draft
4	Visit Hood College library. Find reference materials.		3. Writing history faithfully	
5	Building an annotated bibliography	III. Making a claim and supporting it	4. Use sources to make inferences	Literature review
6	Work in pairs to refine research topics		5. Get writing! Get organized. 6. Narrative techniques for historians	Annotated bibliography: first draft
7	Individual conferences with instructor		7. Writing sentences in history	Topic: updated
8	Graduate school in mathematics: investigation		8. Choose precise words	Outline: first draft
9	Careers in mathematics: investigation	IV. Planning, drafting, and revising	9. Revising and editing	Annotated bibliography: update
10	Poster session for the science building			Poster
11	Field trip to the Smithsonian	V. Some last considerations		Final paper: first draft
12	Professional societies in mathematics			Outline: update
13	Final presentations			Final paper: second draft
14	Final presentations			Final paper, annotated bibliography

Geometric Construction as a Unifying Theme in a History of Mathematics Course

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Introduction

I teach a history of mathematics course that serves as the capstone course for the mathematics major at Ramapo College of New Jersey. This means that the majority of students in the course are seniors and have taken most, if not all, of the courses in the major. The course is offered each fall semester and has as prerequisites, introduction to analysis (our real analysis course) and abstract algebra. While not required, most of the students in the course will also have had our junior-level geometry class.

One of the main issues confronting the instructor of a history of mathematics course is how to deal with the variety and complexity of the possible topics. One can adopt an episodic approach, briefly discussing many different events. Given the richness of the history of mathematics, it is natural for an instructor to focus on a few episodes. The problem with this method is the lack of pedagogical coherence. What do students take away from such a course besides some isolated pieces of knowledge?

The course at Ramapo College strives to cover a typical variety of historical topics (see Appendix A) but also uses a thematic approach that allows students to make connections among the disparate historical events. In particular I use geometric construction as a unifying theme. This theme connects Greek geometry with the geometry of Descartes and the work of Gauss, ending up with a discussion of Galois theory.

Construction has several benefits as a theme in a history of mathematics course. The theme spans more than two thousand years of mathematics, which allows students to see the coherence of the history of mathematics. Furthermore, this theme touches on areas of geometry and algebra, giving the students the opportunity to integrate material from several previous courses. Finally, it can be an entertaining hands-on experience for the students. They can build things! Below I describe the tomahawk, an instrument for angle trisection that I have the students build as part of a homework assignment.

Course Design

From Euclid to Galois, geometric constructability is a unifying theme that we exploit in this course. The syllabus covers many topics besides construction; see the list of the topics covered in the appendix. Geometric construction provides a way of bridging the distance between Greek and modern mathematics. The module on geometric construction covers the following topics:

- Euclid and the origins of Greek geometry
- Compass and straightedge construction in Greek geometry

- Alternative constructions
- Constructible numbers in Descartes
- Constructible polygons in Gauss
- Impossible constructions (e.g., angle trisections)
- Introduction to field theory, including field extensions
- Quadratic extensions and constructible numbers
- Splitting fields
- Galois theory

Let us consider some of these topics in some more detail.

Construction in Greek geometry

In antiquity geometric constructions played an essential role in the development of geometry. The requirements of compass and straightedge as the instruments of geometric construction make for a consistent way of describing the proper content of geometry. Many of the results in Euclid concern constructible geometric figures, circles, parallel and perpendicular lines. All of these results are directly connected to the rules of construction found in Euclid.

The class discussion starts off with the basic constructions, in a sense these are the building blocks of construction; namely, finding the midpoint of a line segment, finding a line through a given point perpendicular to a given line, finding a line through a given point parallel to a given line, and bisecting an angle.

The next step is to consider compass and straightedge constructions of regular polygons. It is important for students to see the historical importance of polygon constructions. The very first proposition in Euclid's *Elements*, the apotheosis of Greek mathematics, is concerned with the construction of an equilateral triangle [3, Vol. 1, pp. 240–243]. From there the class considers the construction of more complex polygons, such as the pentagon, decagon, and others (even the 17-gon).

A particularly simple method for constructing the pentagon can be found in Appendix B. Constructing regular polygons can be done directly using the compass and straightedge, but such constructions can also be performed using constructible numbers, as discussed below. For example, the side length of a regular pentagon inscribed in a circle of radius 1 is $\frac{1}{2}\sqrt{10-2\sqrt{5}}$, which is a constructible number.

Alternative constructions

It is useful to see what happens if one modifies the Euclidean construction rules of compass and straightedge. There are different ways of changing the rules of the game. For example, the Italian mathematician Lorenzo Mascheroni (1750–1800) considered what constructions are possible using the compass alone. Interestingly enough, the constructions that are possible using compass alone are equivalent to those that are possible with compass and straightedge. This assumes that if two points are constructed with the compass then one may consider the line connecting them to be constructed. The interesting article by Hungerbülher [6] gives a proof of this result.

As an example of a construction with compass alone, I have assigned the problem of finding the midpoint of a line segment using compass alone. A more interesting construction problem is that of finding the inverse of a point with regard to a circle. Two points P, P' are said to be inverse with respect to a circle C of radius r if the points are collinear with the center O of the circle and we have $OP \cdot OP' = r^2$. The inverse of a given point can be found using compass and straightedge or by just compass alone.

Van Schooten is a Dutch mathematician who, in 1649, translated Descartes' *Géometrie* into Latin along with commentaries. He also investigated constructions that are possible using only a ruler, but no compass.

Euclidean constructions only allow for an unmarked straightedge, but one of the rules for Van Schooten's constructions is one can copy one line segment onto another line. This requires a marked straightedge. Van Schooten showed that using a ruler one can bisect an angle and bisect a line. A good overview of Van Schooten's ruler constructions can be found in the online article by C. Edward Sandifer [13].

Constructible numbers

Descartes, in his *Géometrie*, first showed how to construct numbers using compass and straightedge. Starting with a particular line segment designated to be our unit length, we define a number a to be constructible if a line segment $|a|$ units long can be constructed using compass and straightedge.

In particular Descartes demonstrated [2, pp. 2–13] that the set of constructible numbers is closed under the operations of addition, subtraction, multiplication, division, and taking square roots. Clearly the set of constructible numbers will include all rational numbers.

The topic of constructible numbers opens up an important connection between geometry and algebra. The set of constructible numbers forms a field that properly contains the rational numbers. Since we can construct irrational numbers only through the square root operation, if c is any irrational constructible number, the degree of the field extension $\mathbb{Q}(c)$ over the rational numbers will be a power of 2. This fact is useful in proving that certain constructions are impossible using only compass and straightedge, for example, the impossibility of trisecting arbitrary angles with compass and straightedge.

To see an example of how to construct the division of two constructible numbers, see Appendix C.

Constructible polygons

Which regular polygons can be constructed with a compass and straightedge? For the Greeks it was clear that one could construct an equilateral triangle, a square, a regular pentagon, hexagon, octagon, and decagon. But what of the other polygons? An open question for the Greeks was whether or not the regular heptagon (seven-sided polygon) is constructible. The inability to construct the heptagon must have certainly led to doubts about its constructability. The answer to this and other questions of which polygons are constructible is answered by Gauss in the last past of his monumental 1801 work *Disquisitiones Arithmeticae* [5, pp. 450–460].

In this work, Gauss proved the celebrated theorem that a regular n-gon is constructible if the number of sides equals a power of two times a product of Fermat primes (including none). A Fermat prime is a prime of the form $2^{2^n}+1$ (the only known such primes being 3, 5, 17, 257, and 65537). Wantzel in his 1837 paper [15] proved the converse of this theorem. I give a proof in class of Gauss's theorem using Galois theory (see Gallian [4, p. 576]).

In our class we will also show that the 17-gon is constructible. This allows me to introduce cyclotomic polynomials and their splitting fields, which makes another connection between the students' knowledge of abstract algebra and geometry. Klein, [10, pp. 19–23] gives a good discussion of cyclotomic polynomials and the construction of the 17-gon.

Impossibility proofs

There were several constructions dating from antiquity that appeared to be impossible, but whose proof of impossibility waited until the 19th century. The impossibility of trisecting the angle, duplicating the cube, squaring the circle, and constructing the heptagon are topics that we cover in class. Let's look in more detail at the problem of trisecting the angle.

While it's true that one cannot trisect an arbitrary angle using compass and straightedge, it is possible to trisect specific angles, such as a right angle. In class we show a counterexample to the trisection of arbitrary angles. We prove the impossibility of trisecting a 60° angle, with compass and straightedge, using an argument from Gallian [4, pp. 399–401].

If we relax the rules for construction then it is in fact possible to trisect angles using relatively simple methods. In class we discuss two trisection methods. The first method involves a marked straightedge. In other words, we may mark off distances on the straightedge, which is not allowed under the usual Euclidean rules.

An even more interesting trisector is the instrument known as the tomahawk. The origins of this device are not clear, but a discussion of its history and other details about it can be found in Martin's text on geometric constructions [11, pp. 20–21]. A model of the tomahawk can be seen in Figure 1. One orients the tomahawk with respect to the angle so that the vertex of the angle lies on the perpendicular through point B, one side of the angle passes through point C and the other side of the angle is tangent to the semicircle. Students are given the assignment to actually build a tomahawk and use it to trisect an arbitrary angle. To construct the tomahawk I recommend that students use cardboard or a thick stock paper. A few ambitious students over the years have made the tomahawk out of plastic. Also, multiple tomahawks can be connected to divide an angle into any odd number of parts. See Appendix D for more details.

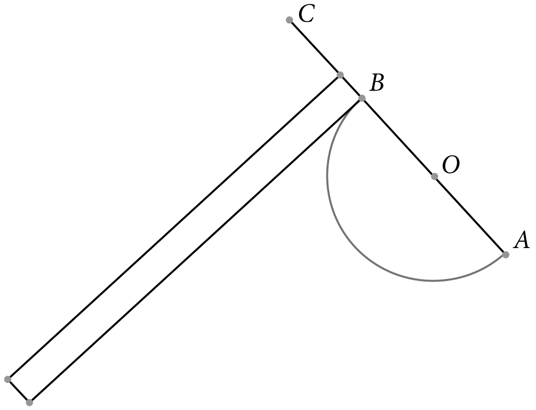

Figure 1. Tomahawk

Galois theory

I give students a quick tour of Galois theory, starting of course with the tragic life of Évariste Galois. We cover the usual topics, field extensions, irreducible polynomials, minimal polynomial, splitting fields, the Galois group, the Fundamental Theorem of Galois theory, and solvability by radicals. As mentioned earlier, I use Galois theory to prove Gauss's theorem on constructible polynomials. As a reference for this material I use the texts by Gallian [4] and Cox [1].

The students in class have previously had courses in linear algebra and abstract algebra, but it is clear that students need basic ideas reviewed. For example, in discussing the degree of a field extension one will refer to the concepts of a basis and dimension from linear algebra. It's perhaps sad, but not unexpected, that students are not able to define what a basis is (even informally). So I need to go over the basic idea of a basis in a vector space and then give examples from field extensions. On the positive side, I think that the concrete way in which vector space ideas are applied in this course will help reinforce those concepts for the student.

For example, we know from Descartes that $\sqrt{2}$ is constructible, hence the extension over the rationals, $\mathbb{Q}(\sqrt{2})$, gives a set of constructible numbers. In this case students can readily see that $\{1, \sqrt{2}\}$ forms a basis for that extension.

I also emphasize the Galois correspondence between subfields of the splitting field and subgroups of the Galois group. This helps students see an application of the concepts they learned in their abstract algebra course.

Resources

As far as resources for this course, I have the students buy Katz's textbook [9] as a general resource. There are several excellent books that discuss construction problems, for example, I would recommend the works of Courant and Robbins [2] and Felix Klein [10]. The delightful book by Dudley [4], shows what happens when math enthusiasts believe, despite all proofs to the contrary, that they have devised a method to trisect angles with compass and straightedge. A well-written article on impossibility proofs is that of Suzuki [14]; also see the article by Alex McAllister in this volume.

As mentioned before, the course doesn't focus solely on the theme of constructability. We also cover Babylonian and Egyptian mathematics; Greek mathematics including Euclid, Archimedes, and Diophantus; Islamic mathematics, focusing on the origins of algebra; Italian algebra of the Renaissance and the solution of the cubic; and the sources of calculus in the work of Fermat, Newton, Leibniz, and Euler. As time permits we may examine the history of additional topics such as continued fractions, an elementary discussion of elliptic curves, and the computation of π.

Assignments

With regard to homework assignments, I give students three types of problems. There are problems where I expect students to perform a familiar calculation by following a particular historical method. For example, I will give students Egyptian *aha* algebra problems using false position, problems taken from the *Arithmetica* of Diophantus, or derivatives calculated using the method of Fermat.

I also assign problems that originate in an historical context but further enrich the students' mathematical knowledge. Such problems may deal with figurate numbers; compute generating functions; or relate to Galois theory.

The third type of problem which I assign asks students to research and write an essay of two to five pages on a particular historical topic. There are generally four to five assignments during a semester. Each homework assignment will have a required essay. Here are some typical essay topics,

- Write an essay on the history of the Pythagoreans (their beliefs, their personalities and their mathematics). You must use at least two sources which were not written for the Internet.
- Write an essay on the history of Zeno's paradoxes. You must properly cite all sources used.
- Write an essay on the history of the concept of function in mathematics.

With regard to the theme of geometric construction, there will be relevant problems on most homework assignments. Here is a short list of such problems.

- Consider a regular pentagon with length of side 1. Let x be the length of a diagonal in the pentagon. Prove that $x = \dfrac{1}{x-1}$. Then solve for x.
- Construct a tomahawk and use it to trisect an angle. You must turn in the tomahawk (with your name on it) and the sample angle that you used for the trisection.
- Given a circle, show how to find its center using only a compass.
- Construct a line segment that is $\dfrac{4-\sqrt{5}}{1+\sqrt{2}}$ inches long.
- Find the minimal polynomial for $\sqrt[3]{1+\sqrt{2}}$ over the rationals. Find the splitting field K of the minimal polynomial over the rationals. This number is not constructible. What impact does that have on the splitting field?

- Compute the side length of a pentagon inscribed in a unit circle. Compute the minimal polynomials for these values, the splitting fields for these polynomials and the corresponding Galois groups.
- Find the minimal polynomial for $\omega = \cos(2\pi/7) + i\sin(2\pi/7)$, the splitting field of this polynomial, the Galois group of the splitting field, and the Galois correspondence of subgroups and subfields.

All of these problems have been assigned and generally successfully completed by the students in the course (generally after some hints on my part).

Lessons Learned

There are many ways to organize a course in the history of mathematics. The use of a theme, such as geometric construction, helps make the history of mathematics appear not as a sequence of random discoveries, but as having a discernible pattern. There are many themes that could be used in this course, for example, continuous versus discrete, the concept of infinity, applications of mathematics, etc.

How well has the theme of geometric construction worked in this course? I have not done a formal assessment, but my experience is that students enjoy geometry because it's visual and has a greater immediacy than more abstract parts of mathematics. Also the topic connects to things they see in linear and abstract algebra. Furthermore, students always find enjoyment in building things, which they get in the tomahawk assignment.

In conclusion, I have enjoyed including this theme in my course and recommend it to other instructors looking for ways of using geometry in their history course.

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Appendix A

Course Content

Here is the course content for our history of mathematics course. The order of the list does not correspond to the order of topics in class. I will vary the order of topics once Greek mathematics has been covered. Generally the course will cover some subset of these topics (but not the same ones every year).

- (I) What is mathematics?
  - (A) A brief overview of the contents and goals of the major areas of mathematics
  - (B) A brief overview of the historical development of these areas
  - (C) The cultural impact of mathematics
- (II) Ancient mathematics
  - (A) Number systems
  - (B) Babylonian mathematics
  - (C) Egyptian mathematics
- (III) Geometry
  - (A) Euclidean geometry
  - (B) Ruler and Compass constructions
  - (C) Impossibility problems: trisecting the angle, duplicating the cube, squaring the circle
  - (D) Descartes and constructible numbers
  - (E) Alternative constructions
    - (i) Mascheroni constructions with compass alone
    - (ii) Rusty compass constructions
    - (iii) Van Schooten constructions with ruler alone
  - (F) Gauss and constructible polygons
  - (G) The 17-gon
  - (H) Tomahawk and angle trisection
  - (I) Galois Theory
    - (i) Separable polynomial
    - (ii) Minimal polynomial
    - (iii) Field extensions
    - (iv) Galois groups
    - (v) Proof of Gauss's theorem on constructible polygons
- (IV) Algebra
  - (A) Diophantus and Diophantine problems up to Fermat's Last Theorem
  - (B) Khwarizmi and the beginnings of algebra
  - (C) Cardano and the cubic equation
- (V) Creation of calculus
  - (A) Fermat
  - (B) Newton
  - (C) Leibniz

Appendix B
Constructing the Pentagon

Consider the specific example of the construction of the regular pentagon. In class we look at three approaches to this problem: the method of Euclid, a simpler construction of Richmond, and the algebraic approach of determining the side length of the pentagon as a constructible number.

One finds the construction of the regular pentagon in Euclid's *Elements*, Book IV, Prop. 11 [3, Vol. 2, pp. 100–102]. Euclid's proof begins by inscribing a 36°-72°-72° triangle in a circle. The ability to construct such a triangle is given in the previous proposition. Overall this is a rather complicated method to construct the pentagon. Thomas Heath makes a comment concerning the proof, that despite its complexity, "I think we shall conclude that the method is nearer to that used by the Pythagoreans, and therefore of much more historical interest" [3, Vol. 2, p. 102]. A simpler approach can be found in a paper by Richmond.

The 1893 article by H. W. Richmond [12] looks at the construction of the heptadecagon, the 17-gon whose construction was first considered by Gauss, but Richmond also gives a simple procedure for the construction of a pentagon. In Figure 2, begin by drawing a circle, centered at O with diameter AG. Construct a radius OB perpendicular to the given diameter. Find the midpoint C of OB. Then bisect $\angle ACO$, with the bisector intersecting the diameter at D. Draw the perpendicular at D, intersecting the circle at E. Then AE is one side of the inscribed pentagon. To see why this is the case, assuming the radius of the circle is 1, then $OC = 1/2 \Rightarrow CA = \sqrt{5}/2$. The fact that CD is an angle bisector and $AD = 1 - OD$ allows us to conclude that $OD/DA = OC/CA \Rightarrow OD/(1-OD) = (1/2)/(\sqrt{5}/2)$. We then find that $OD = (\sqrt{5}-1)/4$. We also know that $\cos 72° = (\sqrt{5}-1)/4$, hence $\angle EOD = 72°$. Since this is the measure of the central angle of the pentagon, AE must be a side of a pentagon.

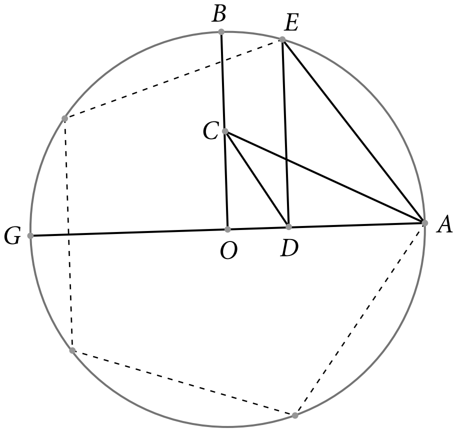

Figure 2. Construction of the pentagon

Appendix C

Division of Constructible Numbers

To give a sense of how the Cartesian construction of numbers works, let us consider the operation of division. If a and b are constructible then a/b is constructible as follows (we may assume that a, b are positive numbers).

In Figure 3, create an arbitrary angle, $\angle AOB$. Mark off segments $OA = a$ and $OB = b$. Place point D on OB such that $OD = 1$. Draw a line through D parallel to AB, intersecting OA at C (note, if $a < 1$ then point C will be on an extension of OA). Then ΔOAB will be similar to ΔOCD (since CD is parallel to AB). Therefore by the ratio of corresponding sides,

$$\frac{OC}{OD} = \frac{OA}{OB} \Rightarrow \frac{OC}{1} = \frac{a}{b} \Rightarrow OC = \frac{a}{b}.$$

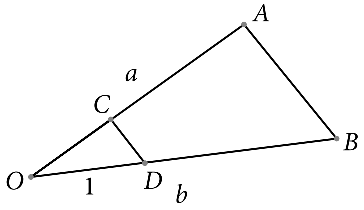

Figure 3. Division construction

Appendix D
Trisection using a Tomahawk

To see how the trisection is performed, consider Figure 4. The trisection is done by orienting the tomahawk so that three conditions obtain:

i. One side of the angle is tangent to the semicircle. In Figure 4, point *D* is the tangent point.

ii. The "handle" of the tomahawk connects points *Q* and *B*. Note, the handle is shown as a single line, in order to not obscure the picture. But in practice students will need to have a handle with a positive width, so as to be able to grasp the tomahawk and maneuver it inside the angle.

iii. Point *C* lies on the side of the angle opposite to the point of tangency.

The proof that the trisection is valid is quite straightforward. Connect points *O* and *D*. Since *D* is a tangent point on the circle, $\angle ODQ = 90°$. Also, the handle is perpendicular to the top of the tomahawk which gives $\angle CBQ = \angle OBQ = 90°$. Further, we know that $OB = OD$ (both are radii of the semicircle) and $OB = BC$ (by construction). That gives us that ΔQBC and ΔQBO are congruent (by side-angle-side) and ΔQOD and ΔQBO are congruent (by hypotenuse-leg, or side-side-side). Therefore we conclude that $\angle CQB = \angle OQB = \angle OQD$, which means that the angle at vertex *Q* has been trisected.

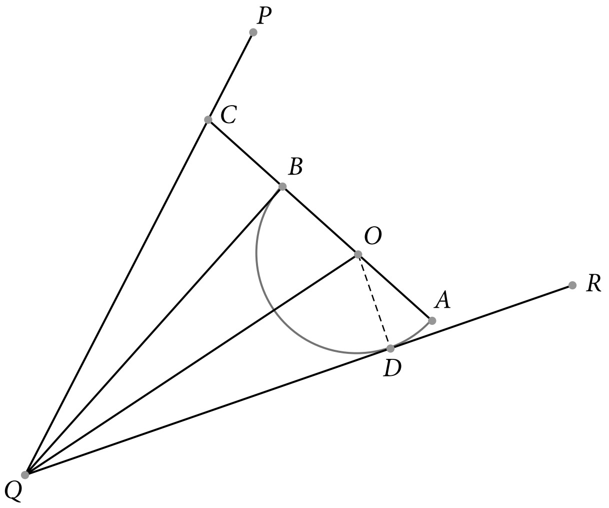

Figure 4. Angle trisection

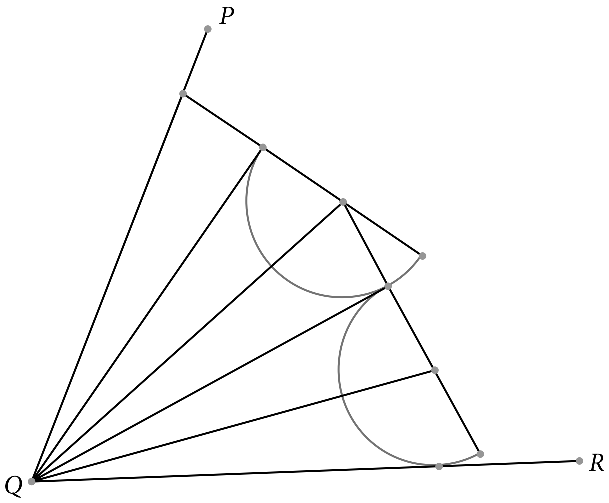

Figure 5. Angle quintisection

Multiple tomahawks can be used to divide an angle into any odd number of equal angles. Figure 5 shows how to quintisect the angle $\angle PQR$ using two equal sized tomahawks.

To summarize, using compass and straightedge, one can divide an angle into a number of equal parts, if that number is a power of two (namely, one keeps on bisecting angles). One can use tomahawks to divide an angle into any odd number of parts. For an arbitrary positive integer *n*, one can write it in the form $n = 2^k(2m + 1)$. Therefore one can perform bisections to divide the angle into *k* equal parts. Then one can use *m* tomahawks to divide each of those *k* angles into $2m+1$ equal parts, thus the entire angle has been divided into *n* equal parts.

II

Courses for Pre-service/ In-service Teachers

Start Talking! A Course Questions Approach to Teaching the History of Mathematics

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Course Description

How can instructors motivate their students not only to intelligibly talk about mathematical and historical ideas, but also to reflect deeply upon such ideas in order to develop their own robust understanding of mathematics' rich traditions and modern disciplinary methods?

The upper division history of mathematics course described in this chapter revolves around a set of Course Questions that can provide instructors with a means for achieving these two critically important pedagogical aims. In relation to these overarching aims, examining pedagogical issues related to secondary mathematics teaching through the lens of the historical development of mathematical concepts and techniques quite naturally becomes an integral theme for the prospective secondary mathematics teachers who are required to take the course. Because the course is also used as an elective for other junior and senior level mathematics majors and minors, however, its primary focus remains on enriching students' mathematical understandings of the content, methods, and standards present in today's mathematics curriculum through a careful study of the origins and development of mathematics in various civilizations. In particular, the course is designed to help students:

- *develop an understanding of the chronological development of mathematical thought* through a careful study of the key landmarks (e.g., people, problems, places) which mark that development;
- *develop an understanding of the forces that shape mathematical ideas*, including the influence of technology, communication, religions, schools, and other social institutions;
- *develop a base of historical knowledge, research skills, and resources* with which to reflect upon and converse about the nature and role of mathematics in today's world and classroom, through a careful study of how mathematics has (or has not) developed in various cultures and time periods; and
- *gain experience in the interpretation of the mathematical activity and thinking of others* through the careful examination of primary source documents.

Two premises about the history of mathematics further inform the course philosophy. First, history of mathematics is neither history alone, nor mathematics alone—both are essential ingredients. Second, history of mathematics involves more than simply viewing the past in terms of its relevance for the present. These premises open up new learning challenges and opportunities for students as we work together to situate mathematical developments of the past within their broader scientific, intellectual, and social contexts. For example, having completed linear algebra as a course prerequisite, students enter with an understanding of certain contemporary mathematical concepts, as well as experience with certain contemporary problem-solving methods. They tend to be less knowledgeable about world historical events, and relatively inexperienced in the practice of historical judgment and research methods. Thus, although neither they nor I can avoid looking through 21st century lenses as we examine the story of mathematics' past together, students can use these same lenses as a means to expand their historical knowledge and skills by examining the work of past mathematicians on its own terms through consideration of questions such as:

- What did these mathematicians think they were doing?
- What were their goals, interests, and methods?
- What did they think they had accomplished?
- What was the broader cultural context of their work?
- How did their cultural context influence their work, and vice versa?

As described in the next section, the overall course procedure has been designed with opportunities and challenges such as these in mind.

Course Design

In addition to the premises mentioned above, my thinking about this course has been guided by the learning theory of *constructivism*, a theory which proposes that all human learning is an active process of (individually and socially) constructing knowledge, rather than merely acquiring it. The instructional strategy that has evolved over the 20 years that I have been teaching the course with these ideas in mind is a modified seminar structure that combines careful reading of secondary and primary sources, active participation in whole and small group discussions, reflective and research writing, oral presentations, and peer evaluation. Details of homework and written projects are given in a later section; the remainder of this section describes the modified seminar instructional strategy itself.

A key component of this instructional strategy is a Course Questions List consisting of a set of summary/reflection essay questions, each of which is tied to one or more of the general course goals. Distributed on the first day of class with the expectation that students will construct a response to each question by semester's end, this list thus defines an explicit set of learning objectives for the course. The Course Questions further serve as the focus for the assessment of student learning; for example, any exam given in the class consists primarily of questions taken directly from this list. A complete list of Course Questions for a sample semester is included in Appendix B.

As an important adjunct to the Course Questions List, Reading and Study Guides (RSGs) tied to the course textbook(s) are distributed on a regular basis. As illustrated in the sample RSGs included in Appendix C, questions on each RSG are framed to support the construction of responses to the Course Questions, while also providing a basis for class discussions of assigned readings. Students are thus given the opportunity to reflect, both individually and collectively, upon the Course Questions over the entire course of the semester. In order to ensure that students take advance class preparation seriously, RSGs are collected at least weekly, reviewed by me for completeness, and returned to students to be placed in a required Course Notebook (described below) to become part of their course grade.

Class time itself is spent primarily in whole-group discussion of historical and mathematical ideas related to questions from the assigned RSG for that day. We simply rearrange desks as needed to allow us all to sit facing each other, and start talking! Because the RSG assigned for each day focuses students' advance reading on very specific questions, all students are able (and expected) to contribute to these discussions. Yet individual contributions can be quite varied, as different students pick up on different aspects of the reading or the RSG questions. Students are thus encouraged to add notes to their RSGs during class discussion (provided they somehow distinguish between their personal reading notes and their notes from the classroom discussion). To open up discussion beyond those aspects of the assigned reading that I have preselected as our focus, RSGs also prompt students to take note of anything they find interesting, useful or perplexing about the reading. Even when students do not volunteer these points in class, their written responses on the RSG provide me with feedback on what they are taking away from the reading, and whether there are questions, concerns, or insights that should be incorporated into later discussions.

After developing a RSG for each assigned reading, I also complete it myself prior to class discussion in order to clearly identify the critical ideas to highlight for each question. Interestingly, I find that these ideas are typically raised by the students themselves in the natural course of discussion. To allow ample opportunity for students to make these contributions, I generally save my own comments about each RSG question until everyone else has shared their thoughts, restricting my initial contributions to clarifying questions, paraphrasing of student responses, and generally directing the overall discussion.

Especially while students are becoming accustomed to class discussions in the early part of the semester, I often have them report out in “round-robin” style, with each person taking a turn to share something from his or her response to a particular RSG question until everyone has shared at least one thought and no one has a new idea left to share. Alternatively, I may ask one student (sometimes a volunteer, sometimes a student of my choosing) to share his or her response to a particular RSG question, with each of the others then taking turns to react to that response—perhaps adding to it in some way, perhaps modifying some aspect of it, perhaps disagreeing with it altogether, but always based on a rationale drawn from the reading that they also share with the group. By giving each student an equal chance to participate, these round-robin strategies minimize the possibility that any one person will dominate discussions. Because I consciously vary the order of turn-taking, students also quickly realize the importance of staying attentive to what their peers have to say. Once they become accustomed to both listening and responding to each other, later discussions tend to flow as a more natural exchange of ideas and questions.

When beneficial for exploring topics related to the readings or for addressing specific course objectives, class discussions of RSG questions are supplemented by whole-group primary-source reading sessions, student or instructor presentations of mathematical exercises that expand on assigned readings, and student reports on background cultural topics. These latter reports play an especially important role in setting the context for our discussion of mathematics from other cultures. During the medieval period, for example, mathematics was studied in four quite diverse cultures: India, China, Islam and Western Europe. Prior to our discussion of the mathematics itself, students are placed into four groups, each of which is assigned one of these regions. Group members then research the distinctive elements of the religion(s) (e.g., Islam) and/or philosophical system(s) (e.g., Confucianism) that were influential within that region in the medieval period, identify any other general cultural or historical features that may have impacted the development of mathematics in that region at that time, and prepare a group presentation. In addition to making each student a class expert on one of these four cultures, their presentations allow us as a class to compare and contrast the impact of different cultures and religions on the development of mathematics during the medieval period.

Each student also presents their own individual research projects (details below) related to mathematical developments of the 18th–20th centuries. Because our 15 weeks together is too short to do justice to over 5000 years of mathematical development, and in consideration of the interests of secondary mathematics teachers, class readings and discussions themselves focus instead on the general development of calculus and associated topics in algebra, analytic geometry and trigonometry through about 1700 CE. Appendix A provides a sample semester schedule showing how this can be accomplished. Consideration of select topics from later periods and more detailed study of particularly illustrative early episodes are included as time permits in a given semester.

Resources

The most essential resource for this course is a good general history of mathematics text. Although an increasing number of general textbooks are available, they are not all created equally. Some are mathematically oriented, with little attention to issues of historical interpretation. Other authors adopt the opposite stance, or aim for some balance between these two poles. Historical treatments of mathematics might also focus heavily on biography or social context, with little attention to factors that are more internal to the discipline (e.g., unsolved problems and standards of proof), or vice versa. For the course described in this chapter, the three most critical features needed in a textbook are (1) a reasonably even treatment of both the mathematical and historical issues related to the topics and time periods to be discussed; (2) that it is written at a level accessible to the students; and (3) a collection of mathematical exercises suitable for use as homework for upper division mathematics student. For these reasons, *History of Mathematics: An Introduction* by Victor Katz is the text that I most often use as our core text, in either the full edition [7] or the brief edition [9]. I have also used Jeff Suzuki's *A History of Mathematics* [13] with good results.

In recent offerings of this course, I have required the two following texts in addition to our central general textbook:

- Jacqueline Stedall's *The History of Mathematics: A Very Short Introduction* [12]
  Reading this superb book during the first week or two of the semester sets the stage for the remainder of the course by giving students a quick overview of the types of questions that historians ask, and the techniques they use to answer them.
- Benjamin Wardhaugh's *How to Read Historical Mathematics* [15]
  Interspersing readings from this text with selections from the core course textbook supports students' ability to read primary sources in preparation for their required research project, while underscoring key historiographical issues raised in Stedall's text.

Other resources that I make available to students through university library services (e.g., reserve, electronic data bases, or interlibrary loan) include collections of primary sources (such as [4], [5], [8]), collections of expository and scholarly articles on the history of mathematics (such as [1], [2], [3], [10], [11], [14], all published by the MAA), and research level journals (e.g., *Historia Mathematica*). The *Dictionary of Scientific Biography* is another valuable resource for student research projects, as are certain web resources on the history of mathematics.

Each of the resources named above can also help the instructor develop a deeper understanding of both the historical development of a particular topic, and the variety of approaches to treating the history of mathematics that exist. A collection of currently available textbooks on the history of mathematics is another especially useful resource for instructors, and a small collection of such textbooks should also be made available to students if the textbook comparison report described below is required.

Assignments & Course Grading Components

While the precise mix of assignments that I use in this course and the weight of each in determining final grades varies somewhat from year to year, the assignments described below have become regular features of the learning and assessment structure of the course.

Participation, Preparation, and Citizenship (PPC)

Each student is assigned a PPC score for the semester, based on in-class participation and engagement levels, including quality of contributions to whole-class and group activities and level of preparedness for discussions (evidenced in part by completed RSGs). On the rare occasion that concerns arise about levels of class participation or preparedness that could adversely affect a PPC score, these are discussed privately with those involved, generally to good effect.

Homework Sets

To consolidate individual understanding of the mathematical ideas encountered in the readings, students complete approximately eight homework sets during the semester. Most homework exercises require them to work in the style of the culture and time period under discussion. Although doing so is one of the more difficult aspects of mathematics history, it serves to deepen their mathematical understanding, while providing them with important insights into the process of assessing and evaluating the mathematical understanding of others.

In addition to mathematical exercises drawn from our textbook, homework sets often include a short reflection or analysis essay tied to the Course Questions List. These essays provide students with practice in responding to such questions, and provides them and me with feedback about their progress in synthesizing course material. All homework exercises and essays are carefully read, commented upon, and scored by me, and students are sometimes asked to revise insufficiently detailed work before a final score is assigned.

Written Course Projects

The goals of the various written projects described below include introducing students to the various resources, periodical literature, and primary source collections that are available for their current and future study of the history of mathematics; increasing their awareness and understanding of a variety of approaches to the history of mathematics and its use in teaching; and providing writing practice and feedback in preparation for their major research project.

Automathography

The idea for this assignment was borrowed from V. Frederick Rickey, who together with Victor Katz ran the NSF-supported *Institute on the History of Mathematics and Its Use in Teaching* (*IHMT*) in the 1990s. In this essay, students (re)introduce themselves to me by telling me about the mathematics and history courses they have taken, their mathematical and historical interests, what they plan to do after graduation, why they signed up for the course, what they expect to get out of it, and any anxieties they may have about it. Assigned the first week of the semester, Automathographies also provide an early alert of any issues with students' formal writing skills that we should address together during the semester.

Important Mathematical Person (IMP) Brief Biography

Again adapted from an idea from Fred Rickey, students are assigned a particular mathematician from the 18th century or later, become acquainted with the basic biographical facts about their IMP by completing a literature search worksheet (see Appendix D), and write a 1–2 page report of those facts. Peer review of the written biography is sometimes incorporated into this project. Due dates for this assignment are early in the semester, and students' assigned IMP may become the topic of their major research project.

Textbook Comparison Reports

For these reports (I typically require two), students select a topic discussed in the required core textbook and one other history of mathematics textbook, read the treatment of the selected topic in both texts, and write a comparison paper in which they critically examine what each textbook has to offer to the study of the history of mathematics.

Literature Reports

A minimum of three literature reports are generally required, to include at least one of each of the following types:

- Expository, based on an article from an approved expository journal (e.g., *Mathematics Teacher*, *For the Learning of Mathematics*), or an approved collection of secondary source articles;
- Technical, based on an article from an approved technical (scholarly) journal dedicated to the history of mathematics or science (e.g., *Historia Mathematica*, *Archive for the Exact Sciences*); and
- Primary Source, based on an approved primary source reading.

The required content of each of these 1–2 page reports includes a complete citation of the article or primary source, a summary of its contents, and a list of questions or concerns that the article or source raised for the student (e.g., things that seemed inaccurate or confusing, or ideas that the student would simply like to know more about). In addition to this common required content, expository and technical article reports also describe the nature of the author's sources (e.g., primary, secondary, recent) and mention some things that were particularly interesting about the author's thesis, while primary source reports include a description of the context and significance of the selected source.

Major Research Project: Paper and Presentation

This project provides students an opportunity to study a particular topic in depth, become the local expert on that topic, and share their expertise with the class and other individuals in the community with an interest in the history of mathematics. It is also an opportunity to exercise and enhance student research, writing and presentation skills. The research project culminates in a written report and a short presentation (≈ 20 minutes).

All research topics must be approved by me early in the semester. A biographically-based project (possibly expanding the IMP Brief Biography) is allowed, subject to certain additional requirements. In particular, a biographical subject must be a post-17th century mathematician, and the paper and presentation must describe the social and historical context in which that person worked, discuss the sources of his/her mathematical interest, provide details concerning his/her mathematical work and describe the reaction of the mathematical community to that work, in addition to providing biographical information. Since many students in the course are prospective teachers, I am also willing to entertain research topics that deal with the history of mathematics education, such as the teaching of some particular topic or set of topics at the secondary level.

To assist students in making consistent progress with their research throughout the semester, a number of "scaffolding" assignments are required, including an initial annotated bibliography, a preliminary written progress report, an in-person research consultation, and an initial draft paper. Students are also encouraged to base their other written projects (e.g., Literature Reports) on a topic related to their major research topic. The final written paper must cite at least four non-encyclopedic, non-web-based, non-textbook sources, at least one of which should be an primary source reading; use of at least one article from a scholarly journal (e.g., *Historia Mathematica*) is also strongly encouraged. Both a Works Cited Page that lists all works directly used in the paper and an Annotated Bibliography that lists all works consulted for the project are required.

Rather than spread research presentations out over several class days, I prefer to arrange one long session that we devote entirely to presentations as a History of Mathematics Mini-conference. This is typically possible on a Saturday late in the semester, and we invite friends, family, and other faculty to join us for the presentations and a Conference Luncheon. A (roughly) equivalent amount of class time is cancelled to compensate students for their Saturday time. Whether presentations are done in a single day or spread out over several, students complete peer feedback forms on all presentations as part of their final project score.

Course Notebook

This assignment has been developed over the years to provide students with a mechanism for:

(a) collecting and organizing RSGs and other course materials in preparation for class discussions;
(b) collecting and organizing material and resources as a possible future reference;
(c) documenting individual engagement with course content throughout the semester;
(d) demonstrating individual growth relative to each of the stated course objectives.

A satisfactory Course Notebook is a requirement for a passing grade of D or better, with details of what it takes for a notebook to be deemed "satisfactory" distributed to students on the first day of class. (Additional requirements for a passing grade of D or better include regular attendance and participation and non-zero scores on all required written projects.) Minimal Course Notebook content includes all completed course work and a cover letter explaining the purpose, arrangement, and content of the notebook. Other materials (e.g., notes on others' research presentations and handouts distributed in class) may be included if they are clearly tied to the course objectives. A score is assigned to each Course Notebook based on its completeness, organization, adherence to technical requirements, the general overall quality of the notebook, and, most importantly, the extent to which the completed notebook demonstrates that its creator has satisfied objectives (a)–(d) listed above.

In most semesters, quizzes and exams (midterm and/or final) are also part of the learning and assessment structure of the course. In these instances, students are allowed to use their Course Notebooks to complete in-class quizzes or exams; as noted earlier, exam questions are also drawn primarily from the Course Question List. In those semesters when I do not give any in-class exams, responses to the course questions are incorporated directly into the Course Notebook; Appendix E provides an example of how this can be done.

Lessons Learned

Perhaps the most important lesson that I have learned about teaching the history of mathematics over the years is that it is easy to become overly ambitious, at the risk of creating an overwhelming workload for the students and instructor alike. As ambitious as my current approach to the course is—and it is ambitious!—its earliest incarnations required even more of students, but with less support. Even the fact that studying the history of mathematics requires more prose reading than they encounter in their other mathematics courses is a challenge for students that I did not initially anticipate. RSGs that help students focus their reading now address this challenge. The beneficial effect that this advanced reading and preparation has had in my history course has been so pronounced that the use of RSGs has also migrated into my other upper division mathematics courses.

Helping students become effective participants in a discussion also requires attention from the instructor. Setting clear classroom participation norms and explicitly discussing them early and often is an important first step in this. Even the act of taking notes is different in a discussion setting than it is in a lecture setting, since there is typically no one writing notes on the board that they can just copy. Helping students learn how to pay careful attention to the comments of their fellow students is therefore critical, not only so that they can stay involved in the discussion, but also because those contributions may be more meaningful, helpful or interesting than anything the instructor has to say! Paraphrasing comments and posing clarifying questions are two of the strategies that can be used in this regard.

Revision opportunities for all homework sets and written projects are now also a standard component of my grading policies, not only in this course but in any course that I teach. This is done in part to provide motivation for students to improve both their grades and their understanding of the material. Beyond this, revision opportunities are provided in recognition of the fact that constructing new knowledge takes time, especially when the domain of knowledge lies outside one's usual comfort zone as is the case for most students in a history of mathematics course. Scaffolding major assignments, such as the major research project of this course, is another important support for ensuring student success in foreign territory.

Clearly, students aren't the only ones working hard in this course: developing RSGs, grading students' original and revised work, and guiding student research via scaffolded assignments takes a good deal of instructor time and effort. But I am constantly amazed by the way in which my students rise to the high expectations set for them in this course. Great class discussions, well-crafted research papers, and engrossing presentations are just a few of the rewards for the investment of time and energy that we all bring to this endeavor. Do I recommend that an instructor new to teaching the history of mathematics start off with such an ambitious set of assignments and requirements? Absolutely not! But I do recommend aiming just a bit on the high side and putting good safety nets in place for you and your students. Then open up your books, re-arrange the desks so that you can all face each other, and start talking!

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9. ———, *A History of Mathematics: An Introduction*, Brief Edition, Pearson, Boston, 2004.
10. ———, editor, *Using History to Teach Mathematics: An International Perspective,* Mathematical Association of America, Washington, DC, 2000.
11. Victor Katz and Constantinos Tzanakis (editors), *Recent Developments on Introducing a Historical Dimension in Mathematics Education,* Mathematical Association of America, Washington, DC, 2011.
12. Jacqueline Stedall, *The History of Mathematics: A Very Short Introduction,* Oxford University Press, Oxford, 2012.
13. Jeff Suzuki, *A History of Mathematics,* Prentice-Hall, Upper Saddle River, NJ, 2002.
14. Frank Swetz et al. (editors), *Learn from the Masters!,* Mathematical Association of America, Washington, DC, 1995.
15. Benjamin Wardhaugh, *How to Read Historical Mathematics,* Princeton University Press, Princeton, 2010.

Appendix A

SAMPLE SEMESTER SCHEDULE

The following schedule includes one week of class time dedicated to student research presentations. Typically, we instead arrange for one long session devoted entirely to presentations as a "History of Mathematics Mini-conference" on a Saturday late in the semester, with friends, family, and other faculty invited to join us for the presentations and a Conference Luncheon. A (roughly) equivalent amount of class time is cancelled to compensate students for their Saturday time, with cancelled class days distributed throughout the semester as most appropriate to allow students to concentrate on upcoming due dates related to their research projects.

Week	Mathematical Focus	Historical Focus	Project Due Dates
1	Numeration Systems	Historical Evidence	
2	Mathematics of Ancient Egypt and Mesopotamia	Social Influences	IMP Topic Requests (must be approved by instructor)
3	Early Greek Mathematics / Euclid's *Elements*	Internal vs. External Stress	IMP Worksheet
4	Euclid's *Elements* continued	Internal Stress	Literature Report: Expository Article
5	Archimedes and Apollonius	Tracing Trends	Research Project: Annotated Bibliography Form
6	Ptolemy and Early Trigonometry	Tracing Trends	
7	Medieval China and India	Cultural Comparisons	Literature Report: Scholarly Article
8	Medieval Islam	Cultural Comparisons	Research Project: Part I of Paper (Expanded IMP Biography or preliminary draft of full paper)
9	Medieval Western Europe / Cross-cultural comparisons	Cultural Comparisons	
10	Renaissance Europe	Cultural vs. Internal Motivation	Literature Report: Primary Source
11	Seventeenth Century Western Europe	Internal Motivation	
12	Calculus Precursors to Newton and Leibniz	Internal Motivation	Research Project: Part II of Paper (IMP Mathematical Contributions or completed draft of full paper)
13	Newton, Leibniz and the Calculus	Internal Motivation	
14	Course Project Presentations	Historiographical Research	
15	Final Exam and/or Final Reflection Discussions	Various	Research Project: Final Paper

Appendix B

Course Questions List for a Sample Semester

(Distributed to students on the first day of class)

RECALL: The following represent major course objectives for the semester. All questions on the Final Exam and a portion of the Midterm Exam will be drawn from this list. Scoring will be based on the following evaluation criteria: *accuracy, use of historical details, completeness, clarity, and presentation/composition.*

We will be discussing material related to the Course Questions throughout the semester. I encourage you to keep a special section in your Course Notebook for notes on these questions. By reviewing and updating this material as we go along, you will be able to pull the final product together more easily.

(1) Outline the historical development of *algebra* from 3000 BCE to 1900 CE. In addition to identifying the major stages of development and individuals involved, identify the factors/forces which either hindered or encouraged development at each stage.

(2) Outline the historical development of *trigonometry* from 3000 BCE to 1800 CE. In addition to identifying the major stages of development and individuals involved, identify the factors/forces which either hindered or encouraged development at each stage.

(3) Outline the historical development of *calculus* from 500 BCE to 1800 CE. In addition to identifying the major stages of development and individuals involved, identify the factors/forces which either hindered or encouraged development at each stage.

(4) Outline the historical development of the *concept of number* through the period ending in 1900 CE, including developments related to negative integers, irrational numbers, and complex numbers. In addition to identifying the major stages of development, identify the factors/forces which either hindered or encouraged development at each stage, especially those with pedagogical implications for teaching in today's classroom.

(5) Explain (i.e., define) each of the following specific factors/forces as they pertain to the historical development of mathematics and provide a historical example illustrating how it has encouraged the development of mathematics. Is there an example where it has instead hindered the development of mathematics? If so, state one.

analogy
consolidation
geographic diffusion
intramathematical diffusion
interdisciplinary diffusion
external cultural needs
external cultural atmosphere
internal usefulness
paradoxes
symbolism
unsolved problems
multiple representations

(6) Who were the three most influential individuals with respect to the development of mathematics in the period we have discussed? Justify your selection.

(7) What were the three non-mathematical events in the history of the period we have discussed which had the greatest influence (positive or negative) on the development of mathematics? Justify your selection.

(8) Identify the three most important mathematical "events" (concepts, problems, techniques, etc.) that occurred during the period we have discussed. (Be specific!) Justify your choice.

(9) Identify three (or more) areas of human activity which have had a significant impact (positive, negative, or both) on the development of mathematics during the historical time periods we have discussed. For each, discuss the impact of that area on the development of mathematics during the period in which it was important.

(10) Compare and contrast Ancient Babylonia, Ancient Egypt, and Ancient Greece in terms of their mathematics, including specific content, favorite topics, strengths/weaknesses, standards of rigor, primary motivational factors, influence of the external culture, and the philosophy of practitioners with respect to the nature of mathematical knowledge.

(11) Compare and contrast Medieval China, Medieval India, Medieval Europe, and Medieval Islam in terms of their mathematics, including specific content, favorite topics, strengths/weaknesses, standards of rigor, primary motivational factors, influence of the external culture, and the philosophy of practitioners with respect to the nature of mathematical knowledge.

(12) Compare and contrast Ancient Greece and 17th century Western Europe in terms of their mathematics, including specific content, favorite topics, strengths/weaknesses, standards of rigor, primary motivational factors, influence of the external culture, and the philosophy of practitioners with respect to the nature of mathematical knowledge.

(13) Compare and contrast the work of Leibniz and Newton in calculus. Is it appropriate, in your view, to consider these two individuals (and not their predecessors) as the inventors of calculus? Justify your response.

(14) Define the term “algorithm”. Discuss the events leading to the introduction of Hindu-Arabic numeral algorithms into Western Europe, as well as the impact of their introduction on the development of mathematics. Are there other instances in the history of mathematics in which the development of algorithms played an important role in the development of mathematics? Reflect upon the relevance of these historical incidents for mathematics teaching.

(15) Discuss the interplay between intuition and logic in the historical development of mathematics. How do these ideas relate to the notion of inductive reasoning and deductive reasoning? Reflect upon the relevance of the historical development for mathematics teaching.

(16) Use historical examples to illustrate the distinction between internal stress and external stress as forces in the development of mathematics (at least two examples of each). Which of these factors has been the more influential? Justify your choice and discuss the relevance of these ideas for mathematics teaching

(17) For each of the mathematicians listed below, describe their contributions to the development of mathematics of their culture and time. Be sure to indicate clearly the culture and time period in which they worked, and comment on the relative importance of their contributions.

al-Khwarīzmī
al-Khayāmmī
Apollonius
Archimedes
Bombelli
Brahmagupta
Cardano
Descartes
Diophantus
Euclid
Eudoxus
Fermat
Jordanus de Nemore
Leonardo de Pisa
Kepler
Mersenne
Napier
Oresme
Pappus
Ptolemy
Regiomontanus
Qin Jiushao
Stevin
Viète

(18) For each of the major mathematical works listed below, identify the author and describe the mathematical content of the work. Be sure to indicate the time and place in which the work was written, and comment on the relative importance of the work. What texts, if any, are as influential or important today as these were during their time? Explain.

Almagest
The Analytic Art
Analyse des Infiniment Petits
Arithmetica
Ars Magna
Condensed Book on the Calculation of al-Jabr and al-Muqabāla
Conics
Coss
De Triangulis Omnimodis
Discours de la méthode
Elements
Introduction to Plane and Solid Loci
Liber Abbaci
On the Domain of Analysis

Appendix C

A sampling of "reading and study guides" (RSGs)[1]

(Based on *A History of Mathematics,* Victor Katz, third edition)

RSG SAMPLE #1

Mathematical Focus: First Concepts **Historical Focus: Societal Influences**
Reading Assignment:
Chapter 1 (pp. 1–28)—skipping or just skimming through the sections already discussed (1.1.1, 1.2.1, 1.2.3)

****

In addition to notes on the following Focus Questions, note anything you find particularly interesting, useful or perplexing.

****

Chapter 1: Egypt and Babylonia

1. What cultural factors or characteristics allowed for the development of mathematics in the two civilizations discussed? How do the motivations for developing mathematics in each compare?
2. Summarize the highlights of the mathematical knowledge of these two cultures [i.e., what did they seem to know?].
3. How would you describe the general character of the mathematics done in these civilizations?
4. Are you surprised by the extent to which mathematics was developed within these civilizations? Why or why not?
5. How might the development of mathematics have been impacted by the fact that mathematics was considered the domain of specially trained priests and scribes? (And is it any different today?)

RSG SAMPLE #2

Mathematical Focus: Square roots & Pythagorean triples in Babylonia
Historical Focus: Reconstructing the mathematics from the evidence

Reading Assignment and Focus Questions

I. **Reread pp. 18–19 (including Figure 1.16) re: Babylonian approximations of square roots.**

Read the paragraph below Figure 1.15 on page 18 especially carefully, and try to sketch out (as a series of steps) the procedure by which the Babylonians may have determined approximations for square roots (according to Katz). How do these steps correlate to the diagram in Figure 1.16 on page 19?

You'll probably need to do some scratch work on this, and should do that on another sheet (but bring it with you to class!). If you're not sure exactly how the procedure suggested by Katz works, please write down **specific** questions or comments about it. (That is, don't just write "Don't know," but be very specific about what part you do or don't understand.)

II. **Reread page 19 (last paragraph)–page 22 (top paragraph) re: Plimpton 322 and Pythagorean triples**

Read page 21 especially carefully, and try to sketch out (as a series of steps) the procedure by which the Babylonians may have determined the entries on Plimpton 322 (according to Katz). How do these steps correlate to the diagrams in Figure 1.18 on page 21?

Again, do any necessary scratch work on another sheet (but bring it to class), and if you're not sure exactly how the procedure suggested by Katz actually works, please write down **specific** questions or comments about it.

---

[1] To promote good discussions, I have found that it is essential that the RSGs that I actually distribute to students include sufficient *space* after each question to allow them to use the RSG to (a) record their pre-discussion responses; and (b) add additional notes during class discussions. Typically, each RSG thus requires 1–2 two-sided pages.

RSG SAMPLE #3

Mathematical Focus: More Greek Mathematics: Euclid – Archimedes **Historical Focus: Tracing Trends**

Reading Assignment:

- **Section 3.9 (pp. 88–90) — Euclid's Data**
- **Introduction to Chapter 4 and Sections 4.1–4.3 (pp. 94–112) — Archimedes**
  We'll focus our discussion on Section 4.3.3, and the difference between *analysis* and *synthesis*.

****

In addition to notes on the following Focus Questions, note anything you find particularly interesting, useful or perplexing.

****

EUCLID'S *ELEMENTS*: The remaining books

Section 3.9: Euclid's *Data* (pp. 88–90)

1. How is this work by Euclid different from his *Elements*?
   What other works did Euclid write, and why did they not survive?

Post-Euclidean Mathematics

As we carry out our study of post-Euclidean Greek mathematics, beginning with Archimedes, keep the following question in mind: *What trends, if any, do we see in Greek mathematics after Euclid?*

Introduction to Chapter 4 (pp. 94–95)

2. No specific questions; just note down any general impressions or questions you may wish to share in class.

Archimedes

3. As you read through Sections 4.1–4.3, keep a record of anything you find noteworthy (e.g., interesting or perplexing) about Archimedes' work (particularly in comparison to earlier Greek work by Euclid and others).

Sections 4.1–4.2: Archimedes, Physics, and Numerical Calculation (pp. 96–103)

4. No specific questions; just note down any general impressions or questions you may wish to share in class.

Sections 4.3: Archimedes and Geometry, wish special attention to Subsection 4.3.3 (pp. 103–112)

5. What is meant by the terms *synthesis* and *analysis*, at least as they appear in Greek mathematics?

Other Vocabulary

- papyrus
- palimpsest
- parchment
- quadratrix

RSG SAMPLE #4[*]

Mathematical Focus: Medieval Islam **Historical Focus: Cultural Influences on Mathematics**

Day 1 Reading Assignment: Chapter 9 Intro (p. 265) and 9.1–9.3 (pp. 266–292)—some of this is review
Day 2 Reading Assignment: Sections 9.4–9.7 (pp. 292–318)

****

In addition to notes on the following Focus Questions, note anything you find particularly interesting, useful or perplexing.

****

PLEASE READ QUESTIONS 6–8 AND THE VOCABULARY LIST BEFORE YOU START READING!
(This will allow you to keep these in mind as you read.)

GENERAL QUESTION

1. As you read (both days), generate a list of what you think were important *texts* for this period and region.

Section 9.2: Decimal Arithmetic (pp. 267–270)

2. What was the role of Islamic mathematicians with respect to the development of the Hindu-Arabic numeral system?

Section 9.3: Algebra (pp. 271–292)

3. Summarize the contributions of al-Khawārizmī to the development of algebra.
4. Why does al-Khawārizmī need six types of linear and quadratic equations (instead of just two, linear and quadratic)?
5. Summarize the HIGHLIGHTS of the contributions made by other medieval Islamic mathematicians to the development of algebra, especially those of al-Khayyāmī. (Do this either by individual mathematicians or by topic—you may need to attach another sheet!)

Sections 9.1–9.6: (pp. 271–318)

6. Summarize additional highlights of Islamic mathematical knowledge (i.e., what did they know?) in the medieval period, noting in particular anything of relevance to calculus or trigonometry.
7. How would you characterize Islamic mathematics during this period in terms of types of problems solved, standards of rigor, purpose of mathematics, etc., in comparison to the other cultures we have considered so far?
8. What do you think are the three most important contributions of medieval Islamic mathematicians?

TO PONDER FOR FUTURE DISCUSSION: Why might the Islamic culture have allowed for the development of mathematics during the medieval period, while Christianity did not?

Vocabulary

- al-jabr
- al-muqābala
- qibla

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[*] Note that this RSG covered the discussion topics for two class days.

RSG SAMPLE #5

Mathematical Focus: Prelude to Calculus **Historical Focus: Internal Motivation**

Reading Assignment: Chapter 15 (including introduction) — pp. 507–539

****

In addition to notes on the following Focus Questions, note anything you find particularly interesting, useful or perplexing.

****

As you read Chapter 15, keep the following question in mind; we will return to it after discussing Newton and Leibniz.

Why are Newton and Leibniz considered the inventors (discoverers?) of calculus, and not one of their mathematical predecessors who did work in this area? Is this a fair reading of the historical facts?

Introduction to Chapter 15 (pp. 507–508)

1. Take a note of anything further that we learn from this reading regarding the importance of analytic geometry in 17th century mathematics.
2. Sketch a copy of Figure 15.1 (p. 508) below for your reference while reading Section 15.1.
   Make sure you know what each labeled segment of the diagram represents.
   - abscissa
   - normal
   - ordinate
   - subnormal
   - tangent
   - arclength
   - subtangent

Section 15.1: Tangents and Extrema (pp. 509–514)

3. *Summarize* the various approaches taken to the problems of tangents and extrema by various individuals who worked prior to N & L.

Section 15.2: Area and Volume (pp. 514–532)

4. What specific area and volume problems were solved prior to N and L? (Give names & dates where possible.)
5. Complete the following definitions:
   a. *Method of infinitesimals: A method for determining areas and volumes in which a given geometrical object is considered to be composed of objects (or figures) of dimension* ______________________________ .
   b. *Method of indivisibles: A method for determining areas and volumes in which a given geometrical object is considered to be composed of objects (or figures) of dimension* ____________________________.

Section 15.3: Rectification of Curves and the Fundamental Theorem of Calculus (pp. 532–539)

6. What does the term "rectification of curves" mean?
   What specific rectification problems were solved prior to the work of van Heuraet?
7. Who were the first people to state the Fundamental Theorem of Calculus? Did any of them prove it?

Appendix D

Important Mathematical Person — Literature Search Worksheet

Your IMP's Name: ________________________________ Your Name: ________________________________

This assignment is intended to provide you with a first library experience in preparation for your paper, to collect some initial background information about your IMP, and to give us a running start on locating additional references for your research, should they not be available locally. In taking notes on your readings, you may make copies of the articles and take notes directly on your copies, if you wish. It should be clear that you did read through the articles, however!

1. Look up your IMP in the Dictionary of Scientific Biography.
   Read the article and take any notes you wish for possible future use; attach those to this sheet.

   The article is in volume _____, pages ____________ , and is written by _____________________________.

2. Find an anecdote related to your IMP in Howard Eves' *Mathematical Circle* books. (These are on 2-hour reserve in our library.) Once you find one (use the indices!), read the anecdote and take any notes you wish for possible future use; attach those to this page.

   Give the volumes and page numbers for the anecdote here: ____.

3. Look for a profile of your IMP in Ioan James' *Remarkable Mathematicians*. (This is also on 2-hour reserve in our library.) Read the profile and take any notes you wish for possible future use; attach those to this page.

   Give the page number for the profile here: ______________________________________.

4. Connect to the St. Andrews history website at www-groups.dcs.st-and.ac.uk/~history.
   Read the biography of your IMP posted there, and take any notes you wish for possible future use; attach those to this sheet.

5. Attach a bibliographic list of books in the CSU - Pueblo Library written **by** and **about** your IMP (if any).
   Here's a format that can be used:

   James, Ioan, 2002, *Remarkable Mathematicians*: *From Euler to von Neumann,* Cambridge: Cambridge University Press.

6. Give a reference for an article about your IMP or his/her work from a scholarly journal (e.g., *Historia Mathematica);* If the article is available in our library, obtain a copy for your future reference; otherwise, print out the abstract and determine its availability via interlibrary loan.

7. List (or attach a list of) one or more references for your IMP not available in our library and not posted on the web.

8. If your research to date has located a full length biography of your IMP, list its title, author, and other bibliographic information here. If unavailable locally, indicate what steps you have taken (or will take) to obtain a copy.

Appendix E

COURSE NOTEBOOK SUMMARY REFLECTIONS — Course Syllabus Excerpt

RECALL: One objective of the Course Notebook is to provide you with a mechanism for demonstrating your individual growth in the six areas defined by the primary course objectives:

1. *developing an understanding of the chronological development of mathematical thought* through a careful study of the key landmarks (e.g., people, problems, places) that mark that development;
2. *developing an understanding of the forces that shape mathematical ideas*, including the influence of technology, communication, religions, schools, and other social institutions, through a careful study of how mathematics has (or has not) developed in various cultures and time periods;
3. *developing a base of historical knowledge, research skills, and resources* with which to reflect upon and converse about the nature and role of mathematics in today's world and classroom through careful reading and discussion of secondary and primary sources, presentations, and reflective and research writing;
4. *developing a deeper understanding of the mathematical content, methods, and standards* of today's mathematics curriculum through a careful study of its origins and development in various civilizations;
5. *gaining experience in the interpretation of the mathematical activity/thinking of others* through the careful examination of original source documents; and
6. *examining pedagogical issues related to secondary mathematics teaching* through the lens of the historical development of various mathematical concepts and techniques

The Course Notebook Summary Reflections listed in this appendix are intended to assist with this demonstration.

The complete set of writing prompts for these reflections appears on pages 2–3 of this appendix, organized by course objective. You may complete some or all of these, but are strongly encouraged to complete at least one writing prompt for each objective for inclusion in your final course notebook.

Evaluation of these essays will be based on the following (mostly qualitative) criteria: *evidence of reflection, clarity, accuracy in use of historical/mathematical details, and presentation/composition.*

NOTE: Although summary reflections, by their nature, can not really be completed until late in the semester, several (most?) of the writing prompts will draw on material that we will be discussing throughout the semester. Thus, you may wish to maintain a section of your Course Notebook in which to keep thoughts, ideas, or other notes on each as them as we go along, and to add to those notes on a regular basis, rather than try to pull everything together at the end of the semester.

Technical Requirements:

- Type each essay using Times New Roman or Calibri Font with font size between 10 point and 11 point (inclusive), 1 inch margins (all sides) and 1.15 line spacing.
- There is no requirement for number of pages, but the nature of the prompts suggests a minimum of 2 pages for most. (In general, quantity does not substitute for quality.)
- Start each essay on a new page, and include a statement of the writing prompt at the top. (An editable version of this document can be found in Blackboard to get you started.)

OBJECTIVE 1: *Chronological development of mathematical thought, important landmarks*

Note: There are three separate prompts for this objective, one of which also relates to one other objective.

- **Writing Prompt 1.A** — Includes response to both parts.
  (1) What were the three non-mathematical events in the history of the periods and cultures that we have discussed which had the greatest influence (positive or negative) on the development of mathematics? Justify your selection.
  (2) What were the three most important mathematical "events" (concepts, problems, techniques, etc.) that occurred during the history of the periods and cultures we have discussed. (Be specific!) Justify your selection.
- **Writing Prompt 1.B** — Respond to at most one part.
  (1) Choose a particular time period and culture that we have discussed this semester. Who were the three most influential individuals with respect to the development of mathematics in that period and culture? Justify your selection of those individuals, and also explain why you choose this particular time and place.
  (2) Archimedes, Newton, and Gauss are considered by many to be the most creative mathematicians of all times (to date). Select five outstanding mathematicians to be placed in the next lower rank of "research mathematicians" and justify your selection. Please restrict your choices to 18th century and earlier individuals only.
- **Writing Prompt 1.C** — Respond to at most one part.
  (1) Describe, compare and contrast Medieval China, Medieval India, Medieval Europe, and Medieval Islam in terms of their mathematics, including specific content, favorite topics, strengths/weaknesses, standards of rigor and proof, primary motivational factors, influence of the external culture, and the philosophy of practitioners with respect to the nature of mathematical knowledge.[2]
  (2) Describe, compare and contrast Ancient Greece and 17th Century Western Europe in terms of their mathematics, including specific content, favorite topics, strengths/weaknesses, standards of rigor and proof, primary motivational factors, influence of the external culture, and the philosophy of practitioners with respect to the nature of mathematical knowledge.[3]

OBJECTIVE 2: *Forces that shape mathematical ideas*

- **Writing Prompt 2** — Respond to at most one part.
  (1) Discuss the influence of religion on the development of mathematics during the time periods we have discussed. Give as many specific examples as you can in support of your own conclusion concerning whether the overall impact of religion has been generally positive or generally negative, identifying clearly the period and culture in which each example is set. How do you perceive the relation between religion and mathematics today? Explain.
  (2) Discuss the influence of educational institutions on the development of mathematics during the time periods we have discussed. Consider in particular who was (or was not) able to study in these institutions, who was (or was not) able to teach in these institutions, and what kind of mathematics was (or was not) studied in these institutions. Give specific examples from various cultures and time periods in support of your own conclusion concerning whether the overall impact of educational institutions has been generally positive or generally negative.

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[2] Note that the last part of this prompt also touches on Objective 2: Forces that shape mathematical ideas.

[3] Note that the last part of this prompt also touches on Objective 2: Forces that shape mathematical ideas.

OBJECTIVE 3: *Developing a base of historical knowledge, research skills, and resources with which to reflect upon and converse about the nature and role of mathematics in today's world and classroom*

- **Writing Prompt 3**

  Reflect upon the ways in which your view of the nature and role of mathematics in today's world (and classroom) has or has not changed as a result of your experience(s) in this course. Is there something that, at the start of this course, you believed about what mathematics is and/or can do which you no longer believe in the same way now? Or, alternatively, has your experience with the history of mathematics in this course reinforced your beliefs about the nature and role of mathematics today? Note that it will be necessary to say something about what your views are in this (both prior to the course, and now if these have changed). You may find it useful to look back at your initial reflection on related matters in Homework Set #1 for this.

OBJECTIVE 4: *Developing a deeper understanding of the mathematical content, methods, and standards*

- **Writing Prompt 4**

  Identify a topic or method generally taught in today's secondary or college mathematics curriculum that you understand differently and/or more deeply than you did prior to this course. Describe exactly how your understanding has changed. Explain the ways in which your experience(s) in this course contributed to this.

OBJECTIVE 5: *Gaining experience in the interpretation of the mathematical activity/thinking of others*

- **Writing Prompt 5**

  Throughout the semester, we have examined various original source excerpts. Reflect upon what you have learned about the interpretation of the mathematical activity/thinking of others from these activities. If there was a particular original source that was especially interesting or important in contributing to your understanding of the interpretive act, describe this source and explain what you think was special about it.

OBJECTIVE 6: *Examining pedagogical issues related to (secondary) mathematics teaching*

Note: There are two separate prompts for this objective, one of which also relates to one other objective.

- **Writing Prompt 6.A**

  Reflect upon the role(s) which the history of mathematics might be able to play in the teaching of mathematics. Are you convinced by any of the arguments that we have encountered this semester in favor of its use in a mathematics classroom that the history of mathematics could be helpful for students? Why or why not? What advantages and disadvantages do you see in its use for those who teach mathematics? If you intend to teach, do you expect to incorporate history of mathematics into your teaching in some way? If so, how? If not, why not?

- **Writing Prompt 6.B** — Respond to at most one part.

  (1) Outline the historical development of the *concept of number* through the period ending in 1900 CE, including developments related to negative integers, irrational numbers, and complex numbers. In addition to identifying the major stages of development, identify the factors/forces that either hindered or encouraged development at each stage, especially those with pedagogical implications for teaching in today's classroom.[4]

  (2) Define the term "algorithm." Outline the major (mathematical and non-mathematical) events leading to the introduction of Hindu-Arabic numeral algorithms into Western Europe, as well as the impact of their introduction on the development of mathematics. Are there other instances in the history of mathematics in which the development of algorithms played an important role in the development of mathematics? Reflect upon the relevance of these historical incidents for mathematics teaching.[5]

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[4] Note that the last part of this prompt also touches on Objective 1: *Chronological Development.*

[5] Note that the last part of this prompt also touches on Objective 1: *Chronological Development.*

From Clay Tokens to Calculus
A Course in the Early History of Mathematics

Charles Lindsey
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Course Overview

Florida Gulf Goast University (FGCU) offers a 3-credit course in the history of mathematics every fall semester, for both mathematics and secondary mathematics education majors entering the upper division (i.e., junior/senior). The course is required for both programs; it has Calculus II as prerequisite, so it can be taken by students just completing the calculus sequence, which includes a substantial population of transfer students. Frequently this is their first mathematics course beyond the calculus sequence. Although our history of mathematics course has much in common with other introductory courses in the subject, there are several special features that make this course distinctive:

- the course is aimed at mathematics and mathematics education majors, who are assumed to have some experience and interest, and who will benefit from a more sophisticated introduction to historical topics
- we pay extra attention to those aspects of the course that will be very useful to future high school teachers, in terms of both historical context and general facility with algebra, geometry, and trigonometry
- we spend a significant amount of time on non-Western cultures, especially developments in areas that high school teachers must be familiar with such as algebra and geometry
- we focus on the "early" history of mathematics, generally understood to run up through the development of calculus
- the course includes substantial discussion of the presocratic Greeks and the origin of the number-magnitude divide, as well as their echoes down through time.

The course, then, serves two main populations: mathematics majors, typically at the point in the program where they are finishing up the calculus/differential equations sequence and beginning to transition into more advanced proof-oriented courses, and secondary math education majors, who typically have done some of the classroom observations required, but have not yet begun their final internships. Both groups include significant numbers of new transfer students, who have usually completed general education (including calculus) at a community college, and need to begin upper-level courses right away but lack the bridge course that is prerequisite for the abstract algebra and real analysis sequences. We also occasionally have graduate students, either M.S. or certificate students, taking the course as a graduate special topics course; most of these are just entering the program and are taking one or two courses at a time as available.

Within the mathematics major, this course serves to address the University's goal of promoting intercultural awareness, so including significant study of non-Western cultures is an important component of the class. More broadly, the course is intended to give mathematics and secondary math education students an overview of the development of mathematics as a human activity that operates within cultural contexts and priorities, and an appreciation of the various modes that have been employed to attack mathematical problems through the ages. Practically speaking, some of the problems and methods used historically can enhance the "toolbox" of techniques available to students. This is especially important for those going on to teach algebra and geometry (and maybe calculus) at the high school level, who truly need a complete mastery of the subject, including the ability to view a problem from multiple perspectives. As noted in the 2001 MAA publication *CUPM Discussion Papers about Mathematics and the Mathematical Sciences in 2010: What Should Students Know?*, "… the kinds of integration of mathematical ideas and connections that are necessary in teaching a coherent secondary program, are unlikely to be obvious to students on the basis of their undergraduate program" [8, p. 33]. Secondary education students, perhaps even more than ordinary math majors, need to have a sense of how mathematical theory and practice evolved into their current forms, especially in the areas that today we lump under the heading of precalculus-level mathematics. Therefore, a special emphasis of this course is on the progression of approaches to thinking about and doing mathematics, as much as on regular techniques and methods. The sample exam review questions, listed below in the Assignments section, show more specifically what this means for our course.

Course Design

The course is a semester-length class, running for approximately 15 weeks, plus a week for final exams. The class usually meets once or twice a week. The overview above lists some main purposes and distinctive attributes of the course; these are implemented through several aspects of the design. In particular, the design of the course is shaped by several key considerations:

- As noted above, the focus of this course is on the earlier part of mathematical history, starting with the archaeological finds from ancient Egypt and Mesopotamia (as discussed in [14] and [11]), up through roughly the late 17th century, when calculus as we know it begins to take shape. This is partly in acknowledgement of the fact that most of the students enrolled have little post-calculus mathematics, and so are ill-prepared to wrestle with more advanced developments. This is also to emphasize the subjects that secondary teachers are likely to be teaching later on, and to make sure their familiarity with those areas is thorough. It is also a simple fact that the subject is far too broad to cover even superficially in its entirety. This approach has the advantage of the mathematics being more accessible to students; the disadvantage is that the time period is, of course, considerably less well-documented. Often my answers to students' questions begin with: "We don't really know, but most of the evidence leans toward…." One obstacle is that students already know, or believe they know, the "correct" way to do these problems already, and so getting them out of the modern mindset is sometimes a challenge.
- Unfortunately, we do not spend much time on spherical trigonometry or ancient mathematical astronomy. I always feel guilty about this, but it does not seem to be a topic that one can do in small doses. We look briefly at some of the Ptolemaic models, and a few general concepts and key questions that concerned ancient astronomers, but not far beyond that. On the other hand, I have put together a short unit on calendar systems, which seems to be neglected in most textbooks. We go as far as studying briefly the Metonic cycle and various ancient approximations to that, the necessity for intercalation and some of the intercalation systems that have been proposed, the continuing (and largely hidden) influence of pre-Julian Roman calendars, Julian day counts, and the like. Some other topics that we generally neglect are logarithms, perspective drawing, navigation, and mapmaking. Good arguments can be made for including each of these, but I have in the end chosen to pay attention to other things.

- Approximately 30% of the course is devoted to Chinese, Indian, and Islamic mathematics. These are all connected, to each other and to European mathematics, and are themselves important features of the history of mathematics generally. We spend some time comparing and contrasting the various approaches taken by these societies to mathematical problems, as well as looking at specific techniques. Here, I think it is important to stick to the evidence when trying to decide whether specific techniques or concepts were developed independently or were borrowed from others. There is a great deal of scholarly-looking misinformation on the internet, and it is important not to jump to conclusions unsupported by evidence. I try to instill in my students a sense of healthy skepticism when encountering online articles that claim to have made major, unacknowledged, discoveries—and there are a great many of them. Although we introduce and very briefly discuss the fascinating Mayan number system and their calendars, we do not spend much time on Mesoamerican mathematics, due to a dearth of both class time and historical evidence.
- The development of algebraic symbolism and systematic processes for solving equations is an important step in the history of mathematics—especially the mathematics that is classified today as "pre-calculus"—and we spend some time on its evolution. Although the evolution of notation is generally presented in bits and pieces in most textbooks, a coherent discussion of notation *per se* is difficult to find, and for this the book [3] by Florian Cajori (1859–1930) is indispensible. Also, in this connection, I have always believed that François Viète (1540–1603) is underappreciated, and so I usually include some additional discussion of his system, as outlined in the *Analytic Art* [12, pp. 11–13]. I feel that it is important for students to understand that the process eveyone learns in high school of solving equations has its origin in a deliberate program to make algebra more like geometry. Under this general heading I include René Descartes (1596–1650) and Pierre de Fermat (1601–1665), and the development of algebraic methods to attack geometrical problems through equations (i.e., analytic geometry). Descartes, in particular, is quite explicit about this [5, p. 216], and this background is important for future teachers of analytic geometry.
- We spend more time than is usual (perhaps too much time) on the presocratic Greeks. I feel that it is essential for students to understand the origin of the number-magnitude distinction and all the related issues concerning indivisibles, commensurability, and their pervasive, long-lasting influence in setting the boundaries for "proper" mathematics, up to perhaps even the present day (see, for example, [1]). Besides, any mathematics curriculum that excludes Zeno's paradoxes has a big hole in it.

Finally, even though the subject matter is almost entirely mathematics that we would consider "below" the calculus level, it is important to remember that the audience here is not non-majors seeking to complete a general studies requirement. The course is aimed at junior and senior math majors and secondary education majors, who at this stage in their education should be going deeper than just learning techniques and applications, especially for topics that they have some previous experience with. Therefore, even though the problems are often elementary by today's standards, I aim at a more thorough investigation of the history of their development. Understanding the historical methods involves understanding the modes and constraints of the times, so I often include excerpts from translations of primary sources, and try to embed students in the notation (or lack thereof) and usage of terms of the period we are studying. For instance, Euclid does not *multiply* magnitudes, he *creates rectangles contained by them*, for the simple reason that magnitudes are not numbers and so cannot be multiplied together. Even when abstract symbols were introduced and used, they were still closely tied conceptually to an underlying object, hence the lengthy delay in accepting negative solutions to equations as "legitimate". Ancient philosophical principles underlie questions like "Is unity a number or not?", which appears as one of Ludovico Ferrari's questions put to Niccolò Tartaglia in their famous 1547 dispute [7, p. 257].

The order of topics (see Course Outline section below) conforms in large part to the textbook I use for the course (Katz's *Brief Edition*, [10]), although most topics are supplemented, some heavily, with readings adapted from other textbooks (e.g., [2], [6]) and additional other sources, of which the most versatile and indispensible is Cajori's seminal work on mathematical notation [3]. The choice of a mostly chronological approach rather than one organized by theme or topic is based on previous experience with trying to integrate thematic units into the course, where the development and elaboration of a specific concept is traced through time. For example, several years ago I took the Pythagorean Theorem and traced its development from the Mesopotamians through the Greeks all the way into the idea of Euclidean norms and modern variants. Although interesting, the combination of themes with chronology proved disorienting for students.

Like many universities and colleges today, FGCU uses a Learning Management System (LMS) to provide a basic online presence and platform for instructors' use; at FGCU we use Canvas. For each of the main topics of the class there is a Canvas module containing a PowerPoint set of lecture notes and images, links to web sites, some supplemental readings, and some extra readings to be used as additional resources. Usually we can do little more than scratch the surface of a particular topic in class, and students frequently have questions that pursue particular issues in more detail (for example, students may want to see how the Chinese *Jiuzhang suanshu* handles different cases of the double false position rule). I have a fairly large number of additional resources, six to twelve for each module, consisting of journal articles or short excerpts from books, posted to enable students to obtain some more detail about a question they might have.

Resources

The resources used for the course, in addition to the textbook, generally fall into four categories: contemporary introductory textbooks and readers, additional useful texts, English translations of primary texts, and websites. There are in addition a number of articles from journals and collections that are more specific to a particular topic. They are usually posted on Canvas as a supplement for students who are interested in pursuing a particular topic in more depth (such as the question of whether the first incommensurable magnitudes were the side and diagonal of a square, or the side and diagonal of a pentagon, for example, as discussed in [13]). Below is a brief description of each type of resource; more extensive lists of the ones I use appear as Appendix C.

Textbooks and Readers

The required textbook for the past few years has been Victor Katz's *A History of Mathematics, Brief Edition* [10], although I also make use of several other commonly-used introductory textbooks (for example [2], [6], and [7]), along with some useful compilations of excerpts in translation from original sources. These readers are often good sources for specific well-known articles and concise commentary. There are many more readers in existence than the ones listed in Appendix C, and they are often a good intermediate source between general textbooks and translations of complete works.

Other Useful Texts

In Appendix C are listed several books that contain valuable additional detail on specific subjects. None of these has the breadth to make a textbook by itself, but each one provides extremely valuable detail and insight on specific topics. Anyone who plans to give more than a cursory overview of one of these areas (the titles make the areas obvious) should consult these. I consider Cajori's *History of Mathematical Notations* to be particularly valuable; anybody looking into the history of mathematics should consider this book as indispensible as a dictionary. If you don't already have a copy, you should get one. Now. You can finish this article later.

Translations of Primary Works

Many public domain translations (into English) of some historically important texts can be found online through Google Books, Scribd, The Internet Archive, HathiTrust, or any of a number of other places. I find these to be quite useful for capturing the thoughts and intent of the original author for the students, and they are essential if you are considering making any significant use of primary sources. The list in Appendix C is restricted to English translations, so, for example, Heiberg's editions of Euclid and Archimedes, although quite valuable, are omitted.

Websites

Appendix C contains a list of ten important and useful websites. Most of these are devoted to some specific topic. Many of them are in large part aggregates of links to other sites. All of them are very helpful, especially in classes now where students tend to turn to the web for basic research rather than the library. Below each listing is a short description of the content of the site. One of the pitfalls with using the web for getting information about historical topics is that there is a great deal out there that is slanted, superficial, exaggerated, or simply wrong. Many sites contain erroneous information and/or mythology that is uncritically repeated despite being debunked long ago, or presented as accepted fact when still a subject of active debate (for example, one can find numerous sites proclaiming the decipherment of the Indus Valley "script", despite the active debate still going on about whether it should be considered a written language at all). These sites listed below are careful to present established facts—with references—and not to claim too much. Again, this list just scratches the surface: I will leave for you, dear reader, the joy of discovering the website that contains all the early issues of *Acta Eruditorum*, including Leibniz's 1684 article on maxima and minima.

Assignments

Typically, the assignments in this course consist of some combination of homework problems, a more extended writing assignment, and midterm and final exams, which generally consist of short answer and essay questions. These have been weighted variously, but usually I weight homework fairly heavily in the course, around 40% of the final average: this is to get students to take the homework problems seriously and to devote some attention to them. At the same time, the homework problems generally require students to use one of the old techniques, occasionally using the old notation, to solve some mathematical problem. This is of course unfamiliar ground for them, and the variety of methods they are exposed to is too broad to expect students to master them all. For this reason, regular homework-type problems are typically not included on the midterm or final exams. The purpose of the homework is to get students to try to get into the mindset of the mathematicians of old, and have them approach problems from a perspective different from what they are used to. Some of the problems themselves are good ones that can be posed to a talented high school algebra or geometry student, and so some of the intent is to give our secondary education students an array of "extra" problems and techniques they can then use with their own classes later on. The various methods developed by Girolamo Cardano (1501–1576), Viète, and Descartes for cubic and quartic equations are good examples of this.

I write my own homework sets. There are usually ten or eleven homework sets assigned during the course of the semester. Each set has roughly 15–20 problems; I generally pick out three or four as "turn-in" problems that students will have to write up and turn in to be graded. The majority of homework problems are adapted or taken directly from the textbook and some of the other textbooks mentioned in the list of additional resources in Appendix C. Some others are made up or taken from other sources, and yet more are taken from primary works such as Fibonacci's *Liber Abaci* (13th century), or the *Jiuzhang suanshu* (2nd century BCE), or the *Ganitasārakaumudī* (15th century). Frequently I need to change units of measure into more familiar forms, but sometimes it is refreshing to use the original units of measure and have students do the conversions; in fact, this is what a good part of the *Liber Abaci* is all about, so it is a good experience. I generally do not insist that geometric problems from Archimedes and Apollonius be done using only the original techniques: these problems are difficult enough even with the apparatus of coordinate geometry available.

While the purpose of homework is to give students practice solving common problems using (usually) older approaches, in my class the purpose of the midterm and final exams is to pull together the great themes of the course, and to look at the big picture of the development and transmission of historical ideas and methods. To this end, these exams consist of short answer and essay questions. Students get a review sheet ahead of time with the important questions so they can prepare and think about the common threads as we progress through the class. As a sample, in no particular order, here are a few of the review questions from a recent class. The first three are from the midterm review, the second three are from the final review questions. These should be useful to help get an idea of the major emphases of the course.

- Explain how the Greek conception of number (as stated in Euclid's *Elements*) differs from our modern conception. Define the two kinds of mathematical objects studied by the Greeks, and describe briefly the difference between them. Why did the Greeks not have a concept of a "number line"?
- Describe the major types of numeration systems used in ancient societies (you should be able to name at least three). Give an example of the use of each type in ancient civilizations. Explain some of the advantages and disadvantages of the various types.
- Describe Archimedes' method for "approximating the value of π." What did Archimedes actually prove? How does his method of approximation differ from earlier attempts? Why did his method continue to be used for so long? [Note: this is for the midterm, so we have not yet studied Liu Hui's method.]
- Compare the Indian and Chinese methods for solving systems of congruences. Which was more sophisticated?
- Describe Viète's contributions to the development of algebra, both in symbolism and in methods. In what way is his concept of "rhetic analysis" important from an historical standpoint?
- Compare the methods for solving polynomial equations in Chinese and Arabic societies, particularly those used by Qin Jiushao (1202–1261) and Omar Khayyam (1048–1131). How are the two approaches different?

Finally, students have some sort of more extended writing assignment during the course of the semester. Although we do not have a formal writing across the curriculum program, written communication is a major part of FGCU's student goals, and we in the math department try to work on this throughout the program with our majors. I have, over the years, tried a few different kinds of writing assignments in this course, with varying degrees of success (see Lessons Learned below). For the past two years, I have tried an alternative, an assignment that I have been calling a "problem analysis." The problem analysis takes a problem or a set of related problems, and asks for a fully-written solution and analysis of the method used, in more depth than is usual with a single homework problem. This exercise is intended to require the depth of thought that would go into an essay, but would require an extended written response in a form math majors are more accustomed to, and one that would be far more difficult to find on a website somewhere. I have assigned two of these exercises each class for the past two years. The appendix below contains one of the assignments, as an example. So far, I have found that students are more willing to put the effort into these types of assignments since they are more obviously related to their major field, and with some direction they generally come around to providing the depth of explanation that is expected. It still takes considerable time to provide proper feedback to a class, but it seems to be working better so far, in the sense that there is some improvement in their ability to write out fully-formed solutions. Perhaps this is a topic for a future research study.

In summary, assignments consist of homework, problem analyses, and midterm/final exams that are short answer and essay based. I tell students the first week that if they keep up with the assignments, this is not really a difficult course to get a good grade in, and if they fail to keep up it will be nearly impossible. This is supposed to be a fun class, and if they get into it they will have fun and learn a lot.

Lessons Learned

Based on my experience of having taught introductory history of mathematics for over 15 years at the undergraduate level, my advice to new instructors is to start with a textbook that leans toward the things you want to emphasize, and mainly follow the textbook the first time around. There is no introductory book that covers everything. I chose Katz's textbook ([9] or [10]) because, in my opinion, his books have the best self-contained coverage of non-European cultures, particularly China, India, and the Islamic world. His coverage of Greek mathematics is less thorough than many others, but the Greek mathematics is the most widely known and is easier to supplement than the others. The main thing is to give some thought beforehand to what you want to emphasize, because it is impossible to do it all, even superficially. Any one of the topics we cover would take a month at least to do it justice, and many could easily be the subject of a whole course. The biggest and most difficult part of planning a history of mathematics course is deciding what to leave out. Once you have been through the course a time or two, you can start to modify the content and bring in more of the supplemental resources available for areas you want to pursue in more depth. One of the things that often surprises even active mathematicians is how much active research is going on in the history of mathematics. Archaeological discoveries are still being made all the time, and there are still thousands of tablets and manuscripts of all kinds sitting in libraries and museums waiting to be read. As you get into the subject you will find many specialized sources that you can pursue at your leisure. Even the introductory textbooks have a good list of references at the end of each chapter; those referenced books and articles are valuable and usually very informative, and most are not difficult to get through interlibrary loan. Some of the best books and articles do not come from the standard mathematics section of the library. Many valuable journals are not well-known to those starting out in history of mathematics (and some are not indexed in MathSciNet): *Isis, Archive for History of Exact Sciences, Historia Mathematica, SCIAMVS*, and many others. If you know any colleagues in history or philosophy, they can probably point you to some other good ones.

I have learned to choose writing assignments carefully. The first few years I taught this course, students were assigned a research paper, usually on a topic chosen from a list given to them. I eventually abandoned this for several reasons: as enrollment increased, the burden of grading and providing feedback, especially on drafts, grew; students did not take the drafting process seriously and ultimately I despaired of getting them to do so; papers on similar topics began to look tediously similar; and the internet eventually squeezed out actual research, so I was faced with teaching basic use of the library at the same time. I gave up, admittedly partly for selfish reasons. The next step was to have students write shorter essays. For a time I assigned two essays of roughly 500 words each, which could be chosen from a list of topics given to the class. While the essays did give the students some practice at writing, ultimately I felt that this was not serving the purpose of enhancing their writing skills. Students are quite capable of writing longer papers; what I wanted, however, was *better* papers, and the structure of the course was not providing the proper feedback/revision opportunities to develop their skills. Ultimately, what doomed the essay approach for me was massive plagiarism, mostly through ingorance and/or delusion. Many students just do not believe that cutting, pasting, and collating paragraphs from websites (or from my own lecture notes, in some cases) is actual plagiarism, as long as you copy and paste from multiple sources. This attitude persisted despite my devoting more and more class time to the discussion of what is or is not allowed (*If you are using the Copy command, then you are, in fact, copying!*). I still feel guilty for having given up, but the alternatives were failing half the class, or spending far too much time not only going over guidelines in class, but also reading drafts and marking up violations.

Some things I used to take for granted with respect to student preparation are no longer true. This is not necessarily bad, just different. High school geometry classes today do much less with proofs or constructions with compass and straightedge than they used to. Usually I ask my class by show of hands who did constructions in high school: for the last few years, it has been much less than half. Some of those need to be reminded what constructions are; many of the students who actually remember them well are from high schools outside the U.S. As a result, I spend a little more time going over the basics of constructions than I used to, and don't pursue them in as much depth. Today, I usually just do enough to get across the idea of a constructible magnitude, and to raise the questions surrounding which magnitudes can be constructed (and why that particular choice of tools, anyway?). Most students have done some proofs in their high school geometry classes, maybe not too many proofs but they know the basics, and seem to be less rigidly wedded to the two-column format than they were several years ago, so that's progress.

The other thing that more "chronologically advanced" faculty have to keep in mind is that almost none of our students have ever used trigonometric or logarithm tables for calculation, and most of them have only experienced tables in their statistics courses, if at all. For them, values of trigonometric functions come from a calculator, just like milk and eggs come from a grocery store. I personally consider the development of tables an important thread in the development of mathematics, so we start from the beginning. This is not a bad thing, it's just different. Fifteen years ago I could assume that my students had at least seen, and probably used, trigonometric tables at least a bit; today I start by trotting out an old trigonometry book and showing them the tables and how they are used. Often I use my 1962 edition of the *CRC Handbook of Mathematical Tables*, since this gives me a chance to show the class some other tables, and have a brief discussion on the purpose of mathematical tables in general.

I have already discussed my struggles with essays and term papers. These are commonly used in other history of mathematics courses, and they can be a useful learning and assessment instrument, so I don't want to tell anyone not to use them. My advice is to think carefully about what you want to accomplish by using them. That's pretty generic advice but it is still sound, and especially so in thinking about something that obviously requires so much time and energy.

The final lesson that I have learned is that there is a very large, very supportive community of teachers and scholars out there who are enthusiastic about the history of mathematics, and who will gladly devote lots of their time, patience, and terrier-like energy to helping out anyone who needs it. There are listservs devoted to the subject, and the MAA even has a rather active special interest group, HOM SIGMAA, devoted to the history of mathematics. I have learned a great deal from my colleagues on the HOM SIGMAA listserv, both from direct questions and just from reading responses to others. Reach out to the history of math community and you will get all the help you need to be successful.

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6. Eves, Howard, *An Introduction to the History of Mathematics*, 6th ed., Holt, Reinhart, Winston, Orlando, FL, 1990.
7. Fauvel, John, and Jeremy Gray, eds., *The History of Mathematics: A Reader*, MacMillan Press, London, 1987.
8. Ferrini-Mundy, Joan, and Bradford Findell, The mathematical education of prospective teachers of secondary school mathematics: old assumptions, new challenges, in *CUPM Discussion Papers about Mathematics and the Mathematical Sciences in 2010: What Should Students Know?*, The Committee on the Undergraduate Program in Mathematics, Mathematical Association of America, Washington, DC, 2001, 31–41.
9. Katz, Victor J., *A History of Mathematics, an Introduction*, 3rd ed., Addison-Wesley, New York, 2009.
10. ———, *A History of Mathematics: Brief Version*, Pearson Education, New York, 2004.
11. Robson, Eleanor, *Mathematics in Ancient Iraq: A Social History*, Princeton University Press, Princeton, NJ, 2008.
12. Viète, François, *The Analytic Art*, Trans. T. Richard Witmer, Dover Publications, Mineola, New York, 2006. Reprint of 1983 edition published by Kent State University Press, Kent, OH.
13. Von Fritz, Kurt, The discovery of incommensurability by Hippasus of Metapontum, in *Studies in Presocratic Philosophy, vol I*, ed. David J. Furley and R. E. Allen, Humanities Press, New York, 1970, 382–412.
14. Wilkinson, Toby, *The Rise and Fall of Ancient Egypt*, Random House, New York, 2010.

Appendix A

Course Outline

Our course is divided into several parts, with the list below showing the main components, each of which occupies roughly a week:

1. Number Systems; Early Egyptian Mathematics
2. Early Mesopotamian Mathematics
3. Early (Presocratic) Greek Mathematics
4. Classical Greek Mathematics (through the *Elements*)
5. Later Greek Mathematics
6. Ancient Astronomy and Calendar Systems
7. Ancient and Medieval China
8. Ancient and Medieval India
9. Medieval Islamic Mathematics
10. Medieval and Early Renaissance Europe
11. Development of Algebra in 16th c. Europe
12. 17th c. Precursors to Calculus (Descartes, Fermat, Harriot, etc.)
13. The Beginnings of the Calculus

Appendix B

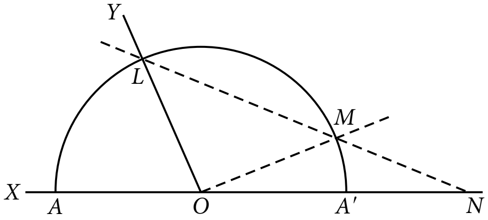
Sample Problem Analysis

In this appendix, I have included one of the problem analysis assignments that was given to a recent class. This was assigned shortly after finishing our study of classical construction problems and the *Elements* of Euclid.

MHF 4404 Problem Analysis 1

This problem analysis will explore some of the methods developed for solving the construction problems of antiquity. You are to give complete answers to all parts of the questions, with figures where needed. Proofs must be complete and organized in a coherent, logical flow.

1. This part concerns a method of trisecting an angle, which is said to be due to Archimedes. In the figure below, angle XOY is given. Draw a circle of any radius r centered at O, intersecting OX at A, OY at L, and OX extended at A', as in the figure. Now, draw a line LN from L down to the base line AA' so that it intersects the circle at point N, and $MN = r$, the radius of the circle.



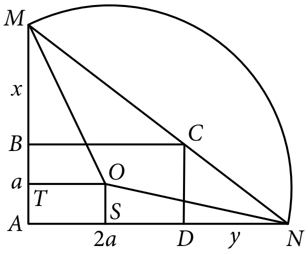
   (a) Show that triangle MON is isosceles, and so $\angle MON = \angle MNO$. Use the exterior angle theorem for triangles (i.e., an exterior angle is equal to the sum of the opposite interior angles) to show that $\angle OML = 2\angle MNO$.

   (b) Show that triangle MOL is isosceles, and hence $\angle OLM = \angle OML = 2\ \angle MNO$.

   (c) Again, by the exterior angle theorem applied to triangle LON, we have $\angle XOY = \angle OLM + \angle MNO$. Use this fact to show that $\angle XOY = 3\angle MNO$ and hence $\angle MNO$ is a third of the original angle $\angle XOY$.

   (d) What part of this construction violates the classical rules about use of compass and straightedge?

2. This part concerns Apollonius's solution to the duplication of the cube problem, which involves inserting two mean proportionals between lengths a and $2a$. In the figure below $AB = a$ and $AD = 2a$. Complete the rectangle $ABCD$, and let O be the center of the rectangle. Now, draw a circle with center O, intersecting AB extended at M and AD extended at N, so that C lies on the line MN. Draw OT perpendicular to AB and OS perpendicular to AD. We will let $x = BM$ and $y = DN$, as in the figure.



(a) Use properties of similar triangles and proportions to show that

$$\frac{a}{y}=\frac{x}{2a}=\frac{a+x}{y+2a}.$$

(b) Since $OM=ON$, triangles OSN and OTM are right triangles with equal hypotenuses. Use this and the Pythagorean Theorem to show that

$$a^2+\left(\frac{a}{2}+x\right)^2=\left(\frac{a}{2}\right)^2+(a+y)^2.$$

Use algebraic manipulations to show that

$$x(a+x)=y(2a+y).$$

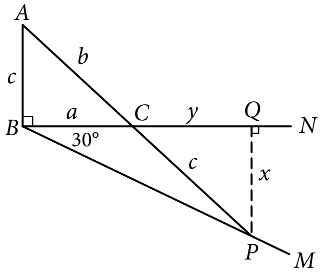
(c) Use the results of the first two parts to show that

$$\frac{a}{y}=\frac{y}{x}=\frac{x}{2a},$$

and hence that $y=DN$ and $x=BM$ are two mean proportionals between a and $2a$, so DN is the side of the doubled cube.

(d) What part of this construction violates the classical rules about use of compass and straightedge?

3. This last construction for duplicating the cube is due to Isaac Newton. Given a segment AB with length c, construct perpendicular BN and then draw BM so that $\angle ABM=120^0$. Let P be the point on BM such that if AP is drawn meeting BN at point C, then $CP=AB=c$. Let $BC=a$, $AC=b$, and $CQ=y$ as in the figure.



(a) Draw PQ perpendicular to BN as in the figure, and let $x=PQ$. Show that triangles ABC and PQC are similar, and that

$$\frac{c}{x}=\frac{a}{y}=\frac{b}{c}.$$

Use these proportions to show that $\frac{x}{a+y}=\frac{c^2}{a(c+b)}$.

(b) Triangle BPQ is a 30-60-90 triangle, so $\frac{PQ}{BQ}=\frac{1}{\sqrt{3}}$. Use this and the result from the previous part to show that

$$a^2(b+c)^2=3c^4.$$

(c) From the Pythagorean Theorem, $a^2=b^2-c^2$. Substitute this into the previous part and simplify algebraically to show that $b^3=2c^3$. Conclude that $b=AC$ is the side of the cube that is double the cube with side c.

(d) What part of this construction violates the classical rules about use of compass and straightedge?

Appendix C

Additional Resources

Textbooks and Readers

In this category are listed several commonly-used introductory textbooks, along with some useful compilations of excerpts in translation from original sources. These readers are often good sources for specific well-known articles and concise commentary. There are many more readers in existence than the ones listed here, and they are often a good intermediate source between general textbooks and translations of complete works.

1. David M. Burton, *The History of Mathematics: An Introduction*, 7th ed., McGraw-Hill, New York, 2011.
2. Ronald Calinger, *Classics of Mathematics*}, Prentice-Hall, Inc. Englewood Cliffs, NJ, 1995.
3. Ronald Calinger, *A Contextual History of Mathematics: to Euler*, Prentice-Hall, Inc. Upper Saddle River, NJ, 1999.
4. Howard Eves, *An Introduction to the History of Mathematics*, 6th ed., Brooks/Cole, Pacific Grove, CA, 1990.
5. John Fauvel, and Jeremy Gray, eds. *The History of Mathematics: A Reader*, MacMillan Press, London, 1987.
6. Victor J. Katz, *A History of Mathematics: an Introduction*, 3rd ed., Addison-Wesley, New York, 2009.
7. Victor J. Katz, ed., *The Mathematics of Egypt, Mesopotamia, China, India, and Islam: A Sourcebook*, Princeton University Press, Princeton, NJ, 2007.
8. David Eugene Smith, *A Source Book in Mathematics*, Dover Publications, Mineola NY, 1959. Reprint of 1929 edition by McGraw-Hill Book Co.
9. Jacqueline Stedall, *Mathematics Emerging: A Sourcebook 1540–1900*, Oxford University Press, Oxford, UK, 2008.
10. Frank J. Swetz, ed., *The European Mathematical Awakening: A Journey Through the History of Mathematics from 1000 to 1800*, Dover Publications, Mineola, NY, 2013.

Other Useful Texts

Here are listed several books that contain valuable additional detail on specific subjects. You can use these (and many others not listed here) to expand your own depth of knowledge, or point interested students toward further reading on topics of interest.

1. Margaret E. Baron, *The Origins of the Infinitesimal Calculus*, Dover Publications, Mineola, NY, 2003. Reprint of 1969 edition published by Pergamon Press, Ltd, Oxford, UK.
2. J. L. Berggren, *Episodes in the Mathematics of Medieval Islam*, Springer-Verlag, New York, 1986.
3. Florian Cajori, *A History of Mathematical Notations*, Dover Publications, Mineola, NY, 1993. Reprint of 2 vol. edition published by Open Court Publishing, La Salle, Illinois, 1928, 1929.
4. Richard J. Gillings, *Mathematics in the Time of the Pharaohs*, Dover Publications, Mineola, NY, 1982. Reprint of 1972 edition published by MIT Press.
5. Sir Thomas Heath, *A History of Greek Mathematics*, II vol. Dover Publications, Mineola, NY. Reprint of 1921 edition by Clarendon Press, Oxford.
6. G. F. Hill, *The Development of Arabic Numerals in Europe, Exhibited in Sixty-Four Tables*, Clarendon Press, Oxford, 1915.
7. Wilbur Knorr, *The Ancient Tradition of Geometric Problems*, Dover Publications, Mineola, NY, 1993. Reprint of 1986 edition published by Birkhäuser Boston.
8. Ulrich Libbrecht, *Chinese Mathematics in the Thirteenth Century*, Dover Publications, Mineola, NY, 2005. Reprint of 1973 edition published by MIT Press.

9. David C. Lindberg, *The Beginnings of Western Science*, University of Chicago Press, Chicago, 1992.
10. Jean-Claude Martzloff, *A History of Chinese Mathematics*, Springer-Verlag, New York, 1987.
11. Otto Neugebauer, *The Exact Sciences in Antiquity*, Dover Publications, Mineola, NY, 1969. Reprint of 1957 edition by Brown University Press, Providence, RI.
12. Kim Plofker, *Mathematics in India*, Princeton University Press, Princeton, NJ, 2009.
13. E. G. Richers, *Mapping Time: The Calendar and its History*, Oxford University Press, Oxford, 1998.
14. Eleanor Robson, *Mathematics in Ancient Iraq: a Social History*, Princeton University Press, Princeton, NJ, 2008.
15. George Saliba, *Islamic Science and the Making of the European Renaissance*, MIT Press, Cambridge, MA, 2007.
16. Glen Van Brummelen, *The Mathematics of Heaven and Earth: The Early History of Trigonometry*, Princeton University Press, Princeton, NJ, 2009.
17. Lam Lay Yong and Ang Tran Se, *Fleeting Footsteps: Tracing the Conception of Arithmetic and Algebra in Ancient China*, revised ed., World Scientific, River Edge, NJ, 2004.

Translations of Primary Texts

This list is restricted to English translations only. Some of them are only available (as far as I know) in print, but some, such as Heath's works, are in the public domain and can be found for free online.

1. Omar Al-Khayyam, *An Essay by the Uniquely Wise Abel Fath Omar bin al-Khayyam on Algebra and Equations*, trans. Roshdi Khalil, Garnet Publishing, Reading, UK, 2008.
2. Archimedes of Syracuse, *The Works of Archimedes*, ed. Sir Thomas Heath, Dover Publications, Mineola, NY, 2002. Reprint of 1897 edition with 1912 supplement published by Cambridge University Press.
3. Boethius, *De Insitutione Arithmetica*, trans. Michael Masi, Editions Rodolpi B.V. Amsterdam, Amsterdam, 1983.
4. Girolamo Cardano, *Ars Magna, or, The Rules of Algebra*, trans. T. Richard Witmer, Dover Publications, Mineola, NY, 1993. Reprint of 1968 edition published by MIT Press.
5. Rene Descartes, *La Geometrie*, trans. David Eugene Smith and Marcia L. Latham, Dover Publications, Mineola, NY, 1954. Reprint of 1925 edition published by Open Court Publishing.
6. Leonardo Pisano Fibonacci, *Liber Abaci*, trans. L. E. Sigler, Springer, New York, 1987.
7. Galileo Galilei, *Dialogues Concerning Two New Sciences*, trans. Henry Crew and Alfonso de Salvio, Prometheus Books, Amherst, NY, 1991.
8. Sir Thomas L. Heath, trans., *The Thirteen Books of Euclid's Elements, Translated from the text of Heiberg*, (3 vol), Dover Publications, Mineola, NY, 1956. Reprint of second edition published by Cambridge Univ Press, 1925.
9. Shen Kangshen, John N. Crossley, and Anthony W.-C. Lun, *The Nine Chapters on the Mathematical Art: Companion and Commentary*, Oxford University Press, New York and Science Press, Beijing, 1999.
10. Sir Isaac Newton, *The Mathematical Principles of Natural Philosophy*, trans. Andrew Motte, Prometheus Books, Amherst, NY, 1995. Reprint of 1848 edition published by D. Adee, New York.
11. Nicole Oresme, *Tractatus de configurationibus qualitatum et motuum*, trans. Marshall Clagett, University of Wisconsin Press, Madison, WI, 1968.
12. Claudius Ptolemy, *Ptolemy's Almagest*, trans. G. J. Toomer, Princeton University Press, Princeton, NJ, 1998.
13. Roshdi Rashed, *Al-Khwarizmi: The Beginnings of Algebra*, SAQI, London, 2009.
14. François Viète, *The Analytic Art*, trans. T. Richard Witmer, Dover Publications, Inc., Mineola, NY, 2006. Reprint of 1983 edition published by Kent State University Press, Kent, OH.

Websites

This list is in no way intended to do more than scratch the surface, but should provide a good starting point. A scratching post, perhaps. Be warned: it is easy to get drawn in and spend an entire day at any one of these without realizing it.

1. John O'Connor and Edmund F. Robinson, *The MacTutor History of Mathematcs Archive*, www-history.mcs.st-and.ac.uk

   The MacTutor Archive, hosted by the University of St. Andrews in Scotland, is arguably the most complete overall resource on the history of mathematics available online. The site is regularly updated, and contains biographies of over 1600 mathematicians, as well as numerous articles, links, and other information about a host of topics. The material on this site is accurate and informative, and is usually a very good place to start looking for information about a specific new topic. I often include pictures of mathematicians in PowerPoint presentations, and I like to link the picture to the person's biography in MacTutor. It provides an easy way for students reviewing the lecture to read more about an individual.

2. Mathematical Association of America, *Convergence,* maa.org/publications/periodicals/convergence

   Convergence is the MAA's journal of the history of mathematics and its use in teaching. *Convergence* contains several parts, including articles both scholarly and pedagogical, compilations of quotations and problems, and a special collection of images called *Mathematical Treasures*. This site is an essential resource for anyone interested in learning more about the history of mathematics, finding classroom resources, getting tips from experienced history of mathematics instructors, or just seeing some very cool historical artifiacts.

3. J. Leichter, "Mathematics and Mathematical Astronomy," wilbourhall.org

   This site is generally found in searches as *Wilbourhall.org* after Wilbour Hall at Brown University. The site itself consists mostly of collections of links to public domain versions of numerous math and astronomy works in Greek, Latin, Sanskrit, and Arabic, as well as public domain versions of translations, links to other online collections, and commentaries and early printed editions of famous works. There are also links to more contemporary versions and newer works that are not public domain, so this cannot always be taken for granted. If one is looking for a set of links to online versions of primary texts and translations, this is an excellent resource.

4. David Joyce, *Euclid's Elements Online*, aleph0.clarku.edu/~djoyce/java/elements

   This site is a complete translation of the thirteen books of the *Elements* attributed to Euclid, with modernization of much of the notation. It contains numerous comments, and each proof has annotations indicating where previous definitions and theorems have been used. Each proposition also has a figure done using a Java applet that allows the user to grab certain vertices and sides and drag them around, to get a sense of what the figure would look like in extreme cases. This is a very thorough site; it usually pops up near the top of any search for online editions of the *Elements*.

5. *The Archimedes Palimpsest*, archimedespalimpsest.org

   This site, originally hosted by the Walters Art Museum, contains complete details of the conservation, imaging, and translation of the Archimedes Palimpsest, which is the only known manuscript containing a copy of Archimedes' *Method*, where he explains his method of discovering area and volume relations "by means of mechanics." This manuscript, discovered in 1906 in Istanbul, was partially translated by Heiberg, and was then lost sometime after World War I. The manuscript resurfaced and was sold at auction in 1998, and since then has been in the hands of conservators and scholars. Using new digital imaging techniques, much of the faded text of the document has been recovered and translated. This site contains the whole story, and includes a complete set of the page images of the manuscript.

6. ProQuest, LLC, *Early English Books Online*, eebo.chadwyck.com/home

   From the home page: "Early English Books Online (EEBO) contains digital facsimile page images of virtually every work printed in England, Ireland, Scotland, Wales, and British North America and works in English printed elsewhere from 1473–1700…." This site contains images of many early mathematics books printed in English, from Robert Recorde's books to Billingsley's English edition of Euclid's *Elements*. Only works in English are included, so many of the original Latin editions of Newton, Barrow, and the like are not included here. It is, however, a great source for original works in English from the 16th and 17th centuries.

7. Dennis Duke, "Six Easy Lectures on Ancient Mathematical Astronomy," people.sc.fsu.edu/~dduke/lectures

   Dennis Duke is a professor at Florida State University, and is a widely published scholar on ancient astronomy. His "Six Easy Lectures" site contains an excellent overview of ancient astronomy, based on Ptolemy's *Almagest*. The site also contains links to animations showing the motions of the planets and moon according to many of the various models proposed by Ptolemy as well as the Islamic astronomers. This site is an excellent resource for someone planning to devote some significant time to ancient astronomy in a class.

8. Jeffrey Oaks, "Bibliography By Topic of the Mathematical Sciences in the Medieval Islamic World," pages.uindy.edu/~oaks/Biblio/Intro.htm

   Jeffrey Oaks is a professor at University of Indianapolis, and is a widely-published scholar in the history of Islamic science and mathematics. His bibliography is very extensive and comprehensive, and contains numerous entrees into the literature on the history of medieval Islamic science and mathematics. This site is an excellent resource for anybody picking up a topic in medieval Islamic mathematics for the first time.

9. McGill University, "The Calendar and the Cloister: Oxford-St. Johns College MS17," digital.library.mcgill.ca/ms-17/index.htm

   This site is dedicated to a famous medieval manuscript, labelled MS17, that belongs to St. John's College, Oxford University. The site is a valuable resource for instructors who want to give students a good feel for the practice of mathematics in Europe during medieval times. The manuscript itself dates from the early 12th century, and is a compendium of works related to science, medicine, astronomy/astrology, and mathematics. The manuscript contains sets of directions for calculating the dates of Easter (the computus), calendars, and other astronomical phenomenon needed for determining the dates of religious festivals. It also contains rules for doing arithmetic using the abacus (counting board), including fraction and multiplication tables, and parts of treatises by Bede, Gerbert d'Aurillac, and Abbo of Fleury. This manuscript nicely sums up most of the details about European calculation practices in the 12th century, as well as being a fantastic exemplar of manuscript artistry.

10. University of California at Los Angeles, "Cuneiform Digital Library Initiative," cdli.ucla.edu

    This is the home page of the Cuneiform Digital Library Initiative, hosted by UCLA (there are a number of other mirror sites as well). This is a very large database of known, studied, cuneiform tablets. Most have some drawings or images from different views; some have transcriptions and/or translations of the tablet's contents. If you want to see lots of actual cuneiform tablets, this is the place to go. The tablets include all different types of content, not just mathematical, but it is well indexed and the search feature allows you to find numerous mathematical tablets, although many of them are fragmentary.

It would be useful at this point to mention one resource that, as far as I know, is lacking: some decent-quality videos that show some problems or examples being worked out. Other areas of mathematics have dozens of YouTube videos posted that show example problems being worked out in detail: a quick search for any calculus topic would find scores of short videos showing problems being worked, with accompanying narrative. Unfortunately, the only videos available for history of mathematics topics are simply recordings of all or part of a class lecture. A set of short videos showing, say, sexagesimal multiplication, operations using Egyptian unit fractions, some key constructions, etc., would be an outstanding additional resource.

Specialized Sources

Finally, here is a sampling of some of the additional books and articles I have used as resources for more specific topics. This is only a sample, and is intended to give the reader an idea of the breadth of material that exists on almost any topic. Again, you can find out a great deal by pursuing backward through the references/bibliography of introductory books and articles.

1. Chris K. Caldwell, Angela Reddick, Yeng Xiong, and Wilfrid Keller, The history of the primality of one: a selection of sources, *Journal of Integer Sequences*, vol. 15, 2012. Article 12.9.8, 40pp. cs.uwaterloo.ca/journals/JIS/VOL15/Caldwell1/cald6.pdf
2. Gillian R. Evans, Schools and scholars: the study of the abacus in English schools c. 980–c. 1150, *The English Historical Review*, 94:370, (1979), 71–89.
3. Steve Farmer, Richard Sproat, and Michael Witzel, The collapse of the Indus-script thesis: the myth of a literate Harappan civilization, *Electronic Journal of Vedic Studies*, 11:2, (2004), 19–57.
4. Menso Folkerts, *Essays on Early Medieval Mathematics: The Latin Tradition*, Ashgate Publishing Co., Burlington, VT, 2003.
5. Roger Hart, *The Chinese Roots of Linear Algebra*, Johns Hopkins University Press, Baltimore, MD, 2011.
6. Jens Høyrup, *Lengths, Widths, Surfaces: A Portrait of Old Babylonian Algebra and its Kin*, Springer-Verlag, New York, 2002.
7. Victor J. Katz, Ideas of calculus in Islam and India, *Mathematics Magazine*, 68:3 (1995), 163–170.
8. G. G. Leibniz, Nova methodus pro maximis et minimis, itemque tangentibus, qua nec fractas, nec irrationals quantitates moratur, & singular pro illis calculi genus, *Acta Eruditorum*, October 1684, 467–473. izwtalt.uni-wuppertal.de/repertorium/MS/AE_1684.html
9. Galina Matvievskaya, The theory of quadratic irrationals in medieval oriental mathematics, in *From Deferent to Equant: A Volume of Studies in the History of Science in the Ancient and Medieval Near East in Honor of E. S. Kennedy*, ed. David A. King and George Saliba, New York Academy of Sciences, New York, 1987, 253–277.
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History of Mathematics Organized by Field of Study:

An Intensive Survey for In-service Teachers

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Course Overview

There are many ways to successfully structure a history of mathematics course, such as by civilization and geographic area or by time period (for example, see [1]). Huntley and Flores highlight a number of different textbooks and course implementations for history of mathematics classes for teachers [10]. Even as far back as 1979 Williamson suggested concentrating mainly on topics important to secondary school [20]. We strongly agree with this sentiment and offer a survey course for in-service secondary mathematics teachers, with the overall goal of having the teachers include history of mathematics in the classes they will teach. The course is designed around the fact that the participants are both students and teachers, and so we will use both terms to describe them depending on which aspect is being addressed. Because the teachers in the course come from a variety of different backgrounds, including lateral entry programs and middle grades instructors, the prerequisite is as inclusive as possible—a course in differential calculus.

The course is two credit hours and it satisfies the required 30 contact hours by meeting all day Monday–Friday for one week in the summer. The condensed timeframe and the audience of teachers lends itself well to pedagogical techniques and assignments that are designed for maximum engagement. The course begins by exploring the role of the history of mathematics in school classrooms including the relationship to NCTM and state standards. We organized the rest of the course by mathematical fields of study in order to help the teachers see direct connections with the classes they are teaching. The fields are number representation and number theory, algebra, geometry, and calculus. The course explores about one field each day, which works especially well for the format. For each field, we survey the history, from earliest definitions to recent results and conjectures, and specifically include non-western cultures and contributing mathematicians from all periods [7]. While the course emphasizes the historical development of the fields, it also exposes the teachers to more recent discoveries so that they can see that the development of mathematics is still a very active area. It introduces many mathematicians from different cultures, so the teachers can appreciate the many different peoples who have contributed to mathematics, ideas they can convey to their students when they are back in their classrooms. The focus is on the history as well as the related mathematics in order to help the teachers deepen their understanding and problem solving abilities and connect to the material. The primary resource for the course is the CD *Historical Modules for the Teaching and Learning of Mathematics* [11].

Course Design

Partly because of the format of the course and partly because of our own expertise, we designed the course with one main topic each day:

Day 1 - number representation and number theory
Day 2 - algebra
Day 3 - geometry
Day 4 - student presentations and precursors to calculus
Day 5 - calculus and final exam
One week later - final project is due

See Appendix A for a general overview of day-to-day activities. In addition, the course website [8] includes more specific details such as the classroom worksheets, handouts, and assignments, and sample final projects from previous semesters. With six-hour days, interactive classes that utilize discussions, computer and web activities, in-class worksheets, and videos are essential. We have access to both a classroom and a computer lab so the class can easily go back and forth. How does this actually work? Quite well—we first present new material on some of the main discoveries in the field of the day using lectures, videos or in-class readings. Many relevant images of people and mathematical objects are included to help the teachers connect to the content and model how they can bring history into their own classrooms. During the lecture portion of class, we pose questions regularly, either orally or in written form, to keep everyone engaged. Next we choose a follow-up worksheet or activity to deepen their knowledge and understanding, often from the CD or another text resource, or an activity that we create ourselves.

To encourage maximum classroom engagement, we often pause briefly to ask the teachers to share their impressions of the material and any prior experiences, beginning with the very first icebreaker activity, where we ask the teachers to discuss their own knowledge of the history of mathematics and to share their experiences of using history in their classrooms. A follow-up activity focuses on justifications for studying the history of mathematics including recommendations from professional organizations such as the MAA and NCTM. Sometimes we ask them to search the web for related information concerning history, people, recent results, or other classroom activities, and report their findings back to the class. It is exciting to share material with teachers that relates to their own classes. Class participation and engagement is outstanding, which makes the long days feel much shorter.

The course highlights mathematical content and ways to incorporate it into the classroom along with the history, development, and people. For example, in the context of numbers, the content includes tally marks, Egyptian and Babylonian number systems, prime, composite and perfect numbers and corresponding theoretical results starting with the Greeks and Euclid's proof of the infinitude of primes. The students also explore Egyptian and Babylonian representations of fractions, the history of irrational numbers, including Cantor's diagonalization and a biography of Cantor, recent results on prime numbers, and representations of the sum of cubes in popular culture. One feature of the structure of the course is that diverse peoples and current conjectures are included regularly. For instance, in the geometry section, we highlight geometry from Euclid, the solution of the Poincaré conjecture, and questions about the geometry of the universe.

Our choices of daily class activities are informed by many sources and our experiences with incorporating history into our own classes. Having already co-taught a history of mathematics class using Burton [4] and Dunham [6] as our primary sources, we bring historical activities into content classes, such as a general education mathematics class for liberal arts students, calculus, linear algebra, modern algebra, geometry, and more. We share these activities during class and on the course website [8] and discuss the benefits and difficulties of bringing them into a variety of classes as well as how we have improved on these over time. The teachers noted that they found the activities that faculty members had previously used in courses especially helpful as a model. As an example, in one activity the teachers complete a worksheet on Georg Cantor and his mathematics (using an "infinite dodgeball" analogy), that had been originally inspired by the *Heart of Mathematics* [3] and *MacTutor's* [15] biography on Cantor, but has been revised significantly after class testing [8].

Students do not have as much out-of-class time to complete extensive assignments as they would in a class that is not condensed. Lecture topics and activities during class are carefully chosen to provide students with both a broad overview and history of a particular field; homework is then assigned that gives the students the opportunity to connect the material from different topics. In the afternoon of the last day of class, the teachers complete a final exam. While it is a bit awkward to hold the final exam in the afternoon of a day in which new material has been covered, it works because the exam focuses on reflective questions that address the connections among the various fields we explore, as well as how to incorporate the ideas from the course into their classrooms. We find this is an effective way to assess our course learning goals. The final project for the course is a lesson plan due a week after the class ends, which allows adequate time for reflection and the creation of a quality product that the teachers can take into their own classrooms.

Resources

- **CD.** The CD entitled *Historical Modules for the Teaching and Learning of Mathematics* [11] serves as our primary course "text." The CD contains many modules of historical content with classroom activities and some guidance for incorporating them into the classroom. We use the CD in the same way that other teachers would use a text—by presenting modules to the students during class or by asking them to look at or complete specific modules for homework. The teachers engage with the CD themselves, by looking through, analyzing, and presenting an activity from it that was not already covered in class (see Assignments below). We and our students both find the CD to be a great resource.
- **Supplemental Readings.** To complement the CD, we choose a variety of other print sources to share with the teachers. These include sources aimed at educators, like *Math Through the Ages: A Gentle History for Teachers and Others* [2] and *Who? How? What? A Strategy for Using History to Teach Mathematics* [21]. The students complete activities from *Algebra Activities from Many Cultures* [13] and *Geometry Activities from Many Cultures* [14], which are designed for use in school classrooms, and then reflect on and discuss their experiences as a student completing the worksheet as well as whether they would use it in their own classes.
- **Web Resources.** We share many different web resources in class, including history of mathematics sites like *Convergence* [5], *MacTutor* [15], *Mathematicians of the African Diaspora* [19], and *Earliest Uses of Various Mathematical Symbols* [16]. The teachers use these sites to explore biographical information, historical content, and activities. The class watches NOVA's *The Proof* video and discusses the online teacher's guide [17] for it. We sometimes direct them to websites with very specific content, like *Approximating Pi* [9], *Inside the Archimedes Palimpsest* [12], and a Witch of Agnesi applet [18] and the teachers share many additional websites during class.
- **Course Website.** The course website lists daily topics and activities, webpage links, references, applets, classroom worksheets, lesson plans, and more. As our students suggest additional links and material in and outside of class, and design their own historical lesson plans, we also add these to the site. This helps develop an ownership in the material and a sense of community surrounding the use of history in the classroom. While we could have used a password protected course site like Moodle, we purposely used a site that these teachers, and the general public, could access after the course concluded [18].

Assignments

As the course primarily serves secondary school teachers, the assignments are intended to give the teachers the opportunity to consider how to incorporate the history they learn into their own classrooms. This is done both informally and formally—during class discussions, and in assignments like a journal of daily reflections or a formal lesson plan.

- **Reflective Journal.** The teachers keep a daily journal where they discuss historical concepts from each of the class days that expanded or deepened their prior knowledge and how this content may be used in a classroom setting. This assignment is intended to have the teachers connect the history they learn to the content they are teaching and to places in the curriculum where the introduction of this history would benefit their students.
- **Internet Searching on History.** One of the objectives of this course is to give teachers experiences in finding resources and historical information that they can use in their classrooms. The teachers gather information on assigned historical topics and mathematicians using the internet. Next they share this information with the class and comment on the resources they found. The teachers would then have good websites and hints on strategies for finding these websites for future use.
- **Historical Classroom Activity.** The teachers choose a specific NCTM or state standard that pertains to their own classes. They connect that standard to history of mathematics content via a classroom activity. The students are not required to create the historical activity themselves, but some do, while others find activities that someone else had created. They share these activities during class, and there is time for follow-up class discussion. This assignment occurs after the second field of study, so that the teachers have already seen models (and reflected in their journals) of many different ways to incorporate history of mathematics into classrooms, but it is still early enough in the course that this is a beginning step to more in depth assignments that are to follow.
- **CD Activity.** This assignment has the teachers interact with parts of the CD [11] not actively discussed in the course. Each teacher chooses and describes an activity from the CD and its relevant history, the NCTM standards that are covered by the content of the activity, and benefits and possible stumbling blocks of incorporating the activity into classrooms. They also specify where the activity will fit into the high school mathematics curriculum.
- **Timeline.** Since our approach to the historical content is not chronological, the teachers may not fully understand the temporal development of the mathematical discoveries and the people involved in these discoveries. Therefore, the teachers create a timeline in a format of their choosing that displays the mathematicians who are discussed in the course. We also require them to find mathematicians that are not mentioned in the class to add to their timeline along with a couple of sentences about the accomplishments of each of them. They have class time to work on this assignment towards the end of the week-long summer session as a review activity, and they share their timelines with each other.
- **Final Exam.** Presenting history in topic format makes it more difficult to draw connections between the different topics during the same time period. Rather than testing specific historical details, the final assessment asks the teachers to discuss connections between the historical development of different topics. One example question asks them to describe the connections between algebra and geometry in the history of the Pythagorean Theorem. Another question requires them to relate the history of mathematics to their classrooms.
- **Final Project: Lesson Plan.** This assignment is a major component of the course. The teachers are required to complete a formal lesson plan on a historical topic of their choosing. The plan includes an abstract, objectives, standards addressed, student prerequisites, lesson outline, assessment strategies,

handouts and answer keys, and an annotated bibliography of direct and related content. The teachers have a week after the last class meeting to complete the project. These lesson plans are intended to be self-contained and classroom-ready and are shared with the other teachers on the course website [8].

Lessons Learned

The course is organized by topic and we feel this is an excellent approach for our situation. Many different mathematical topics in many different time periods can be covered in a two credit hour course that meets in one week. The Program Standards for Initial Preparation of Mathematics Teachers (NCATE/NCTM, 2003) requires knowledge of the historical development in number and number systems, algebra, Euclidean and non-Euclidean geometries, calculus, discrete mathematics, probability and statistics, and measurement and measurement system for secondary mathematics teachers, and also includes the recommendation for knowledge of the "contributions from diverse cultures." With only thirty contact hours, we choose depth over breadth so we do not cover all of these topics. We do not regret this decision as it allows us to connect historical content with more recent and current work. If we were teaching a three-hour course, we would organize it in a similar manner, and the extra time would be spent looking at the history of additional recommended fields.

The CD [11] is our primary text and we and the teachers feel that it is a great resource for this class. It provides many good examples of units containing specific history and how to use each in a mathematics classroom. However, for a three-hour course, and perhaps even for our two-hour course, we might add a second primary resource—a more traditional history of mathematics textbook that would reduce the need to supplement the CD with additional content.

The one week course in the summer serves our teachers' time constraints and professional development needs very well, but places some limitations on the types of assignments we can use, as there is not much time between when one day's class ends at 4:30 P.M. and when the next day's class begins at 8:30 A.M. So we choose tasks with quick turnaround times, such as reflections and web searching, in lieu of assignments that required the teachers to complete extensive research or readings. If the course were spread over a longer period of time, we would include additional research projects and presentations on topics not covered in class.

Our course is designed for mathematics teachers and one of the main objectives is for the teachers to include mathematics in the courses they teach. While some of our class time is lecture-based, we provide many in-class activities for the teachers to engage with the historical content as they would similarly do with their students. Having a class of teachers results in quite interesting discussions with diverse perspectives on the mathematical content and history.

Overall, we like the topic-based approach and would use it as a general approach to teaching history to other populations of students, such as pre-service teachers and other undergraduate mathematics majors. One drawback of this approach is that it could miss some connections between topic areas and might not give the students an overall chronology of events. We remedy this by explicitly pointing out the relationships between the history of the different areas and by assigning the timeline project. Organizing the course by field of study allows us to cover a broad range of the history of mathematics while exploring the progression of fields our students teach in their own classrooms and, perhaps most importantly, we and the students enjoy the course.

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Appendix A
Course Outline

The names in parentheses are people (or cultures) we discuss during class activities. For more information, see the course website [8].

- Day 1
  - Previous experience with history of mathematics.
  - Why history in the classroom?
  - History of number systems (Egyptian, Babylonian), prime numbers (Euclid of Alexandria, Eratosthenes of Cyrene), rational numbers (Egyptian, Babylonian), and irrational numbers (Babylonian, Greek, Georg Cantor). Pythagorean triples and sum of cubes in popular culture and modern results (G. H. Hardy and Edward Maitland Wright, David Wilson, Randall Rathbun, Cristian Calude, Elena Calude, and Michael Dinneen).
  - Reflection on representations of mathematicians in society and bringing mathematicians into classrooms (Benjamin Franklin, David Blackwell).
  - Introduction to CD.
- Day 2
  - History of solving one and multiple linear equations (Egyptian, Babylonian, Chinese), quadratic equations (Islamic).
  - History of the Pythagorean theorem and Pythagorean triples (Plimpton 322, Pythagoras of Samos), Scarecrow theorem in popular culture.
  - The movie *The Proof* and discussion (Andrew Wiles, Goro Shimura, Yutaka Taniyama).
  - History of solutions of cubic equations and introduction of complex numbers (Giralamo Cardano, Rafael Bombelli).
  - Solutions of the quintic equation (Évariste Galois).
- Day 3
  - Collect and discuss the historical classroom activity.
  - History of early geometry knowledge (Babylonians, Egyptians, Greeks (Thales of Miletus, Pythagoras of Samos, Plato, Euclid of Alexandria), Chinese, Africans). Shape and size of the earth (Thales of Miletus, Eratosthenes of Cyrene, Abu Rayhan al-Biruni).
  - History of Euclidean geometry. Folding, proving and even walking the sum of the angles in a triangle (Pythagoras of Samos, Euclid of Alexandria, Walter Whiteley). Using and proving the Pythagorean theorem (Babylonian, Chinese, Pythagoras of Samos, Euclid of Alexandria). Water wheel video.
  - History of non-Euclidean geometry (Menelaus of Alexandria, Girolamo Saccheri, Johann Lambert, Carl Friedrich Gauss, János Bolyai, Nikolai Ivanovich Lobachevsky, Eugenio Beltrami, Felix Klein). Non-Euclidean geometry in art and popular culture (Maurits Cornelius (M.C.) Escher, Daina Taimina, Sarah Greenwald).
  - Shape of the universe (Eudoxus of Cnidus, Aristotle, Claudius Ptolemy, Apollonius of Perga, Hipparchus of Rhodes, Aristarchus of Samos, Nicolaus Copernicus, Johannes Kepler, Sir Isaac Newton,

  Bernhard Riemann, Albert Einstein, Carl Friedrich Gauss, Nikolai Lobachevsky, Karl Schwarzschild, David Wilkinson, Max Planck).
  - Web searches on geometry accomplishments (Girard Desargues, René Descartes, Pierre de Fermat, Blaise Pascal, Leonhard Euler, Carl Friedrich Gauss, Bernhard Riemann, Felix Klein, David Hilbert, Donald Coxeter, Shiing-shen Chern, Benoit Mandelbrot, Kenneth Appel, Karen Uhlenbeck, Frank Morgan, Carolyn Gordon, Grigori Perelman).
  - Examples from the history of geometry education (Pythagoras of Samos, Plato, Greece, England, Italy, United States). Related research publications and quotations (Sebastian Le Clerc, Alva Walker Stamper, Jennifer Bergner, David W. Stinson, Joseph Malkevitch, Nathalie Sinclair, Mohammad Yazdani).
- Day 4
  - CD assignment presentations.
  - Precursors to calculus. Zeno's paradox (Zeno of Elea). Inscribed and circumscribed polygons (Archimedes of Syracuse, Hypatia of Alexandria). Area of a parabola (Archimedes of Syracuse). Series and the history of π (Egyptian, Mesopotamian, Archimedes of Syracuse, John Wallis, James Gregory, John Machin, Gottfried Leibniz, Johann Lambert, David Bailey, Peter Borwein and Simon Plouffe) as well as related open questions (L. E. J. Brouwer, Émile Borel)
  - Analytic geometry (Apollonius of Perga, Bonaventura Cavalieri, René Descartes, Pierre de Fermat)
  - Create a timeline of mathematicians
- Day 5
  - Collect and discuss the reflective journals.
  - Continue history of calculus (Isaac Barrow, Sir Isaac Newton, Gottfried Leibniz) and history of calculus notation.
  - Disseminating calculus (Maria Gaetana Agnesi and the witch of Agnesi).
  - Introducing rigor in calculus and modern analysis (Karl Weierstrass, Georg Cantor).
  - Revisiting representations of mathematicians and ethnomathematics.
  - Current research problems related to topics covered.
  - Final Exam.
- One Week Later
  - Final Project Due.

Creating the History of Mathematics through Creative Assignments

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Course Overview

A background in the history of mathematics should be a key component of a mathematics degree, particularly a mathematics education degree. Schools that offer history of mathematics often have only one course that must service both populations. The course described here is a survey course intended for both secondary mathematics education majors as well as mathematics majors. In order to make this course productive for both audiences, the course is built around a variety of different types of assignments that each student can tailor to their major. This allows mathematics majors to delve deeply into various topics while allowing education majors to gain experience with different types of assignments and modes of presentation.

Most often students enroll in the course their junior year, with integral calculus being the prerequisite. It is designed to give the student a broad overview of the history of mathematics and its cultural interpretations while including a substantial amount of mathematics. With this in mind, the course first looks at counting with a brief overview of the number systems from some of the more well-known cultures: Egyptian, Babylonian, Greek, culminating with the Indo-Arabic system. The course then follows a fairly standard progression from Greek geometry and number theory, through the devolvement of algebra and algebraic notation, solutions of the quadratic and cubic, and the work on series and quadrature. This leads to the development of the calculus and the priority dispute between Newton and Leibniz. At the end of the course, selected modern topics such as abstract algebra, linear algebra, non-Euclidean geometry, or graph theory are added based on time, student background, and interests. (At the start of the course students fill out information sheets that include questions about what upper-level mathematics courses they have taken and their mathematical interests so modern topics can be chosen that best fit the class makeup.)

Course Design

I have two goals for the course. The first is to engage students in deep discussions about the material and to challenge their view of mathematics, sciences, and education. The second goal is for the students to present their knowledge in as many different ways as possible.

To engage the students in discussions and challenge their worldview several different approaches are used. These include outside readings and historical problems that present mathematical methods of different cultures and times. Students are asked to master the techniques and discuss the reasoning behind each approach. The mathematics is continually presented in the wider context of the scientific and even socio-political world of that time. Examples of this are the Cardano-Tartaglia episode and the method of obtaining a scholarly position, Galileo and the Catholic Church, and the Newton-Leibniz debate and its ramifications. However, the activity they seem to enjoy the most is working textbook problems from different time periods. The students marvel at what was expected of students at different times in the past.

The purpose of tackling these historical problems and approaches to mathematics is to get the students talking. To do this effectively, during class discussions all opinions are welcome as well as all opposing opinions. I encourage this by posing a thought-provoking question or playing devil's advocate. I then take a seat with the students and become just another member of the class. I ask questions of the person speaking, I share my own thoughts, I model changing my mind, and I challenge them to question their own perceptions and preconceived notions. Teachable moments arise when someone raises a question that allows us to explore a different way of thinking. I often adjust an upcoming assignment to allow new topics that have arisen through discussions to be explored more fully. Midterm and final exams are essay, and incorporate these "organic" topics. Students are told this up front so they stay involved in the discussion and take notes.

The second goal is to allow the students to present their knowledge in many formats. For future educators, I want them to gain experience with as many different modes of presenting material as possible. By working with different mediums, they gain the background and confidence needed to use these methods in their own classrooms. Secondly, and this ends up being very important pedagogically for the students, presenting their research in different ways allows them to leverage their strengths as well as work on their weaknesses. For example, poster presentations develop their organizational and visual skills as well as their ability to answer questions verbally. Creating timelines allows them to interact with new technology and to organize their knowledge logically and efficiently. Building a physical model allows the students to creatively explore the concepts from class kinesthetically. Writing assignments, which are now required in many curriculums, leverage and foster research and writing skills. All these assignments allow students weaker in computation to shine, and forces the more mathematical students to work their language and creative skills. See the assignments section for a complete list of the assignments used.

The course was initially taught in a four-week intensive January term. I feel that this arrangement is a wonderful setting for teaching a history of mathematics course. Since the students are taking no other courses, the history is their focus for the whole month and they can be asked to put in much more time and energy. Each four-hour course meeting is broken up into various activities: working problems, discussions, projects, videos, presentations and more: thus the longer class period is not a burden. We move from one type of activity to another with little lecture. This variety of modes of discovery is not changed when this course is taught during the regular semester, but each course meeting will usually only employ a short lecture mixed with working problems and then a portion of a larger activity that may cover several days.

Resources

The resources used may change depending on your library's holding. Head to your library to see what books and videos are in the collection and what on-line journals are available. Check to see what old texts or documents might be in your school's archives or special collections. Incorporating original sources is an invaluable approach, and if those sources can be viewed directly, the impact on the student is that much richer. See several of the sources in this volume dedicated to the use of original sources in the history of mathematics classroom.

Text

As the primary course text, I use *Great Moments in Mathematics before 165*0 and *Great Moments in Mathematics* after 1650 [4, 5].

Both volumes are by Howard Eves (1983) and are still available through the Mathematical Association of America (MAA). These two slim paperback volumes were originally written as high school texts. However, I think most high school students would be hard-pressed to handle more than the first few chapters. The chapters are short and direct, so one or two can be covered easily in a class period and additional readings can be given without burdening the student. The exercises are comprehensive and at the appropriate level for mathematics majors while varied enough to allow the students to explore different ways of thinking and problem solving and they always provoke discussion. The text is a bit dated, but this itself can become a type of original source, allowing for interesting discussions of how societal and educational norms change over time.

Supplemental readings

Over the years I have found the historical articles in *Math Horizons* perfect supplements to any course. For history of mathematics I keep a file of these articles and give the class an article or two a week. They are short, a joy to read, and once again thought provoking. Individual article pdfs can be downloaded from the MAA website.

Web Resources

My students are introduced to the MacTutor site of biographies of mathematicians [6]. This site provides detailed biographies of mathematicians as well as historical overviews of many mathematical topics. In particular I point out to the students that at the end of each article the author provides other useful references. The Mathematical Association of America's online journal *Convergence* and its images page *Mathematical Treasures* is a, excuse the pun, treasure trove of ideas for using the history of mathematics in the classroom [3].

Science and Technology Museum websites offer a wealth of information, images, and teaching ideas. Object groups of mathematical and scientific items housed at the Smithsonian [9] is particularly useful with over 30 mathematical object groups of interest to mathematics educators. The Smithsonian Digital Learning Lab [10] is a very useful resource that allows educators to build their own image library drawn from Smithsonian and outside sources such as *Convergence* that can be shared with students.

I also use various educational websites such as the "Magic Gopher". This site has a turbaned gopher guess your secret number after a string of numerical computations. It is a great way to get the students to analyze a problem mathematically while having fun and thinking about what they can do with their own students. I use it very early in the course when we are talking about number systems. But it also allows me to get them looking at what is available on the web that they could use in their teaching. Finding other interesting mathematical or pedagogical sites is part of an assignment, described below.

Videos

I show several videos in class. Near the end of the course we watch the Nova special *The Proof*. This prompts students to internalize what they have seen all semester: that mathematics is a dynamic, evolving discipline. The students enjoy discussing what it means to "be" a mathematician after that video. There are many documentaries available on mathematicians. In particular, the Walters Art Gallery documentary or NOVA special on the Archimedes Palimpsest are very worthwhile. The 2016 film *The Man Who Knew Infinity* about Ramanujan is well done. Finally, Bill Dunham's *Great Courses* series of lectures on great theorems is a pleasure. The list of good videos is always growing, so see what is available and on the web.

Supplemental readings for the educator

I have found the following texts very useful for providing additional information or problems for use in class.

A Contextual History of Mathematics, Ronald Calinger [2]. A compressed history of mathematics that focuses on the social and political context.

Classics of Mathematics, Ronald Calinger [1]. Focused histories of the key topics in the history of mathematics.

Philosophers at War: The Quarrel between Newton and Leibniz, Alfred Rupert Hall [7]. This volume details the famous dispute.

Mathematical Expeditions: Exploring Word Problems across the Ages, Frank J. Swetz [11]. As the title would suggest, this is a great source for problems.

Assignments

A variety of different modes of expression for graded events are used in this course. These include:

Problem sets

As mentioned above, the Eves texts were chosen in part for the exercises. Select problems are turned in for a grade. Often I will have students exchange papers and evaluate them while we talk about the problems. This exercise has pedagogical benefits as it generates discussion and allows students to compare methods. But care must be taken to not make students uncomfortable exchanging papers. Students simply discuss each other's approach (they do not grade each other's work). Often some correcting of the mathematics happens during this process.

Article review

This assignment is done in the first quarter of the term to get the students thinking about their term paper. Students pick a topic of their own interest, do a JSTOR search (this gets them on to JSTOR early) to find an article, read it, and write a review of it. I tell them to imagine they are reviewing this article for a magazine and I hand out a nice review pulled from the MAA reviews as an example. I like to assign reviews as opposed to summaries or reports because the students must write more engagingly than they would if they were simply reporting to me. More importantly, they have to really think about what they read, the style it is written in, do the arguments hold together, who would want to read this, etc. In short, they must form an opinion on the article and effectively express that opinion. It also allows me to start addressing writing concerns and styles in advance of their term paper.

Poster presentation

Students must make a poster on a topic in the history of mathematics. It does not need to be one that was covered in class, and about half the students will choose a new topic (too often Fibonacci). They can either make a physical poster, a virtual poster on the computer, or a simple webpage. For one hour during class, we have presentation day. Students split their time between standing near their posters to answer questions and viewing the other posters. A comment sheet posted next to each poster/computer allows students to give comments to the author. I have found the comments very insightful and use them to help in my grading.

Short biographical presentations

A list of about 30 mathematicians that are relevant to the course, listed in the order they will appear in the course and the approximate date at which that presentation will be expected in class, is given out the first day. The second class meeting the students sign up. A few select scientists such as Galileo, Copernicus, and Kepler are included. Students then present a ten-minute PowerPoint presentation on their person of interest. When this course is taught in the J-term, two presentations are given a day: one at the start of class and the second after the break. These are meant to be general overviews of the person and their contribution to mathematics (and science). Often I will give the first biographical presentation on the first or second day of class in order to model what is expected.

Comparative biography search

Related to the biographies mentioned above, are two key persons are not included in the list students choose from. These are Pythagoras and Euclid. (You may include others.) Half the class is assigned Pythagoras and the other half Euclid. Each students does a search on their assigned person to see how many stories they can find that are different or contradictory, how many resources actually have references, and what are those references. A long discussion takes place in class, started by students sharing what they have found. The idea is for them to realize that for ancient personalities, much of what we have is legend that came about comparatively recently. Often a story originates with one source that is then picked up and used by others until it becomes "knowledge". One example is the Pythagoreans' sacrificing 100 oxen when they discovered irrational numbers. We also discuss being a discerning reader: how can there be that much information on Pythagoras's life when no records were left behind? I actually hope that a student finds those "unique" references that talk about Pythagoras's wife and kids! Then we can really have a great discussion. I want them to learn to question everything.

Annotated Timelines

The students enjoy making timelines and they are a great project for in the classroom. There are several online timeline creators and the students must use one of these for this project. These programs allow branching of timelines, interesting ways to visually present data, as well as annotation bubbles or drop-down windows for details. I do not tell them which program to use but let them explore the different options and find the one that they like the best. In the end approximately six to eight different platforms are used which the other students will then view. The students must choose a specific topic for their timeline such as the history of Greek mathematics, women in mathematics, or another biography. Again about an hour is allotted in class for timelines to be presented. You may want to reserve a computer lab for this. And like the posters, a comment sheet is place next to each work station for student feedback. These timelines are useful for the prospective teachers who can use them in the classroom as class projects. With the use of technology in the classroom such as iPads or Chromebooks, having school children create timelines is an effective and fun way to have children engage in research early on.

Website review

Students are asked to pick a topic they would teach once they have their own classroom and find a website that has activities related to the historical topic they choose. The site can be for educators and provide activities they might use in a class if they are education majors, or it may be a site of general interest if they are not. They then need to turn in a one- to two-page review of the website similar to the article review. This allows them to start to build their own library of useful websites. One requirement is that they address the legitimacy/reliability of the site: Can you tell who wrote it? Are references given? I then collect all the URLs and give the complete list to the class for their future reference. For mathematics majors, this assignment is modified to reflect what could be used in an undergraduate mathematics course. I ask them to find a mathematical website, possibly expository in nature, which would have been interesting for them to have seen during one of their previous mathematics courses.

Historical Lesson Plan

For the education majors, I ask them to pick a topic we have covered and write a unit plan for a K–12 class. They must include the actual lesson plans, resources, worksheets, activities etc. to cover 2–3 class periods. For non-education majors, I let them design their own project. For example, I have had students build an historical model such as a cycloid, do a lesser paper or book report, or any other topic they are interested in. If they propose something interesting I am all for it. They usually have something they would like to do but have never had the chance.

Physical Models

Most research indicates that kinesthetic learning is optimal for capturing the students' attention and providing for deep and lasting learning. Students often report that the activities in which they create a physical model of a concept or a historical device were the most rewarding portion of a course. Students can build replicas of mathematical, scientific, or engineering devices, or they can build models that display a mathematical or scientific concept. For example, an abacus can be made using a shoe box, wires or string, and buttons or beads. A simple sextant can be made using the circular cardboard found under frozen pizzas, string, and a washer. A wooden model can be made to show that the brachistochrone is the path of quickest descent. Students can use their imagination to develop a plan to build such a model or use pre-existing plans, many of which can be found on the internet. One resource specifically for creating models in the classroom is *Hands on History: A Resource for Teaching Mathematics* [8]. There are numerous images of historical devices available on the internet that can inspire projects (as well as simply be used in class to enhance lecture). As mentioned above, the *Mathematical Treasures* portion of the MAA online Journal *Convergence* offers hundreds of images of historical books and objects [3]. While living outside of Washington DC I did research on historical devices at the Smithsonian National Museum of American History. Images and descriptions of many of the devices owned by the Smithsonian, as well as other museums, are available online. Smithsonian mathematics object groups include Adders, Calculating Machines, Kinematic Models, Spherometers, Dividers and Compasses, Ellipsographs, and much more [9].

There are options for how models can be used in the classroom. Models such as the abacus or Napier's Bones that come up directly during the course of the semester can be made in class with students breaking into groups. These group activities can be as quick as twenty minutes in the case of the home-made abacus or sextant, or the whole class period for slightly more involved projects. Alternatively, groups or individuals can pick from various projects and work on them outside of class over the course of a week or two. An hour can be set aside as with some of the other assignments described above for students to display their creations and view others. Care must be taken in preparing for the construction of physical models, whether they be done in class or outside. Materials need to be gathered, either by the instructor or the students, clear step-by-step directions need to be given, either by the instructor or in writing, and enough time allotted. It will always take longer than expected. For whole class projects, make the object yourself ahead of time to make sure it is easy enough to do in class. Having a completed model helps the students see what is expected and know it is possible to make with the material given. For projects outside of class, do not assign builds that require the use of machinery more dangerous than a simple hand saw and hammer and nails. If students are picking individual projects, have them turn in a simple proposal ahead of time so you can make sure they are not biting off more than they can chew.

If having students build a physical model is outside of what you are prepared to do, either in class or out, the idea of using physical models is still very powerful. Students who are interested can make a model as an alternative to the term paper or lesson plan. If you are creative, you can build your own model to bring to class. If your school has a 3-D printer, there is an increasing library of plans available online. Check with your department or library, there may well be old teaching devices from the 19th or early 20th centuries hiding in storage. One item that might still be in a storage room at your institution is spherical chalk globes for demonstrating spherical geometry or wood or plaster models of surfaces. Dust off these historical items and give them new purpose as original sources.

Term paper with presentation

This is a traditional term paper and carries a fairly weighty grade. Students must submit a topic proposal and preliminary reference list early on. I am very liberal about what constitutes a good topic, as long as there is history and mathematics involved and good resources available. Depending on the class, I sometimes require a rough draft be submitted. Correct citation and a comprehensive reference list with only minimal web sources is required. Students must trade papers with a partner about a week before the final due date to get feedback. (They show me the marked-up paper and this constitutes a few points towards their grade.) Fifteen minutes can be allotted in class about a week before the papers are due so students can talk face to face with their partner about their papers.

The 10 minute accompanying presentation counts for 10%–20% of the term paper grade. They do not have to do a traditional presentation, though most do. Alternatives students have used include a short documentary to show for the presentation portion or a graphic novel style slide show or book. Presentations are usually spread out over the last 2–4 class periods.

Midterm and Final as writing assignments

Given the emphasis on discussion in this course, my midterm and final are both essay. I pull the questions almost completely from topics we discussed in class, even if there was no reading assigned on it. I tell them this early on, so they take notes during the discussions. For example, I ask them to outline the different cultures and languages that an ancient Greek work might pass through before we can read it in modern English translation. These questions require the student to focus on the connections brought up during class and internalize them. Often our discussions during class grow very organically, so I have new topics available each semester. Students write and answer two essay questions for possible inclusion on the midterm and the final. They get a few points for doing this and I tell them that I will use the "juicy ones," as I call them, on the test. And I do! They must think about the material from a different prospective than they are used to and cull through all we have looked at and decide what is important and has depth. Basically, they have to put themselves in the position of instructor, which makes them a partner in the course as opposed to simply a consumer. Often they write problems that are harder than I would ask.

Alternative Assignments

Occasionally a student will request an alternative assignment. Depending on your goals for the course and how important any one assignment is to your purposes, allowing a student to do an alternative assignment can be very powerful for the student, and for you. If this course is taught to a wide range of students, allowing everyone to choose their own style of final project allows students to incorporate their interests and creative abilities into their study of the history of mathematics. It gives the students control over their learning and allows them freedom of expression. Perhaps most importantly, the presentations are so varied that the students do not start to lose interest as often happens when they are asked to sit through twenty or more student Power Point presentations. In the past I have had students who chose to do a play/skit depicting an episode from history; a dance performance depicting a concept from geometry, graph theory, or combinatorics; an art installation based on a mathematical or ethnomathematical concept; a detailed book report on a major historical reading; and a graphic novel style paper and presentation. However, the best alternative assignment a student has done was a twenty-minute documentary on Cardano which she then showed to the class. It was well done with music, images, and voice narrative. She was well on her way to being the next Ken Burns. For alternative projects, I require that the student meet with me to discuss the idea to ensure that the project is on a comparable level to other projects or the regular assignment. A short outline of their research and a reference list is still required for any alternative project.

I enjoy having both education and mathematics majors in the course. This combination of students makes the discussions and problem sessions more interesting because pre-service teachers and mathematics majors often approach mathematics from different perspectives and with different goals and strengths. Depending on the assignment, I will purposefully either mix the two groups or keep them separate. For problem solving or discussion groups, I have mixed groups so a variety of opinions and approaches are used. For group projects I group by major so the students will have a shared career goal they can focus on while they research and plan.

Lessons Learned

I have taught several different types and styles of history of mathematics courses and use many of these techniques in all of those courses (as well as in many of my straight mathematics courses). But what makes this course different is the variety of graded events used. Everyone learns differently, and these varied approaches allow students to play to their strengths at some points and to work on their weaknesses at others. I find that students, especially the education majors, appreciate the variety of activities as well as all the ideas it gives them for their own teaching. They leave class with a basic list of resources, a handful of activities already tried and ready to go, and a few displays (their poster, timeline, etc.) that they can use in their very first teaching job and include in their teaching portfolio. However, it is easy to pack too many types of activities into the semester. Trying new types of assignments often takes more time for the students outside of class, and more time in class to discuss and present than you might realize. The mistake I most often make is not giving students enough guidance. I like to allow students to be creative and I am often impressed with what they accomplish. However, many students are not (yet) self-reliant enough to take a general description and run with it. These students will then produce a minimal product and complain that they did not know what was expected of them. Give enough guidance ahead of time as to what you expect, but not so much that they do not have room to explore and create. An example of prior student work is often very effective and inspiring for them. This will save you needing to stop class later to clarify once you see that the students are off track. Be ready to adjust as you go along. Once you have tried a new activity, you will know for the next time how to fit it into your course most effectively. So my suggestion is to do only a few new assignments or activities each semester until you have a core set of assignments that fits well with your interests and goals.

As mentioned earlier, I enjoy having both education and mathematics majors in the course. However, this also presents the main challenge faced in teaching this course. At times the pace is too fast for one group or too slow for the other or the material is too hard for one group while too easy for the other. Since the mathematics majors usually have not seen the history before, their interest is piqued, but they often want the material to be more challenging. I try to have a variety of problems available; this allows students to go above and beyond the assigned problems. During problem sessions in class, I encourage education and mathematics majors to work together. This allows the sometimes less proficient education majors an added resource and the mathematics majors the chance to hone their knowledge. The alternative assignments allow students to pursue topics of their own interest. The discussions are the key to keeping all the students engaged. Like all courses, it is a balancing act. I encourage the students to keep me informed of what they are finding interesting so I can adjust the focus and depth as the course progresses. By making the students a partner in the course and giving them a variety of assignments and choice in their topics, they take away not only a strong background in the history of mathematics, but a deep understanding of the connections within mathematics, between mathematics and science, and mathematics and the wider world.

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Appendix A
Course Outline

Sample Course Outline: 14 week, 4 credit version. Note, for a 3–4 week "J-term" course, each 3–4 hour class meeting would correspond to roughly one week in the full-term version.

Week	Content	Chapters from Eves	Major assignments due & Special Events*
1	Introduction, number systems, Babylonian and Egyptian mathematics	1, 2	
2	Measurement and Greek mathematics	4, 5, 6	Website review I
3	Axiomatic method and Euclid	7, 8	Comparative biographies
4	Archimedes and trigonometry	9, 10	Article review I
5	Seed of algebra	11, 12	Poster Presentations
6	Indo-Arabic number system and the abacus	13, 14	Making and using abacus
7	Solving equations and review	15	Midterm exam or equivalent
8	Algebra and solving equations	16, 17, 18	Website review II
9	Logarithms and the birth of modern science	19, 20, 21	Make and use Napier's Bones
10	Analytic Ggometry and the approach to calculus	22, 23	Article review II
11	History of calculus	25, 26	Historical Lesson plan
12	History of computing	42, 43	Term paper draft or outline
13	Optional topics such as History of Abstract or Linear algebra	24, 27–41	Timelines due and timeline presentation day
14	Review for final, Presentations		Term paper and final presentations
15			Final exam or equivalent

*Daily biography presentation and weekly problem sets not included in this table.

From Student to Scholar:

Integrating Undergraduate Research in the History of Mathematics into a Mathematics History Course

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Course Overview

Studying the history of mathematics can be a valuable and enriching experience for a mathematics student. I believe that actively working in the field, however, is even more valuable. This article will describe my attempts to incorporate some "High-Impact Practices"—notably active learning and the incorporation of undergraduate research into the curriculum—into a History of Mathematics Class at Central Washington University (CWU) through a series of writing assignments designed to lead students to do their own original scholarship in the field.

This course, simply titled History of Mathematics, is available to anyone who has completed the "Sets and Logic" course which at CWU serves as an introduction to proof and as a bridge course. The majority of the student audience in the History of Mathematics class consists of future teachers. The course is required of all Secondary Mathematics Education majors. The primary goal of the course, therefore, is to cover the historical background and context of the standard high school mathematics curriculum. The secondary goal is to introduce students to the process of doing their own scholarship in the history of mathematics.

This goal is accomplished by incorporating undergraduate research through a series of assignments and projects designed to introduce students to the world of scholarship in the history of mathematics. According to informal polls taken on the first day of class, roughly half of the students enter the class unaware that the history of mathematics is an active research field. Furthermore, almost none of the students has any experience with the field – they have read neither primary nor secondary sources in the discipline, and have never considered pursuing the study of the mathematics history themselves. With this in mind, all major assignments are designed to give students a window into the discipline of the history of mathematics.

From a broader context, the goal of incorporating student scholarship answers a call from the Council on Undergraduate Research. This national organization, founded in 1978, recently elucidated five "strategic pillars" [3] comprising the most important goals of the organization. The first of these pillars calls for students to engage in research activities in their coursework.

> **Pillar 1. Integrating and Building Undergraduate Research into Curriculum and Coursework**—focusing on building curriculum that is both research-based and supports undergraduate research as an outcome of curriculum [3].

Course Design

The class meets for fifty minutes for three days a week over a 10-week quarter. The list of topics to be covered is outlined before the start of term and given to the students. The course proceeds chronologically, spending four weeks on Babylonian, Egyptian, and Greek mathematics, one transition week on the mathematics of the non-Western world, and five weeks on the history of mathematics from about 1200 CE to the twentieth century. Details are given in Appendix A.

While the broad topics covered are chosen by the instructor, students are expected to act as scholars and to take considerable ownership of the direction of the course from the first week of class. This is accomplished through a method of reading and response in the following way. Each day, students are assigned a short reading (about 10 pages) from one of our two primary course texts. Students are then required to send an email "response" to the reading of about 250–500 words to the instructor before class the next day. The content of these responses can be flexible, but students are encouraged to try to treat the text as if they were scholars in the field, and to note what they found interesting, what they found confusing, what they still want to know more about, and what they believe could be improved about the text.

The instructor reads all emails before class, and synthesizes common themes and important questions; the classroom discussion and lecture then focus on the material about which the students were most interested in discussing. In practice, the instructor "copies and pastes" interesting comments and questions from the students anonymously and without attribution to a two-page text document, copies of which are passed out to students at the beginning of class. Class discussion then consists of working through these questions and comments together.

Several of these questions are invariably small matters that can be explained quickly and easily by the instructor. Inevitably, however, many submitted questions lend themselves well to class discussion. For example, after reading about the development of algebraic notation in *Math through the Ages*, one student wrote "I am curious as to whether or not you believe that the algebraic notation that we use today is the most practical of the forms that have been used by various people in the past. Is it possible that we are making things harder on ourselves because our notation is more cumbersome than an alternate form?" Rather than answering this question immediately myself, I posed the question to the class, leading to a spirited discussion of the mathematics we had studied to date, the notation my students had been taught, and whether as future teachers they could consider teaching alternate notation to their own students. Throughout class, the students are expected to be engaging with the course material both as new learners and as critical scholars.

As a practical aside, I have spoken to many mathematics professors who feel uncertain of how effectively to lead a full-class discussion, as this is not part of many faculty member's own educational background. To the faculty member aspiring to do this well, I strongly recommend spending a few consecutive days in a class offered in a different department. Talented philosophy professors, for example, generally excel at this, and one can learn many ideas and methods by example in a short time by carefully observing their methods.

The two major exceptions to the instructor-led full-class discussion concern group work and short student presentations

Group work

About five times during the term, students complete group activities, ranging in duration from about 10 minutes to about 30 minutes. One popular activity is a mini-Primary Source Project from the TRIUMPHS project [9], which presents students with an image reproducing a Babylonian clay tablet. Without any information about the structure of Babylonian numerals, student groups are asked to decipher the tablet and to explain the mathematics it contains (the full assignment is reproduced below). Initially dubious that they are capable of doing so, most groups find that they can complete the task; moreover, by the end of their work, they seem to grasp Babylonian numeration quite deeply (see Appendix B). Other group work involves the exploration of other primary texts, and the completion of a series of tasks concerning differing notation throughout history.

In keeping with the course goal of introducing undergraduate research and incorporating scholarship, each of the group work exercises is designed to scaffold a research skill. The Babylonian tablet exercise, for example, is an introduction to the sometimes daunting task of finding a direction to begin when faced with a large amount of complex information. The benefit of a close reading of an original text reinforces another (self-evident) skill. Students are explicitly told, for each group project, the scholarly skill I hope they will improve through its completion.

Student Presentations

One of the course texts is William Dunham's *Journey through Genius* [3]. On selected days, students are given the task not just of reading the proofs of the theorems it contains, but also of being able to recreate them. During class the next day, a student is (pseudo-)randomly selected to present a theorem. The student is allowed to ask for help from the class if they get stuck, but must stay at the board and outline the major ideas before sitting down. The combination of a student presenter and the class participation results in deep student engagement, and seems to aid learning as well. Assessment of student group work and presentations is included in the "class participation" component of the grade.

Resources

Required course texts

1) *Mathematics through the Ages*: *a gentle history for teachers and others* (2nd expanded edition), by William P. Berlinghoff and Fernando Gouvêa [1].

   For my audience of future teachers of mathematics, I have never found a better primary course text than this book. The opening section, *A History of Mathematics in a Large Nutshell*, provides a readable and short sketch of the entire history of the field, and provides a nice outline and launching point for further discussion. The second half of the book contains short vignettes of the history of almost every topic the students will someday teach; the students recognize this, and generally pledge to keep the book for the rest of their professional lives, with plans to refer to it often.

2) *Journey through Genius*, by William Dunham [3].

   By providing detailed proofs of theorems, described according to the historical methods of proof, Dunham's book provides the perfect counterpoint to the Berlinghoff and Gouvêa text. The students find this reading more difficult, and a large part of class time the day after one of these readings is often given to working through one of the proofs together. While challenging, the students learn a tremendous amount from this effort, and they usually find the process very satisfying.

Other assigned readings

1) An excerpt from Chapter 1 of Benjamin Wardhaugh's *How to Read Historical Mathematics* [6].

   This reading guides the reader through two short primary sources that present familiar mathematics in unfamiliar settings. I assign it as reading early in the term, and have groups work through questions concerning the text during class.

2) Clemency Montelle's *A 'Symbolic' History of the Derivative* [4].

   The history of notation is a very important part of the history of mathematics, but it is not treated at length in many sources. This marvelous chapter from an MAA collection traces some important issues in the history of notation in calculus, while also providing a nice treatment of the development of calculus itself.

3) A few pages of one of Euler's papers (in translation).

   Over 100 of Euler's papers have been translated and posted on the Euler Archive [9]. Euler's writing is friendly and insightful, and he is my favorite author for students to read carefully in a primary source. Instructors should look through the list of available papers, and choose one that matches their own or their students' interest.

Vital for the instructor

There is also a family of resources without which I could not teach the course as effectively. These *sine qua non* works include

1) Benjamin Wardhaugh's *How to Read Historical Mathematics* [6].

   This book is packed with useful information, historical insight, and a wealth of examples from which to draw.

2) Jeff Suzuki's *Mathematics in Historical Context* [5].

   Before class, I usually spend some time in Suzuki's volume, which quickly and beautifully provides cultural and historical background and context for a region and time period. There are few books from which I've learned more about mathematics and its many connections with culture.

3) Glen van Brummelen's *The Mathematics of the Heavens and the Earth* [2].

   I borrow deeply and widely from this text when I teach the history of trigonometry. While too detailed to be read easily by most of my students, I could not teach without it.

Assignments

Four types of assignments are used for assessment: homework, quizzes, writing assignments, and reading responses. Homework is given sporadically about once per week, and short quizzes (designed to be completed in 15 minutes) are given regularly, also once per week. The majority of student effort, however, goes to the two other categories of work.

Reading Responses

Reading responses fulfill a crucial function in the class in the form of developing the habits of mind of scholarship in the students, and also closely guide my own lesson planning. The students are expected to complete a reading before each class meeting. We proceed chronologically through history, timed so that we can spend the last two class periods on the twentieth century. The first 60 pages of *Mathematics through the Ages* consists of a short "History in a Nutshell", and the rest of the book consists of vignettes focusing on the history of specific topics in mathematics. On a typical day, students may read a few pages of the nutshell history and one of the vignettes. About once per week, we read a chapter of *Journey through Genius* appropriate to our current period in history.

After each reading, students email a "reading response" to the professor. These were mentioned above, but since they form such an integral part of the course, I would like to share more details here. The responses give students a chance to critically assess the text as would a scholar in the field, and also to privately share any thoughts, concerns, questions, and confusions that they have about the chapter. These must be sent to me, as the instructor, at least an hour before class, and I spend the hour before class reading the responses and choosing which parts to discuss during lecture. There has never been a day when their comments and questions were insufficient to use as a basis for a 50-minute class. Their comments are insightful, and their questions are often subtle. Consider a few excerpts from responses from our reading on trigonometry. (Note among other things the several instances of poor grammar or spelling errors above. This bothers me not at all. There is a place for formal writing in my class, but I do not require this from the email responses, which are supposed to stand in for a conversation. Once students come to believe this, I find that the amount of writing they do increases.)

- I was not surprised to learn that Euler had a great influence on understanding how the sine graph looks like since he has had a great influence on geometry and we use geometry consistently with trigonometry.
- My only comment/question for this section was it took so long for people like Hipparchus and others who worked with "trig functions" to use a circle with an arbitrary radius r ?
- I was surprised that it was not until Euler that Sine was recognized as a function from angle-space to ratios. The more I learn about the history of mathematics the happier I am that I am learning mathematics in the 21st century.
- I don't know if I am just challenged in this way or what, but I still don't understand after both readings why we have to use half the chord of twice the angle.
- Or why that is more accurate I guess. Also they kept referring to spherical triangles. Did they think that the world was spherical at this time period? Or were they just looking at the sky as a a two dimensional circle and using the sine formula that way?
- The text talks about how Ptolemy, in his book, talks about "spherical triangles". Is this an early use of non Euclidean geometry?
- How much more information would we have by having the table that Hipparchus made?

I copy anonymous excerpts like these into a Word document, photocopy them, and distribute them to the students—they really seem to enjoy hearing the thoughts of their classmates concerning the reading. Note that many of these comments are multi-faceted: in the antepenultimate bullet point above, the student is asking about contemporary knowledge concerning the shape of the earth, and about what assumptions were made about the "shape of the sky", in addition to being rather fuzzy on why spherical triangles may be different than planar ones. The insight I get about the students' understanding from these responses is invaluable, and I have begun to adopt this model in other classes.

Students are graded on their responses only in that I count the number of responses they send over the quarter. If a student's responses are of very low quality (e.g., too short, no evidence of deep thinking at all), I let them know via email, and the quality usually soon improves.

Writing Assignments

It is in the writing assignments that much of the work to introduce students to the world of historical scholarship takes place. There are three assignments, and each is set up to draw students deeper in the field. I cannot take credit for these assignments, however. I use slightly modified versions of assignments that Fernando Gouvêa shared with me years ago. (Gouvêa tells me that he in turn adapted his first writing assignment from one created by Fred Rickey[1].) I am thus the grateful heir to a lineage of talented historians and teachers.

The premise of the first assignment is fairly standard—it asks students to write a biography of a mathematician. A list of good candidates is passed around class, and each student chooses one. What differentiates the assignment from that offered in some other classes, however, is the detailed requirements regarding background reading that the students are given. The students are instructed, in a series of steps, to search their

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[1] Fred Rickey, to the best of my knowledge, sprung fully formed from the head of Zeus when the world was young.

textbooks for references to the mathematician, and to search the university library, either for biographies of, or for books by, their mathematician. They then must read the biographies on both the St. Andrews site [7] and on Wikipedia, and note any differences between these (students often come to class quite proud of finding differences between the biographies, which does wonders for their skills in critical reading). Students then go to the *Dictionary of Scientific Biography*[2] to find more scholarly information. Only after taking notes from all of these sources are they allowed to write.

The second writing assignment lets the students interact directly with scholarly literature in the history of mathematics. They are given the task of finding a peer-reviewed article written on mathematics history, of reading it, and of summarizing and explaining it in two pages. This second assignment, therefore, puts the students directly in the role of a researcher scouring the literature and seeking to understand it. I would offer one important word of caution, however. I have learned that I must specify good sources for articles, and that the students must send me the article they plan to use and have it approved. It is too easy for students to be attracted to an interesting article on the golden ratio, say, which mentions a bit of history. I do not allow articles of this type, but rather require that articles be written with history as their primary focus. (A notable side-benefit to this assignment is that I, as the instructor, receive nice summaries of many articles I wish I had time to read myself!)

The instructions for the third writing assignment are written in the voice of our textbook author, Fernando Gouvêa. In the assignment, Gouvêa asks students to write a vignette for his (wholly fictitious) new book, *More Mathematics through the Ages*. Students are asked to write something very much like the vignettes in the existing book, with the difference that their mathematical topic must come from the university (and not high school) curriculum. The assignment carefully describes the length, content, style, and even the margins in which the vignette must be submitted. Although this assignment was written for his students, I find that my students are quite enthusiastic when I tell them that their textbook author has sent me this request directly. At the point that they begin this assignment, they have become fairly well grounded in the basic methods and sources in the history of mathematics, and their final works are usually quite impressive.

Grading

The writing assignments and the reading responses are, to me, the core of student work in this class, are the grades are weighted accordingly. I based 20% of the final grade on each of the three writing assignments and the reading responses, with 10% of the final grade based on homework and in-class participation.

Lessons Learned

I have learned that students are remarkably receptive—even eager—to engage with the scholarly community. The history of mathematics presents a wonderful opportunity for such engagement. Reading peer-reviewed research in history presents its own set of challenges, but overall I suspect it is easier for students than reading modern research in, say, operator algebras. The end of term is quite rewarding, as students are justifiably proud of what they have accomplished.

After experimentation and some failures, I would offer the following regarding the major activities in class:

- *Oral presentations*: It is helpful to establish, very early, an environment in which students can present at the board, fail, and not be judged for their failure. I have seen much deeper student engagement once the students feel "safe".

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[2] The *Dictionary of Scientific Biography* (*DSB*) is another highly useful source in the history of math and science. My own university library, for reasons unknown, discarded their original copy two years ago. Since the books are necessary for my class, I acquired my own set. I would note, for the benefit of readers whose library does not possess a copy, that I was able to secure a used copy quite inexpensively on a well-known auction website.

- *Reading responses*: The only thing that seems to be required for these to succeed is that the instructor makes a sincere effort to read all emails and to refer directly to them in class. I have found considerable time savings in sending an email to the entire class from my Gmail account, and asking students to send their responses in reply. Gmail automatically puts their responses together into one conversation thread, speeding up the reading process for the instructor.
- *Writing Assignment 1*: If you can make time, require students to have a short (10 minute?) meeting with you a week or two before the paper is due to go over the notes they have taken on the various sources they consulted. Knowing that the instructor is interested in their notes dramatically improves their quality.
- *Writing Assignment 2*: Be sure to require all students to submit the article they plan to summarize to you for your approval. I have found this to be absolutely essential. It is easy for a student to find an inappropriate article (on, say, the appearance of the golden ratio in nature) and to base their paper on it.
- *Writing Assignment 3*: I offer this option for all of the assignments, but it's most useful here—students may submit a rough draft 10 days before the assignment is due for comments from me. Because assignment 3 is the most intimidating to them, presenting this option goes a long ways to reducing anxiety and improving quality.

Throughout the course, I make it clear that I expect them to act as beginning scholars in the field of the history of mathematics. This in turn requires me to treat them as such—to quickly and respectfully read their work and their emails and to respond to them professionally. The resulting classroom environment is well worth the effort.

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9. The Euler Archive. eulerarchive.maa.org.

Appendix A
Course Outline

With the exception of Week 1, the class meets three days per week. The topics covered on each of these days are briefly given here.

Week 1: Introduction; pre-history of mathematics

Week 2: Systems of Numeration; Egyptian and Babylonian mathematics; introduction to Ancient Greece

Week 3: Geometric constructions; detailed quadrature of the lune; Pythagorean theorem; reading from Euclid

Week 4: Greek number theory and the infinitude of primes; Archimedes; mathematics of Ancient China

Week 5: Indian Mathematics; history of trigonometry; the Arab World

Week 6: Fibonacci; historical and cultural discussion of late modern Europe; early algebra

Week 7: Algebra and the cubic formula; early probability; Descartes and analytic geometry

Week 8: Isaac Newton; the calculus wars; intro to 18th-century mathematics

Week 9: Leonhard Euler; Survey of the 19th century; Cauchy and Weierstrass

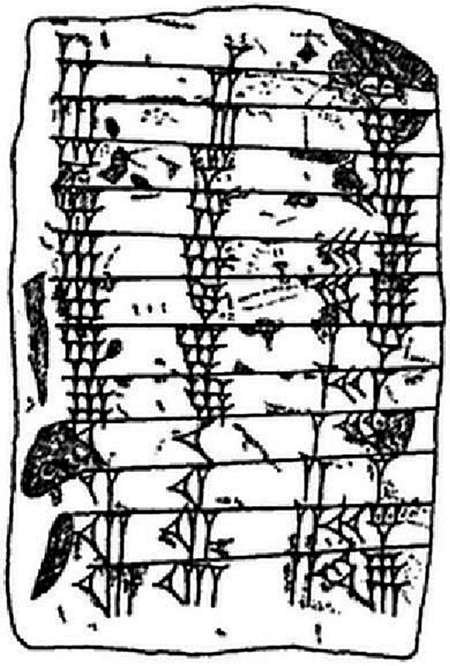
Week 10: The 20th century; history of computing; great Theorems of the 21st century

Appendix B

A Primary Source Project for in-class group work

Reading an old tablet

The picture below is a recreation of a Babylonian clay tablet. Study it, and answer the following three questions:



1) How do Babylonian numerals work?
2) Describe the mathematics on this tablet.
3) Write 72 in Babylonian numerals.

An Activities-Based History of Mathematics Course for Pre-service Secondary Teachers

Christine Latulippe
Norwich University

Course Overview

The history of mathematics course described here engages students through a combination of hands-on discovery learning, collaborative group work, and informal student presentations, providing context to what they're learning in other classes, and helping them gain appreciation for the mathematics we know and use today. This junior-level survey course is taught on a 10-week quarter system, with a prerequisite of multivariable calculus. It is a required subject matter preparation course for students seeking licensure to teach middle school or high school mathematics ("pre-service secondary teachers"), and can be used as an upper-division elective course for mathematics majors and minors. The course described here puts students into the role of math detective, includes playing games and solving problems as people of earlier eras did. A variety of in-class activities and out-of-class assignments support students as they acquire a broad overview and awareness of the mathematicians and mathematics of the past, and develop a critical view of historical resources.

Course Design

Four goals guided the development of this course: 1) pace the course to give a true overview of the history of mathematics, 2) prepare and present lessons relevant to classroom teachers, 3) emphasize the "who" of the history of mathematics to help students understand mathematics as a human endeavor, and 4) have students solve problems the way earlier mathematicians solved them.

To address the first goal above, a division of the 10-week course into thirds resulted in approximately three weeks focused on numbers and arithmetic, three weeks focused on geometry, three weeks focused on algebra and trigonometry, and one week for final student presentations. In each section of the course, the content theme was addressed across time and place, and included information about important players, innovations, and texts.

In-Class Activities

This class is routinely scheduled to meet twice a week for 75 minutes each time, providing large blocks of time to really dig into ideas from the history of mathematics. Students were encouraged to be present at every class meeting, where they engaged in activities which modeled the type of hands-on learning that is more common in middle and high school teaching than in the university classroom (goal 2). In-class explorations were designed to complement out-of-class readings and assignments, and to support goals 3 and 4 above. The activities used to teach the history of mathematics included hands-on discovery learning, collaborative group work, and informal student presentations. Rather than utilizing these instructional strategies individually, these pedagogical tools were regularly intertwined to maximize their benefit for varied student learning styles.

Collaborative Hands-on Learning

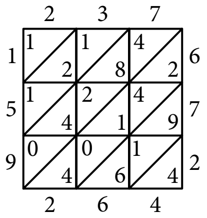
Use of manipulatives in class supported the second goal of capitalizing my ability to present lessons relevant to classroom teachers. During the Numbers and Arithmetic portion of the course, students spent a full class period in "Base Five Land" where all counting and arithmetic had to be done using base five. Adapting a common elementary school mathematics manipulative, students used individual popsicle sticks, bundles of five popsicle sticks, and baggies containing five bundles of popsicle sticks as a hands-on illustration of adding one and exchanging to the next place value. This was a powerful introduction to bases used in different cultures historically (e.g., base sixty for the Babylonians), and a remarkable lesson to students about why and how our own base ten place value system works. During the Geometry portion of the course students spent a class period working through a packet of geometric constructions using compass and straightedge, which for some was the first time they had used a compass. After students participated in this construction activity, they often commented on Euclid and Archimedes with newfound appreciation. During the Algebra and Trigonometry portion of the course, students created their own simple paper slide rules by following directions found in a CD published by the MAA called *Historical Modules for the Teaching and Learning of Mathematics* [3]; some students had never heard of slide rules!

Students played many games in class, most of which were created for K-12 classrooms, but sometimes still proved challenging to college students, and always modeled quality classroom resources. For example, on the first day of class, students were introduced to famous mathematicians through a "Go Fish" or "Authors" style game from *Historical Connections in Mathematics: Resources for Using History of Mathematics in the Classroom, Volume II* [7]. By playing this game, students learned basic facts or anecdotes about 13 mathematicians, including Euler, Hypatia, Banneker, and Gauss. Throughout the semester, this game was revisited by passing out cards and using them to assign groups (e.g., "all the Eulers get together and form a group"). As part of their out-of-class mathematician biography assignment, students were offered extra credit to create a game card for their mathematician that matched the style and design of the cards in the original game. I now have a complete laminated deck of "Mathematicians" game cards created entirely by students; they love to see their work in such a fancy format, and it's a great extension of the original game.

Collaborative Discovery with Mini-Presentations

Mini-presentation activities involved students examining and synthesizing new information in a short period of time (no more than one class period) and then conveying that information to others. One such activity during the Numbers and Arithmetic portion of the course (adapted from [11], and supporting goal 4) put students in the role of "math detective", where they were asked to determine how a particular multiplication algorithm or tool works. Each student group was given the same example multiplication problem computed using a different algorithm or tool (Egyptian multiplication, Russian peasant, lattice, Sacchieri, Hindu, and Napier's rods/bones). Below is an illustration of the lattice or gelosia multiplication algorithm and corresponding activity instructions.

> You are a math detective trying to figure out how an ancient culture multiplied. Study the multiplication example for 237 × 672. When you know how to multiply as in the example, show how to multiply 129 × 1,482.



Product = 159,264

Be ready to explain the process to your classmates.

a. How does it work?
b. Is this method of multiplication easier or harder than the algorithm for multiplication that you have learned? Explain.

In a parallel activity during the Geometry portion of the course, student groups worked through different "proofs without words" (e.g., [5]) of the Pythagorean Theorem, including President James Garfield's 1876 proof. Each group received an image, and directions which stated: "Study the diagram(s) below. Determine how it/they prove the Pythagorean Theorem. Be prepared to explain the 'proof without words' to your classmates." Many students had only memorized $a^2+b^2=c^2$ and were thinking about the Pythagorean Theorem visually for the first time with this activity; a visual representation combined with an algebraic formula can go a long way in teaching this famous theorem to their own students someday.

Another mini-presentation activity had students read, summarize, and present biographies of assigned mathematicians within one class meeting. To support goal 3 and expand the class collection of mathematicians, these biographies focused on mathematicians no one had studied for their biography project, and who weren't heavily discussed (if at all) in *Fermat's Enigma*. Using the MacTutor History of Mathematics website [6] and other resources, I have created two-page biographies of a variety of mathematicians, which student groups receive in class. Each of three mathematicians was studied by two different student groups, who created posters and shared highlights of their mathematician's life with their classmates. Two very different looking posters about the same mathematician often conveyed the exact same student-selected highlights, which became the important personal notes that most students remember about those mathematicians. For example, when teaching in the spring, May birthdays include Florence Nightingale, Maria Agnesi, and Bertrand Russell, so these three mathematicians were presented on during a May class day. Women mathematicians are often neglected with student biography projects, so another trio worthy of study is Mary Somerville, Emmy Amalie Noether, and Sofia (Sonia) Kovalevskaya. While secondary mathematics topics don't always align well with popular mathematicians, birthdays or themes are one avenue for introducing middle or high school students to mathematics as a human endeavor.

Resources

There were two assigned texts for the course described here. *Math Through the Ages: A Gentle History for Teachers and Others* [2] is an incredibly readable text, full of questions to assign to students and suggested additional resources. As a non-historian of mathematics, this text was far less overwhelming (and less expensive) for me and my students than a more traditional calculus-text-sized history of mathematics textbook. *Math Through the Ages* begins with a 60 page "History of Math in a Large Nutshell", and continues with 25 sketches (5–8 pages each) about various mathematical topics like Negative Numbers, The Story of π, and Solving First Degree Equations. Students were assigned to read about half of the "nutshell" introduction and about half of the sketches. These readings complemented what we were doing in class, with in-class activities supplementing, rather than repeating the text. Twenty-five percent of students' grades came from problem sets (with most problems drawn directly from *Math Through the Ages*) and reading assignments. Appendix A provides an outline of course assignments and readings, and Appendix B provides course point allocations.

Students read all of *Fermat's Enigma* [8] at a rate of about one chapter per week. *Fermat's Enigma* tells the story of Fermat's Last Theorem from Pythagoras to Andrew Wiles, neatly conveying the historical context of the theorem at key points in its lifespan. The tale of Fermat's Last Theorem is an adventure, and *Fermat's Enigma* reads like a mystery novel. Multiple students have reported that they were going to pass the book on to a relative or friend to read because they enjoyed it so much. Students were assigned 3-5 reading questions per chapter as homework, and the questions assigned were inspired by Loe and Rezac [4]. Some reading questions can be answered directly from the text (e.g., Chapter 1: "What is the difference between mathematical proof and scientific proof?"); some questions require a little bit of outside research and reflection (e.g., Chapter 3: "What is the Fundamental Theorem of Arithmetic? Algebra? Calculus? Which of these have you seen proven before? Do you understand each of these?"); some questions are mathematical (e.g., Chapter 6: "Prove the following using mathematical induction. For all positive integers n, $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$"); and some questions inspire student creativity (e.g., Epilogue: "There are two limericks in the epilogue. Pick a mathematical topic and compose an original limerick about it."). See Appendix C for the full list of assigned *Fermat's Enigma* reading questions.

We watched a number of movies in class, beginning with *The Proof* [9], a 60-minute NOVA documentary about Andrew Wiles and Fermat's Last Theorem. Watching this movie early helped provide students with a preview or road map to guide their reading of *Fermat's Enigma*. Another great NOVA film is *Infinite Secrets* [10], about the Archimedes Palimpsest, best viewed during the Geometry portion of the course. *Project MATHEMATICS!* [1] is a series of videos produced by Tom Apostol at the California Institute of Technology, intended to explore high school mathematics topics. There are three titles in the series, each containing three approximately twenty-minute long films. From this series, *Early History of Mathematics* provides a review of the first few weeks of the course and a preview of geometric ideas to come. *The Theorem of Pythagoras* and *The Story of Pi* are both relevant to teaching the history of geometry. *Sines and Cosines Part II* (trigonometry) and *Sines and Cosines Part III* (addition formulas) are the best examples in the series of computer animation being used to illustrate advanced mathematical ideas, and students have wished out loud that they had seen these videos in high school when first learning about trigonometry. Media and movies help university students connect with course content, and they also show pre-service secondary teachers high quality resources available for use in their own classrooms. VHS copies of the *Project MATHEMATICS!* series are allowed to be copied freely for educational purposes and can often be obtained for free from a local NASA Educator Resource Center.

Assignments

Out of Class Assignments

In an effort to emphasize the "who" of the history of mathematics (goal 3), students were required to research, write, and present a biography of a mathematician. Students were provided a list of about 40 mathematicians to choose from but they were welcome to choose a mathematician not on the list. Students had approximately four weeks to complete the project, and a major resource for most students was the online MacTutor History of Mathematics Archive [6]. Instead of students presenting their biographies to the entire class, they gave "mini-presentations". I would go through my class roster and create three rotations of groups where nobody overlapped from rotation to rotation (if Jasper and Claire were in a group together the first time, they would not be together again that day). Then students would present to just their small group, and listen to their groupmates. After the allocated time was up, students would rotate to the next group and begin again. This presentation style had three main advantages: 1) it was less stressful for students than being in front of the whole class; 2) it was easier for students to listen to just a few people sitting close to them, than to a whole class worth of presentations; and 3) most importantly, it was a clear example of how practice makes a presentation smoother and easier in subsequent iterations—this was a helpful point to encourage students to practice for their larger final course presentation.

The other course project was a *Fermat's Enigma* timeline. As students read *Fermat's Enigma*, they were instructed to make a note of mathematical ideas and mathematicians discussed in the book in order to create a timeline in a format and size of their choosing that included the mathematics and mathematicians discussed in *Fermat's Enigma*. Directions for this assignment were fairly open, but timelines were expected to be well organized, free from errors, and professionally presented. For full credit, students needed to include a thorough collection of mathematicians and mathematical ideas, including the mathematician that they researched for their mathematician biography, regardless of whether that person is mentioned in *Fermat's Enigma*. Student timelines have varied in form from spreadsheets made in Microsoft Excel to science fair poster boards and six feet of butcher paper. Students have submitted a small case full of mathematicians' business cards, a creative scrapbook of mathematicians and their achievements, a timeline in proportions similar to the Rhind Papyrus, and a student-created YouTube video timeline. One student submitted a very important scroll, complete with aged edges, in an unmarked paper bag, much like the Archimedes Palimpsest being left at the Walters Museum in a simple duffle bag. The full *Fermat's Enigma* Timeline project grading rubric can be found in Appendix D.

The final course project was a group presentation and student-created board game. Students chose their own groups of 3-4 students, and selected an era in the history of mathematics, or an area of mathematical study that was of interest to all group members. The final deliverables for this project were an annotated bibliography of presentation resources, a 20-minute in-class presentation, and a student-created board game related to the same topic. As checkpoints along the way, groups needed to declare their topic and submit minutes from the first group meeting at about the 4-week mark, and submit a presentation outline at about the 6-week mark. Annotated bibliographies were due during the last week of the course, at the same time as presentations, and needed to include at least six properly cited references with a one paragraph summary of each. Students were provided with a short list of project topic suggestions, including "the History of… (calculus, statistics, number theory, etc.)", Pascal's triangle, cryptography, mathematics of a particular culture, and famous unsolved problems. Additional topics that students have proposed and presented on include mathematical feuds and "deaths and suicides in mathematics". Presentations have varied from more traditional PowerPoint slides to a Who Wants to Be a Millionaire? style game which presented information through audience engagement, and five toga-wearing students who presented on Ancient Greek mathematics without any technological help. The final group project also included a history of mathematics board game that teaches or reviews some component of the history of mathematics, preferably related to their presentation topic. For easier storage in their future classrooms, students were encouraged to limit their board games to the size of a typical file folder. On the day of the final exam, we had a potluck lunch and "math history carnival" where students played each other's games. All final project rubrics can be found in Appendices E, F, and G.

Midterm and Final Exams

For their take-home midterm exam, students were allowed to use mathematics references and discuss exam questions with their classmates and instructor. I assigned midterm questions which complemented in-class activities and problems found on the problem sets, and which could continue teaching students. For example:

a. Who is René Descartes? State one interesting fact about René Descartes.

In 1637, Decartes' famous book "Discours de la methode…" was published, including three appendices.

b. Express the number 1637 using each of the following numeration systems:
   i. Egyptian hieroglyphics
   ii. Roman numerals
   iii. Babylonian cuneiform
   iv. Mayan numerals

Question 1 on the in-class final exam reflected goal 3 of emphasizing the "who" of the history of mathematics. It read:

a. List ten mathematicians you have learned about during this quarter. Include first and last name; spelling must be accurate enough to be recognizable.
b. For five of the mathematicians you have listed, provide one fact about their lives or mathematics.

Reflecting each content portion of the course, below are three sample questions from the in-class final exam, drawn from the history of mathematics; these questions support goal 4. Although the underlying mathematical content of these questions would be appropriate in a middle or high school classroom, the reading comprehension required to translate the rhetorical algebra can sometimes make them difficult for university students to begin working on.

Numbers and arithmetic. From Robert Recorde's *The Whetstone of Witte*, 1557:

> A captain marshalls his army in a square formation. When the square is of one size, he has 284 men too many. But when he rearranges them in a square one man more on a side than before, he lacks 25 men. How many men does he have?

Geometry. From the *Nine Chapters on the Mathematical Art,* date unknown:

> A square, walled city of unknown dimensions has four gates, one at the center of each side. A tree stands 20 paces from the north gate. A man walks 14 paces southward from the south gate and then turns west and walks 1775 paces before he can see the tree. What are the dimensions of the city?

Algebra and trigonometry. From Fibonacci's *Liber Abaci*, 1202:

> Two birds start flying from the tops of two towers 50 feet apart; one tower is 30 feet high and the other 40 feet high. Starting at the same time and flying at the same rate, the birds reach a fountain between the bases of the towers at the same moment. How far is the fountain from each tower?

Lessons Learned

For a survey course, the pacing and division of topics into thirds worked well, and planning in advance helped to not become bogged down in any one area of history (goal 1). To teach a comparable course on the 15-week semester system, I would add one week to each of the current topics, and allow two weeks for the history of calculus. Calculus is relevant to high school teachers, and it is content that mathematics majors and minors alike have at least one year of experience with by their junior year, providing them something to connect the historical ideas to.

I was determined to assess students on mathematics and historical ideas that were separate from the memorization of names and dates, and felt that take-home exams would be a more appropriate format for the assessment questions aligned with this philosophy. The first few times I taught the course, it was frustrating to find students using the internet to look up and solve many of the historical problems on their take-home exams (famous problems and solutions are easy to access online). For the last time I taught the course, students sat for an in-class final exam, and after about 20 minutes of independent work they were allowed to complete the exams in groups. Supporting the fourth goal for the course, this exam format nicely allowed for students to be working through famous problems stated in their original form, without modern resources.

Originally, the final project included a research paper to be submitted on the day of student presentations. The quality of the papers that students submitted was disappointing, but because the papers were being handed in on the last class day, there was no real opportunity to help students improve. Without moving the project earlier in the semester to allow time for re-writes, I began including checkpoints in the project grading rubric like group meeting minutes and a project outline, to be sure that there were fewer surprises on the last days of class. Requiring an annotated bibliography instead of a full research paper supports students' research and writing skills, but allows their presentation to be their final synthesis of knowledge, and reduces my grading load substantially. A board game that other students will play also encourages students to do more careful editing and fact-checking than an essay that only their instructor will read. In the future, students will evaluate each other's games as they play them at the last class meeting.

Students' *Fermat's Enigma* timelines are one of my favorite assessment items from the course. The open style of the assignment directions without a model or template can be difficult for some students at first, but after reviewing the scoring criteria together, I suggest a range of timeline formats from writing an essay to creating an interpretive dance, as long as the criteria are met. More than the format, the quality and thoroughness of the timeline are key to doing this project well, and in past classes students have risen to the challenge and enjoyed the project. This assignment is an important example of differentiating the mathematics classroom, which is valuable for college-aged students, but also models the pedagogy for pre-service teachers (goal 2).

A very important lesson learned the first time I taught the course was to bring my camera. I have no record—apart from memory—of Gerolamo Cardano and Niccolo Tartaglia appearing in the classroom to participate in a final presentation about mathematical feuds. Now I take pictures of everything!

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APPENDIX A

Course Outline

Week	Tuesday	Thursday
1 Numbers	Read *Ages* pages 5–14	No Class — Holiday
2 Numbers	Read *Ages* pages 65–70 *Fermat's Enigma* Intro & Ch. 1 DUE	Read *Ages* pages 79–82 * Your Mathematician selected by this date
3 Numbers	Read *Ages* pages 85–90 *Fermat's Enigma* Ch. 2 DUE Select Group topics and dates by this date	Read *Ages* pages 93–98 Read *Ages* pages 73–76
4 Geometry	Read *Ages* pages 155–160 *Fermat's Enigma* Ch. 3 DUE Problem Set #1 DUE	Read *Ages* pages 14–32 Your Mathematician DUE & mini-presentations in class
5 Geometry	Read *Ages* pages 107–110 *Fermat's Enigma* Ch. 4 DUE Group Presentation Minutes/Action Steps DUE	Read *Ages* pages 139–144 Take Home Midterm DUE
6 Geometry	Read *Ages* pages 163–166 *Fermat's Enigma* Ch. 5 DUE	Read *Ages* pages 113–118
7 Algebra	Read *Ages* 37–42 *Fermat's Enigma* Ch. 6 DUE	Read *Ages* pages 121–124 Group Presentation Outlines DUE
8 Algebra	Read *Ages* pages 127–130 Problem Set #2 DUE *Fermat's Enigma* Ch. 7 & Epilogue DUE	Read *Ages* pages 185–190
9 Algebra	*Fermat's Enigma* Timelines DUE	Group Presentations
10	Group Presentation Annotated Bibliography DUE Group Presentations	Final Exam in Class
Final		History of Mathematics Board Games Final Course Meeting

APPENDIX B

Course Assignments and Point Allocations

Attendance and participation	50 points	~8%
Problem Sets and reading assignments	150 points	25%
Your Mathematician biography	75 points	12.5%
Fermat's Enigma timeline	75 points	12.5%
Midterm and Final (75 points each)	150 points	25%
Final Group Presentation and Board Game	100 points	~17%
Course Total	600 points	

APPENDIX C
Fermat's Enigma Reading Questions

Your answers may be typed or neatly handwritten. Please use complete sentences and proofread your work in order to limit spelling, grammatical, and typographical errors. Your answers should reflect having *read* the chapter(s) and also having *thought* about the chapter(s).

Preface and Chapter 1

1. What is a perfect number? Defective? Excessive? What open problem hasn't been solved involving these numbers? Why are these numbers discussed? Who was interested in these numbers? Why?
2. Who discovered the principles of musical harmony? Describe the relationship between harmony and rational numbers.
3. What is the difference between mathematical proof and scientific proof?
4. State Fermat's Last Theorem. How is it different from Pythagoras' theorem?

Chapter 2

1. The Birthday Problem: The "birthday problem" is an example of the counterintuitive nature of much of a particular branch of mathematics. What branch of mathematics? Who worked in this area? Why is the birthday problem counterintuitive?
2. Describe Fermat's contributions to the area of calculus. Were you aware of this?
3. What is number theory and why is it important?
4. What importance is attached to the proof by contradiction?
5. Prime numbers greater than 2 can be separated into two categories. Describe them. What special property do primes in the first category have? Who proved this?

Chapter 3

1. Who was Monsieur Antoine-August LeBlanc? Describe the contributions made to the solution of Fermat's Last Theorem by Monsieur LeBlanc.
2. What is the Fundamental Theorem of Arithmetic? What is the Fundamental Theorem of Algebra? What is the Fundamental Theorem of Calculus? Which of these have you seen proven before? Do you understand each of these?

Note: not all of these are mentioned in *Fermat's Enigma*. You may have to do a little independent research to answer this question.

3. What did Lamé and Cauchy overlook in their pursuit of the proof of the theorem? Who pointed this out to them? What other contributions to mathematics did these three distinguished mathematicians make?
4. You have been completing a list of mathematicians for your timeline. Give the names and life spans for five of the people you have on your list so far.

Note: Not all of this information is in *Fermat's Enigma*. You may have to do a little bit of independent research to answer this question.

Ex: Pierre de Fermat 1601–1665

Chapter 4

1. How did Fermat's Last Theorem save a man from suicide? How did this man inspire others to attempt to prove the theorem?
2. What is the original "liar's paradox"? Why is it a paradox? What is Gödel's related statement and how is it related to the liar's paradox?
3. Describe the effects of World War II on the public perception of mathematics.
4. Why is Euler's conjecture a good example of why absolute truth is needed in mathematics?

Chapter 5

1. How did Frey and Ribet contribute to Wiles's work?
2. In Chapter 5, Singh introduces modular forms and comments on their high degree of symmetry. Then he discusses several examples of symmetry in mathematics. Comment on or explain at least one example of symmetry in a way that would be understandable to a child. Give or illustrate your example using a simple figure that is different from a square.
3. What was Shimura's philosophy of goodness and how did it affect his outlook on the purpose of mathematics?

Chapter 6

1. Describe Wiles's research process/strategy. Compare this to your own process/strategy when working on a difficult problem.
2. Prove the following using mathematical induction. (Studying Appendix 10 may help.)

   For all positive integers n, $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$.
3. You have been completing a list of mathematicians for your timeline. Give the names and life spans for ten more of the people you have on your list so far (these should all be different from the people you submitted with your Chapter 3 questions).

Note: Not all of this information is in *Fermat's Enigma*. You may have to do a little bit of independent research to answer this question.

Ex: Pierre de Fermat 1601–1665

Chapter 7 and Epilogue

1. State Fermat's Last Theorem. For how long did Andrew Wiles work on it? For how long have mathematicians worked on it? Why is it important?
2. Do you believe that Fermat really had a proof to his theorem? If so, do you think that Andrew Wiles's proof was the same as Fermat's proof?
3. There are two limericks in the epilogue (one by F. Gouvêa who is a co-author of your textbook for this class). Pick a mathematical topic and compose an original limerick about it. A limerick is five lines long where lines 1, 2, 5 rhyme with each other, and lines 3, 4 rhyme with each other.

   If you prefer, you may instead compose an original haiku about some mathematical topic.
4. Work on your timeline.

APPENDIX D

Fermat's Enigma Timeline Rubric

As you read *Fermat's Enigma*, make a note of mathematical ideas and mathematicians that are discussed in the book. You will create a timeline in a format and size of your choosing that includes the mathematics and mathematicians discussed in *Fermat's Enigma*.

Below is a general grading guideline. Additional points may be deducted for grave errors ("Leonhard Euclid") or added for outstanding achievements.

Professionalism and Neatness

Work is neat, clear, and easy to read; errors are minimal.	Work is not carefully presented; errors are somewhat distracting.	Work is sloppy and careless; errors are predominant.
15 points	8 points	0 points

Organizational structure

Chronology and organization of timeline is sensible/clear to an outside observer	Timeline is mostly easy to follow; some information appears out of a sensible sequence	Events/people not placed in a recognizable sequence/order; it is difficult to know what information goes together.
10 points	5 points	0 points

Thorough collection of mathematicians and mathematical ideas

including "Your Mathematician"

40–50 mathematicians and mathematical ideas listed/addressed	30–39 mathematicians and mathematical ideas listed/addressed	20–29 mathematicians and mathematical ideas listed/addressed	19 or fewer mathematicians and mathematical ideas listed/addressed
25 points	20 points	15 points	0 points

Informational presentation of mathematicians and mathematical ideas

Besides appearing in *Fermat's Enigma*, why do they appear on your timeline?

All or nearly all people and ideas are presented with pertinent information like dates, fields of study, country of birth, major contributions, etc.	Approximately half of people and ideas are presented with some form of pertinent information	Few or no names or ideas on your timeline have any associated information presented
25 points	15 points	0 points

APPENDIX E

Final Group Project Rubric

- 100 points total; all students in one group will receive the same score/grade.
- Please let your instructor know *immediately* if a group member is not cooperating or actively participating in the preparation of your Group Presentation.

Meeting Minutes — 1 due per group (5 points)

Group member names, presentation topic, and meeting date clearly listed . . . . 1 point

List, group member by group member, of what each person . . . . 4 points

1. has already done, and 2. will do next to prepare for the presentation

Outline — 1 due per group (10 points)

Group member names and topic clearly listed . . . . 1 points

At least four references listed . . . . 2 points

Professionalism . . . . 2 points

typed or neatly handwritten; free of spelling, grammatical, and typographical errors

Planning, progress, and collaboration

5 points available—your outline will be assessed points as follows:

* Clear indication of planning, progress towards final project, and collaboration . . . 5 points

* Outline of mathematics is there, but weak/no logistics, or vice versa. . . . . 3 points

* Reading your outline gives me no hope that you will be able to pull together a project within the next month—please see me! . . . . 0 points

Note: This is a preliminary outline that should serve the purpose of getting you started on the right track for your presentation. If some portion of the information on your outline changes between submitting your outline and your final presentation, that is okay.

Annotated Bibliography — 1 due per group (25 points)

Minimum of six references cited correctly using APA or other accepted citation style . . . 9 points

Descriptions of sources . . . . 12 points

1 paragraph (3–5 sentences) clear, thoughtful, summative description of each reference

Formatting and professionalism . . . . 4 points

free of spelling, grammatical, and typographical errors;

12 point font and 1″ margins, double spaced

APPENDIX F

Group Presentation Rubric

1 per group (40 points)

Preparation and collaboration . . . . 6 points

- Evidence of advanced planning, and rehearsal
- Timing is good; 18–22 minutes used—not too long or short
- Significant participation by all group members

Presentation and professionalism . . . . 6 points

- Audible voice level and eye contact with audience
- Appropriate dress, language, posture
- Overall comfort and confidence with material and presenting
- Fluent/smooth use of props, posters, technology, etc. (where applicable)

Organization & Flow . . . . 8 points

This is not a series of mini-presentations. This is one coherent group presentation.

- Presentation has a clear beginning/introduction, middle, and end/conclusions
- Group members transition smoothly between presenters and between ideas
- Material is presented in a way that is understandable, and would be of use/interest to a typical mathematics major or high school mathematics teacher

Content. . . . . 20 points

- Mathematical ideas presented are clearly understood by presenters
- Historical ideas related to the topic are presented clearly, and in a manner relevant to the topic
- Presentation provides a good balance of mathematical and historical ideas

APPENDIX G

Board Game Rubric

1 per group (20 points)

↓ Category Score →	5	4	2	1
Accuracy of Content	Math and math history content are accurate. All problem cards and game-related content made for the game are correct.	Math and math history content are mostly accurate. Game cards and game-related content contain minor errors.	Math and math history content are somewhat basic for intended audience. Cards and game-related content contain distracting errors.	Math and math history content are too basic for intended audience. Game cards have numerous mathematical errors.
Neatness and Organization; Professionalism	The game presented is neat, clear, organized in a fashion that is easy to read and follow.	The game presented is neat and organized in a fashion that is usually easy to read and follow.	The game is mostly neat, organized, and professional. It may be lacking some "polishing" elements.	The game appears sloppy and unorganized. It is hard to follow the direction of the game.
Game Directions/ Rules; Clarity and Accuracy	Rules are written clearly enough so that all players can easily participate. Rules are on a separate card from game board. No typos or errors.	Rules are written, but one part of the game needed slightly more explanation. Rules are separate from game board.	Rules are written, but people had some difficulty figuring out the game. Rules may be attached to back of game board.	The rules are not written, or contain so many errors that they are distracting.
Game is Complete and Visually Appealing	Game is made on file folder that is decorated inside and outside including all supplies for play. The game is ready to play.		Game is sloppy or haphazard. Game is missing more than one of the components needed to play.	Game is incomplete and unattractive.

A History of Mathematics Course Focused on Number, Operation, and Solving Equations, supplemented with Biographies and Videos

Diana White
University of Colorado Denver

Course Overview

This course focuses primarily on the history of mathematics prior to calculus, with the greatest emphasis on the development of number, operation, and solving equations. It is cross-listed as a mathematics course at the advanced undergraduate and beginning graduate levels. The prerequisite is a first course in calculus, and this course is not a prerequisite for any other courses. The course is required of students in the mathematics education track of the mathematics major, and an open elective for other mathematics majors and mathematics minors. Approximately two-thirds of the students enrolled in this course are pre-service or in-service teachers. All students complete the same assignments, though specific topics are sometimes left to student choice.

Course Goal

The course balances providing an overview of a vast subject with depth of coverage. The higher geometry course at our institution includes a significant amount of historical content, and the students in the mathematics education track of the mathematics major must also take that course. In addition, given the large number of pre-service teachers in the course, as well as the emphasis on number, operation, and algebra at the primary and secondary level, the course topics seem particularly pertinent to their future work.

The course has four primary goals: (a) increase students' mathematical content knowledge, (b) increase students' historical understanding and appreciation of mathematics as an evolving discipline, (c) increase students' communication skills and ability to give and receive feedback, and (d) increase students' ability to learn independently.

Goal 1. Mathematical content knowledge

Students increase their awareness of and skill at making connections between various areas of mathematics. For example, we study the ancient geometric methods for completing the square and how these have evolved into something often taught as solely an algebraic concept. Students increase their knowledge of historical methods and approaches to mathematics, with a particular focus on arithmetic and algebra. For example, students study how ancient civilizations computed the area of a circle and what approximation of π this would yield, as well as how a variety of societies presented and solved linear and quadratic equations.

Goal 2. Historical understanding and appreciation of mathematics as an evolving discipline

Students develop an understanding of the complex interplay between mathematics and the social milieu in which it developed (i.e., mathematics as a socio-cultural endeavor). As one example, we focus on how the Egyptians and Babylonians developed mathematics to meet societal needs. Students develop an appreciation for and understanding of the contributions of some prominent mathematicians. We also study numerous mathematicians, with a focus on their lives, the influence of their society on them and their work, and their accomplishments within that framework. Furthermore, students have an exposure to and develop an appreciation for working from primary historical sources. For example, we have used primary source materials on π from *Pi, A Source Book* [4], a primary source project on graph theory by Barnett [2], and primary sources on Pascal's triangle from the books by Smith and Struik [19, 21].

Goal 3. Communication and feedback

Students increase their skill at communicating mathematics orally and in writing according to disciplinary specific conventions, to include appropriate structure, notation, and vocabulary. We accomplish this via feedback on homework and verbally in class, as well as on small group presentations. Students develop the ability to provide meaningful feedback on the oral and written work of others, and to accept such feedback on their own work. They gain this experience by providing peer feedback on biography papers, as well as on small group presentations. Finally, in order to enhance their ability to work with others to understand mathematics, students regularly work in small groups at their seats and at the board on mathematical problems or on understanding mathematics from a text.

Goal 4. Independent learning

Students develop facility with aspects of reading a mathematical textbook, to include ensuring that students understand each provided step, recognize when details are omitted, and be able to supply them. They expand and refine their ability to find information on mathematical and historical topics, as well as to critically analyze historical sources. This is accomplished via a biography assignment that requires specific types of sources, such as the *Dictionary of Scientific Biography* [9] and through comparisons of different sources of information on Gauss, Galois, and Blackwell.

Course Design

The course meets twice a week for 15 weeks, for 45 hours of class meeting time. During the first half of the semester, we intersperse the biographical aspect of the course with the mathematical content. During the first 3–5 weeks, we spend approximately three weeks of class time studying ancient Egyptian and Babylonian number systems, as well as their arithmetic and algebraic methods. We use material from the texts by Burton [7] and Bunt [6]. Some semesters we have included material on Plimpton 322 from Robson [14, 15]. During this time, students also get started on their biography assignments, a 5–8 page paper on a mathematician of their choice. An in-depth description of this assignment can be found in a prior article by the author [22]. In preparation for these, we also complete the Galois and Blackwell assignments; see Appendix E.

During the next few weeks, we study quadrature of the lune and squaring the circle, as a prelude to learning about constructible, algebraic, and transcendental numbers. We follow the treatment in *Journey through Genius* [10] closely, which assumes some major results that are beyond the scope of the course, such as the fact that π is a transcendental number. Students work on substantive mathematical problems from the unofficial problem set that accompanies [10]. We have also included some original source material on π [3].

During the third quarter of the semester, we study the development of algebra, using the textbook by Burton [7] as well as [10]. Specifically, we spend several class periods on material from the *Arithmetica* [8], several on Cardano and solutions to cubic equations, and at least one on ancient Arabic methods such as completing the square.

During the final quarter of the semester, we study sizes of infinity from original source materials and sometimes from [10]. We have on occasion also studied the origins of graph theory from original source materials [2]. Students are also responsible for completing an individual book assignment, where they read an approved popular book on the history of mathematics and write a structured review.

Resources and Assignments

This course uses a combination of traditional texts, supplemental readings, and videos. The primary text is *A History of Mathematics*, by David Burton [7]. This is a traditional textbook with most sections having the general form of historical narrative interspersed with mathematical explanations, followed by a mathematical problem set for the student to complete. The secondary text is *Journey through Genius,* by William Dunham [10]. This book is written to be accessible to a popular audience and is quite lively, with historical information interweaved with mathematical explanations. While the students generally prefer Dunham's book to the more traditional text, we use both extensively.

Reading Assignments

To help focus students' reading in *A History of Mathematics* by David Burton [7], we use a list of reading prompts to guide and focus student reading toward our main course themes. Over time, we have also created reading prompts for other readings. Some of these prompts spur discussion in class, and other times students come to class prepared to discuss the prompts in small group "round-robins", partner presentations, or whole class discussions. At times the students' responses to these prompts are collected, with the students receiving full credit if they clearly had notes and had attempted each prompt thoroughly.

Mathematical Content Assessments

Students submit approximately six different sets of mathematical problems from the Burton text [7], a problem set that accompanies *Journey through Genius* [10], and occasionally from *The Historical Roots of Elementary Mathematics* [6] and *Trigonometric Delights* [13]. The problems focus on topics related to the main course themes of number, operation, and solving equations. The course has no written exams. Some semesters we have a few pre-announced five-minute definition quizzes at the beginning of class to ensure that the students have memorized a key definition related to the material for that day.

Galois and Blackwell Assignments

These assignments are used during the first third of the course, as a prelude to the biography assignment, details of which can be found in [22]. The goal of these assignments is for students to explore how different sources can vary widely in both factual information and tone, as well as to expose the students to a modern mathematician from an under-represented group. For each assignment, students have guiding questions that are found in Appendix E. Students respond in writing to these questions, share with a partner or small group during class, and then submit their work.

For the Galois assignment, students read a chapter on Galois from the well-known book *Men of Mathematics* [3], written for a popular audience. They compare that portrayal to information contained in "Genius and Biographers: The Fictionalization of Everiste Galois" [16], a scholarly article addressing the accuracy of various portrayals of Galois.

For the Blackwell assignment, half of the class reads an article [17] from the *Notices of the American Mathematical Society* composed of brief first-hand accounts of him as recollected by colleagues, collaborators, and former students. The other half reads a more standard biography, "Dr. David Harold Blackwell, African American Pioneer" [1], from *Mathematics Magazine.* In class, students share their findings with a partner who read the other article. After we discuss both articles as an entire class, I share information with them about the American Mathematical Society and the Mathematical Association of America.

Most semesters we have a similar assignment related to Gauss, based on the article and websites by Hayes [11, 12].

Videos Assignments

Throughout the course, students watch three to five videos outside of class. Each video provides an overview of the history of mathematics in a given area that aligns with our course goals. Some videos are professionally produced, for example the British Broadcasting Corporation (BBC) video by Marcus du Sautoy, *The Story of Maths, Volume 1* [18], which discusses ancient Babylonian, Egyptian, and Greek mathematics. Others are lectures, for example on polynomial equations and calculus, from a course at New South Wales University, Australia, by N. J. Wildberger [23]. His lectures are based on the book by Stillwell [20].

For the aforementioned videos, all students watch the same video and answer the same set of questions, which we in turn debrief during class. Specific examples of these assignments can be found in Appendices C and D. However, some semesters I have incorporated a written assignment that they submit individually that is based on a video of their choice from Wildberger [23]. Each student chooses a video presenting material that correlates with his or her mathematical experience level.

Lessons Learned

I was asked to teach a course on the history of mathematics when I was a first year tenure-track faculty member. I had never taught it before, and in fact had essentially no knowledge of the subject. However, I lacked the courage and wisdom at that time to convey this to my department chair and associate chair. I consulted with the faculty member who had taught it before me, and this was of some help. However, his syllabus and materials were not suitable for a novice junior faculty member who was completely new to the subject. In addition, I felt the book he suggested was too advanced, given the modest prerequisite of only a first course in calculus. After a few years, I switched to the text by Burton [7], which provided more structure and support for both the students and me. As a novice instructor, I wish that I had known the importance of a solid text that fit both my background knowledge and the mathematical maturity of my students. I would caution novice instructors that some of the most historically accurate and well-respected texts amongst history of mathematics scholars may not be the ones that are most readily used by a junior faculty member or by someone who is new to learning the history of mathematics.

The variety of student mathematical levels in the course continues to present a challenge. However, I have used it to the course's advantage by incorporating a good deal of peer learning. I do have to be careful to ensure that students with weaker or less experienced mathematical backgrounds do not let the others carry the load. To do so, techniques that I use include asking the least experienced students to write for the groups, by telling groups to have each group member explain their conclusions in turn, and by intentionally asking specific students to explain what they have learned to me in their small group setting.

History of mathematics is a vast subject, and a single course addresses but a snippet of what could be done. As such, each instructor has both the liberty and the responsibility of choosing a course that balances the student backgrounds, program needs, and instructor strengths and background in the discipline. My course has evolved considerably as my own knowledge and skill evolved, and it will continue to evolve over time.

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Appendix A

Course Outline

Week 1	Egyptian Number System and Arithmetic
Week 2	Egyptian Arithmetic and Algebra, Video 1
Week 3	Babylonian Number System and Arithmetic, Galois Assignment
Week 4	Babylonian Arithmetic and Algebra, Blackwell Assignment
Week 5	Greeks, Archimedes and Estimating Pi, Biography Assignment Due
Week 6	Greeks, Video 2
Week 7	Quadrature, Algebraic and Transcendental Numbers
Week 8	Quadrature, Algebraic and Transcendental Numbers
Week 9	Algebra, Diophantus
Week 10	Algebra, Cardano Assignment
Week 11	Algebra, Video 3
Week 12	Original Source I, Euler and the Konigsberg Bridge
Week 13	Original Source II, Sizes of Infinity
Week 14	Original Source II, Sizes of Infinity
Week 15	Final Exam Week

Appendix B

Cardano Assignment

If I had to select one reading for this course on solving equations, I would select this one. This assignment requires students to work through Chapter 6 of *Journey through Genius* in detail, filling in missing steps of the mathematics, and writing up a nice explanation to each of the guidelines below. Students read and respond to the guidance and questions provided below, having time to start the work during class, and then submit a write-up after they complete the work outside of class.

a. A Horatio Algebra Story: From p. 133–142 (up to where the section "Great Theorem" begins), your goal is to summarize the basics of the key players and their roles and inter-relationships with regard to the solution of the depressed cubic. Go into more detail on Cardano, as he is the primary focus of the chapter. Ensure you can define a depressed cubic (good quiz question), as well as discuss the role of public challenges in mathematics.

b. Great Theorem — p. 142–146. Your task is to work thought this in detail, practicing in depth the skill of reading and thinking deeply about the mathematics in a text.
   i. Translate the paragraph after the theorem into modern algebraic symbols and match it up with the equation on the bottom of p. 145 (at the end of the proof of the theorem).
   ii. Ensure you see where each of the volumes of the decomposed cube comes from. Describe this in your own words, perhaps using a diagram and pointing to the various parts.
   iii. Work out on your own the algebraic steps at the top of p. 144.
   iv. Figure out how you can derive formula (*) on p. 144 geometrically from the cube. That is, figure out how that formula represent appropriate volumes. Explain this.
   v. Pay attention to how equation (*) represents the equation of a depressed cubic, and hence the rationale for the substitution in the middle of p. 144. Since m and n are fixed, this becomes an equation of two equations and two unknowns (but not linear equations!) Notice how solving for one of the variables and plugging it into the other equation gives a sixth order equation. How does this help us?
   vi. Follow the rest of the algebra to solve for t and hence u and then x. Why was the negative square root discarded? Need it be discarded?
   vii. Be able to explain what is meant by "solution by radicals" or "algebraic solution" (good quiz question).
   viii. Notice that the overall method here was to lower the degree of the equation that needed solving from a cubic to a quadratic, and then to use that information to solve the cubic. This is a specific example of a technique commonly used by mathematicians, both computationally and in proofs; namely, to modify the problem to one for which you already have the solution/knowledge, and then use that to extrapolate back to the original problem to get a solution/proof.

c. Further topics on solving equations:
   i. Explain why it is "easy" to find the rest of the solutions to a cubic equation after one is found.
   ii. Explain why Chapters 12 and 13 of *Ars Magna* are superfluous to us today, as well as why they were needed then. Ensure you connect your reason with the geometric perspective of the time.
   iii. Work through the algebra of depressing a general second-degree equation and understand how this is a way to derive the quadratic formula.

  - iv. Work through the algebra of depressing a cubic. Notice that the substitution is very carefully chosen to get rid of the x^2 term.
  - v. Explain why complex numbers were developed in response to solving cubic equations as opposed to quadratic equations.
  - vi. Describe the two main steps of solving a general quartic.
- d. Epilogue:
  - i. Note that a general quintic (5th degree) equation cannot be solved by radicals. What parts of the procedures for solving the cubic and quartic equations do work?
  - ii. Look up (if you do not recall it) the intermediate value theorem and be able to explain why it guarantees that a quintic must have a real solution.

Final piece of assignment: Make up your general cubic and solve it by following the steps of the proof in the theorem. This should involve, first, depressing it, then using the Cardano procedure step-by step. That is, you are not being asked to just plug in the appropriate information into his formula; rather, you're being asked to go back through the entire derivation (depressing it and solving it), in the case of your specific example. Hint: you need to make sure that Cardano's procedure, as modified slightly in the book, will work for your cubic. If you accidentally get the square root of a negative number at any point, then you need to choose another cubic.

Appendix C

Video Assignment: The Story of Maths

This assignment is based on 40 minutes of the 4-hour "Story of Maths" video collection, a BBC Documentary by Marcus du Sautoy. Students generally complete it within the first two weeks of the course, and it focuses on ancient Egyptian and Babylonian mathematics. The assignment questions, which students submit written responses to, are:

1. Discuss the highlights of the following mathematical ideas that arose in the Egyptian part of the video: number system, bodies as a basis for measurement, Rhind Papyrus, fractions, geometric series, pi, golden ratio, Pythagorean Theorem/Right angle triangles, Moscow Papyrus, and calculus.
2. Discuss the highlights of the following mathematical ideas that arose in the Babylonian part of the video: number system, right angle triangles, quadratic equations, Plimpton 322.
3. What are some similarities and differences between Egyptian and Babylonian mathematics as presented in the video? Note: this part of the assignment is asking you to synthesize information, so take some time to think deeply here.
    - Number system
    - Algebra
    - Proof
    - Why geometry arose
    - Types of documents that remain
    - Impact on modern day mathematics
    - Right angle triangle

Respond to the following questions in more detail, using full sentences, paragraphs, etc.

4. Summarize this video, or key parts of it, in 3–5 sentences to a friend or family member who is college educated but not a math or science major.
5. What are 2–4 things that you learned from this video that you found particularly interesting or surprising (and why)?
6. Connect this video to at least two things that we've done in class.
7. Connect this video to something you have learned in another class. If possible, connect it to both a math class and a non-math class.
8. What are 1–2 questions that this video evokes?

Appendix D

Video Assignment: Polynomial Equations

This video is from a recorded history of mathematics class, based on [20] taught by N. J. Wildberger at New South Wales University, Australia. The assignment questions, which students submit written responses to, are as follows:

1. Discuss some challenges to the development of mathematics during this timeframe.
2. Discuss some influences that the Renaissance had on the study of mathematics.
3. When/who introduced some of the key symbols in mathematics? Make a list for yourself—he gives a nice overview.
4. What key advances did Viete make to mathematics?
5. What was the common way in which quadratic equations arose? What quadratic does this give rise to? How does this relate to Cardano?
6. Write your own example of completing the square, as al-Khwarizmi would have done it. Explain your work.
7. Explain how the Babylonians would find a square root. Try to illustrate with a specific example. (Connect to Newton's method!)
8. Explain how the Greeks would calculate a square root using geometric methods.
9. Discuss some limitations involving straightedge and compass constructions that arose in the video.
10. You can skip the first 6 minutes of Part 6b if you'd like.
11. Explain Vete's approach to solving a cubic.

Appendix E

Galois and Blackwell Assignment Questions

Students are required to submit typed answers to these questions, and each response should be at least a paragraph. In class, students debrief these questions in small groups, often using a round-robin format, with a whole class debrief to follow.

Questions from Bell's book [3]

1. Describe the overall tone of this chapter.
2. What are the main emphases of this chapter?
3. How does this make you feel emotionally as you read it?
4. What questions does this article leave you pondering?

Questions from Rothman's paper [16]

1. How does the style of this article (way in which it was written) differ from the chapter from E. T. Bell?
2. What do the goals of this paper seem to be? How does that compare with the goals of the chapter from E. T. Bell?
3. Describe at least three ways in which Bell seems to take the liberty to omit or modify relevant information. How does each of these change your perspective on Galois?
4. What are the sources on which Rothman relies for his article? Briefly describe each. Does he depend more on some than others? If so, any thoughts on why?
5. How does this paper make you feel emotionally? How does that compare to the other Galois article?
6. What did you gain from reading this paper?
7. What questions does this article leave you pondering?
8. Do you think that the literary license that Bell took is acceptable? Why or why not?

Questions from Blackwell articles [1, 17]:

1. What was the style of your article? What approach did it take to conveying information about Blackwell?
2. What were some key milestones professionally in Blackwell's life?
3. What were some key personal events in his life?
4. What is your sense of Blackwell as a person?
5. What were some of the obstacles and challenges that he had to overcome?
6. What influence did Blackwell seem to have on others?
7. What questions does this article leave you pondering?
8. What have you learned from this assignment that will help you write your own biography?
9. What are two other questions that could have been on this list, but were not?

Appendix F

Sample Reading Prompts

Sample Reading Guidelines—Burton Chapter 1

1.1: Be able to explain several historical methods of counting, as well as how and why civilizations needed these methods.

Be able to discuss ancient need for the concept of number.

1.2: Describe how the historical context of Egypt led to the development of the Egyptian number system.

Describe the impact of papyrus on Egyptian hieratic numeration.

Describe multiplication in Greek alphabetic numerals.

Review Roman numerals as needed—be able to convert to/from base 10 to Roman numerals for the integers between 1 and 3999.

1.3: Describe cuneiform and how it impacted Babylonian writing
Describe the Babylonian positional number system, including why it is not a true place value system.

Sample Reading Guidelines—Burton Chapter 2

2.1:
1. Describe the impact of Napolean's invasion of 1798 on our knowledge of ancient Egyptian mathematics.
2. Describe the Rhind Papyrus, the Rosetta Stone, and their impact on our knowledge of ancient Egyptian mathematics.

2.2:
1. Be able to explain and apply the "double and add" method for multiplication.
2. Be able to explain the Egyptian method of division.
3. Describe some key features of fraction decompositions in unit fraction tables.
4. Be able to explain and apply the splitting method and Fibonacci's method for decomposing a positive rational number.

2.3:
1. Describe and apply the method of false position.
2. Describe and apply the method of double false position

III

General Education Courses

A Mathematics History Course for Liberal Arts Students

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Course Overview

Mathematics Through Its History is an activity-based elementary mathematics history course intended primarily for non-science students at my small, comprehensive, liberal arts-focused university. The title of the course is intended to carry two meanings: we attempt to study mathematics throughout its (early) history and we study mathematics by approaching it via its early history. The course focuses on the historical development of counting, number systems, arithmetic, geometry, and algebra—that is, school mathematics—and also includes a few more modern topics. Mathematics Through Its History is usually offered during our 3 ½-week May Term, when students take just one course. This gives us lengthy class periods for activities and explorations. However, it has also been taught during our regular 13-week semester. Another special feature of the course is that I originally designed and taught it with mathematics majors, the majority of whom intended to become mathematics teachers, and mathematics majors continue to participate in the course as "co-teachers," tutors, and graders. Both my university's May Term and my use of co-teachers will be discussed in more detail below.

The prerequisite for Mathematics Through Its History is placement (via exam or SAT/ACT scores) into the lowest level courses students may take to fulfill my university's mathematics requirement. Students with a good understanding of intermediate algebra can usually achieve this placement. University general education requirements include a mathematics course, a laboratory-based science course, and a third course in mathematics or science. Mathematics Through Its History satisfies this third requirement and is selected by students from a wide range of disciplines who prefer mathematics to science and/or who have an interest in history and culture. Most students in the course also take a finite mathematics course, either before or after they take Mathematics Through Its History.

One audience the course has not attracted is future elementary school teachers, who have their own two-course sequence of mathematics courses at my university. However, the idea of using the history of what is now school mathematics—number systems, arithmetic, geometry, and algebra—in general education mathematics courses certainly could and should be extended to courses for future (and in-service) teachers. Students in the Mathematics Through Its History class invariably reflect on their own elementary and secondary mathematics educations and often suggest themselves that elementary school teachers especially would benefit from seeing mathematics through the lens of its history. Future secondary mathematics teachers at my university have an opportunity to take a higher-level mathematics history course (see the article, "A

Combined Number Theory and Mathematics History Course for Mathematics Majors and Minors," in this volume) and to participate in the present course as teaching assistants.

Mathematics Through Its History has been popular with students, who seem to find it challenging but enjoyable. The course has been even more rewarding for my mathematics major co-teachers, several of whom have gone on to teach high school mathematics or to serve as graduate teaching assistants in mathematics masters and PhD programs. One of them went on to design and teach a mathematics history course for very bright fifth and sixth graders in the Johns Hopkins Center for Talented Youth summer program. Some have even returned to campus as guest presenters in the course after graduating from the University of Redlands.

Course Design

As noted above, Mathematics Through Its History covers the historical development of counting, number systems, arithmetic, geometry, and algebra—that is, school mathematics—along with some additional fun and accessible topics, such as Fibonacci numbers, perspective drawing, Pascal's Triangle, Euler's Formula, the Königsberg Bridge Problem, and the Four Color Theorem. It has followed a different schedule and included different topics each time it has been offered, but is organized roughly into units on:

- counting and number systems in various (primarily early) civilizations
- algorithms for elementary arithmetic from various times and places
- mathematics of ancient Egypt, Mesopotamia, and China
- the Pythagorean Theorem in various cultures
- mathematics of ancient Greece
- circles and π in various cultures
- newer developments (medieval to modern).

A more detailed outline appears below as Appendix A.

Besides the usual important objectives of students becoming better thinkers, problem-solvers, readers, writers, and speakers, my goals for students in the course include:

- understanding and appreciating elementary mathematics more deeply
- believing, based on the evidence of human experience, that there are many ways to solve mathematical problems and develop mathematical ideas and methods
- seeing how mathematics influences culture and how culture shapes the development and practice of mathematics
- recognizing mathematics as an important human activity motivated both by curiosity and by practical application.

I have never in another mathematics or mathematics history course seen students achieve the second outcome so successfully. We try to teach students at all levels that there are multiple ways to frame concepts and solve problems, with limited success. Studying a variety of real-life historical and/or cultural approaches to a mathematical topic with which students already feel quite familiar seems to be more convincing than presenting multiple approaches with little or no historical or cultural context, especially to a topic new to students.

Although Mathematics Through Its History has been taught during a traditional academic semester, I much prefer to offer it during our one-course May Term, which means it meets for nearly 3 hours per day (including breaks), 4 days per week during a 3 ½-week term. Class sessions nearly always begin with worksheet activities, followed by a combination of interactive lectures and hands-on activities, such as learning to use an abacus, learning to use a Chinese counting board, exploring Pythagorean figurate numbers with "pebbles" (candy), doing puzzle proofs of the Pythagorean Theorem, doing straightedge and compass constructions, and constructing the Platonic solids to discover Euler's formula. In general, activities are designed so that students discover results rather than simply review or practice them. Students also have nightly reading and homework. During the early years of the course, I gave short daily quizzes with a few "big quizzes" and a "final quiz," but I have since dispensed with the daily quizzes. The main purpose of the daily quizzes was to make sure students remained engaged and did not fall behind. However, I have found that, especially during the May Term, it is difficult for students *not* to be engaged with the course! Grades are based on quizzes and exams (50%), daily homework assignments (25%), and in-class activities and worksheets (25%).

As noted above, class sessions usually consist of a combination of interactive lectures, worksheet activities, and hands-on activities. There are far too many interesting topics and activities for just one course. As a result, I have probably not used exactly the same set twice in 16 years. However, the course has generally followed the outline given in the preceding section, with "newer developments" nearly always including mathematics of medieval India, Islam, and Europe (mathematics of the Maya, Inca, and Aztecs having been studied earlier in the course), together with a few highlights of later European mathematics.

A more detailed outline of a 3-credit class offered during our 14-day May Term, meeting for 2 ½ hours per class session (not counting breaks) appears below as Appendix A. This schedule has also been used for a 3-credit class offered during our 13-week semester, meeting for a total of 2 hours and 40 minutes per week, where one class day during May Term is roughly equivalent to one week during the semester.

Ideally, student "co-teachers" for the course would be trained at least a semester ahead of the offering of Mathematics Through Its History with which they will assist. In reality, they often learn material "just in time" to present it to the class and/or help students with homework. The majority of my co-teachers have been mathematics majors who are prospective high school mathematics teachers; nearly all have studied mathematics history with me previously in a sophomore-level mathematics history course for mathematics majors and minors. In fact, I initially designed and prepared the course with a team of mathematics majors during summer sessions for which the students were paid and during academic year sessions for which students earned academic credit. Pairs of students from the design team then presented individual units of their choosing during the first offering of the course, and mathematics majors continue to do so today.

Resources and Assignments

As noted above, students are assigned reading and homework exercises corresponding to each class day. The reading comes from a variety of sources and is provided to students on electronic reserve through our university's library. Once I have identified desired sources, the library staff obtains copyright permissions and pays any fees (within reason). Many in-class worksheets and activities are also provided electronically. My favorite in-class worksheets are those that allow students to discover and explore ideas, including several by Claudia Zaslavsky [4, 5] and Beatrice Lumpkin [2]. For reading, I like to expose students to a variety of authors and sources, and I strive to make reading selections that are both accessible and thought-provoking. For homework problems, especially more mathematical homework problems, I am partial to the two *Historical Modules for the Teaching and Learning of Mathematics* [1] to which I contributed, *Negative Numbers* and *Lengths, Areas, and Volumes.* (For additional resources, see Appendix C, below.)

I write many of the homework problems myself. These include a fair number of short-answer questions about the reading, designed to help ensure that students read carefully, along with a few questions about historical issues that require longer answers. However, most homework problems require students to practice using historical mathematical methods, albeit almost invariably with a modern twist. For a selection of problems of the latter type, see Appendix B.

One assignment consists of writing one's "mathematics autobiography." This is an assignment adapted from the popular book, *Overcoming Math Anxiety,* by Sheila Tobias [3], in which students are to relate their experiences with and attitudes toward mathematics at home, school, work, and play throughout their lives. The experiences reported by women students, especially, have improved dramatically over the years I have been teaching the course.

Lessons Learned

In teaching a general education course like Mathematics Through Its History, the instructor has the pleasure of picking and choosing among many interesting and workable options for class activities and for outside-of-class reading and exercises. I change the activities and topics with nearly every course offering, according to my interests and those of my student assistants. One must be realistic about the number of topics chosen. A constant caveat for teaching a course like this one is that the students who populate it work at different speeds, some very slowly and some very quickly. Allowing students to discover results for themselves and developing understanding rather than settling for rote manipulation takes a lot of time for most students. However, it always pays to have extra challenges in mind or on hand for the quicker students in the class.

A 4-credit course with increased contact time could provide deeper study of existing topics or it could include a greater variety of topics and activities. (Ideas for additional topics appear at the end of Appendix A.) A 4-credit course, especially one in which contact time is not increased, might include individual or group projects, culminating in papers and/or presentations.

Finally, some of the worksheets and activities I use in the course are based on those of Mary Harrigan, who wrote and collected them for her course, The Mathematical Experience, offered at Nazareth College in Rochester, New York. Harrigan shared her course materials with me during the National Science Foundation funded MAA Institute in the History of Mathematics and Its Use in Teaching, in which we both participated as members of the 1996-97 cohort.

References

1. Katz, Victor, and Karen Dee Michalowicz (eds.), *Historical Modules for the Teaching and Learning of Mathematics,* MAA, 2005.
2. Lumpkin, Beatrice, *Algebra Activities from Many Cultures*, J. Weston Walch, 1997.
3. Tobias, Sheila, *Overcoming Math Anxiety,* W. W. Norton, 1978; updated paperback version, 1995.
4. Zaslavsky, Claudia, *Multicultural Math: Hands-on Math Activities from Around the World,* Scholastic Books, 1994.
5. Zaslavsky, Claudia, *Multicultural Mathematics: Interdisciplinary Cooperative-Learning Activities,* J. Weston Walch, 1993.

Appendix A

Course Outline

An outline of a 3-credit class offered during our 14-day May Term, meeting for 2 hours and 50 minutes (about 2 ½ hours, excluding breaks) per class session follows. This schedule has also been used for a 3-credit class offered during our 13-week semester, meeting for a total of 2 hours and 40 minutes per week, where one week during the semester is roughly equivalent to one class day during the May Term.

Day 1 (May Term) or **Week 1** (semester)
  Number systems from various cultures to include ancient Egyptian, ancient Chinese, and ancient Mesopotamian numbers, along with the later Maya civilization: bases 10, 20, and 60

Day 2 or **Week 2** (etc.)
  Simple, multiplicative, and positional number systems
  Addition and subtraction in various cultures to include the Chinese counting board and abacus

Day 3
  Arithmetic algorithms from various cultures, ancient to modern, with focus on addition, subtraction, and multiplication

Day 4
  First 50-point quiz
  Early river civilizations (Egypt, Mesopotamia, China, India)
  Ancient Egyptian mathematics: multiplication and division by doubling, unit fractions including division of loaves (or rounds) of bread, all from the Ahmes Papyrus

Day 5
  Ancient Mesopotamian mathematics, to include deciphering a Mesopotamian tablet and discovering Mesopotamian fractions
  Introduction to ancient Greek mathematics

Day 6
  More Greek mathematics: Thales' shadow measurement and Pythagorean pebble arithmetic (triangular, oblong, and square numbers)

Day 7
  Pythagorean Theorem: puzzle proofs, applications in various cultures
  Video: "Pythagorean Theorem"

Day 8
  Second 50-point quiz
  Greek geometry from Euclid's *Elements:* axiomatic system, construction with compass and straightedge, triangle properties and congruences

Day 9
  Golden Ratio and Fibonacci numbers
  Circles and π: estimates of π

Day 10
  More Greek mathematics: geometry and "geometric algebra"
  Circles and π: area of circle via pebbles, pizza slices, rope
  Video: "Story of π"

Day 11

Spheres and cylinders, including Archimedes' favorite theorem about them
Eratosthenes' prime sieve and measurement of the earth

Day 12

Third 50-point quiz
Mathematics of ancient and medieval India
More geometric algebra: al-Khwarizmi and completing the square
Medieval Islamic mathematics: negative numbers, algebra
Omar Khayyam's poetry and mathematics

Day 13

Topology: Platonic solids and Euler's formula, map coloring, Möbius strips

Day 14

Review for final examination (group presentations of review problems)
Video: "Early History of Mathematics"

Day 15

Final examination (100 points)

Alternate topics

Topics in addition to or instead of those listed above have included volumes of Egyptian pyramids and truncated pyramids, theories on the development of numbers and of writing via ancient Mesopotamian clay tokens, the method of false position, more about the development of algebraic notation, Maya calendars, magic squares, Pascal's Triangle, perspective drawing, early North American arithmetic texts, the Königsberg Bridge Problem, networks and Euler paths and circuits, knot theory, the Division Algorithm and identification numbers, cryptography, and women in mathematics. Future topics might include recreating Mesopotamian tablets with clay or play-doh, symmetries and tiling in various cultures to include Moorish Spain, Japanese origami, probability in early Native American games, and voting methods.

Appendix B

A Selection of Mathematical Homework Problems

Day 1

In *Africa Counts,* Claudia Zaslavsky wrote that the markings on the Ishango bone comprise a period of almost six (modern) months. What evidence did she have for this assertion? (*Hint:* How many marks are in each of the three columns? What is the total number of marks?)

Show how to represent the number 692 using Mesopotamian, Egyptian, Chinese, Maya, and base 5 number systems.

Day 2

Show how to use the Chinese counting board to compute 955 – 729.

Show how to use the Chinese abacus to compute 955 – 729.

Day 3

Show how to use the method of equal additions to compute 955 – 729.

Day 4

Use Egyptian doubling to compute (a) 18 × 25, (b) 26 × 33, (c) 21 × 85, and (d) 59 × 105.

Use Egyptian division to divide 174 by 6, and to divide 124 by 16.

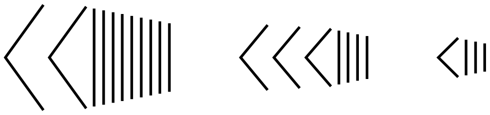
Write the fractions 3/10, 7/10, and 9/10 as sums of distinct unit fractions and, if you like, the fraction 2/3. Show how to use unit fractions to divide 3 round loaves of bread equally and fairly among 10 persons.

Day 5

Convert the Mesopotamian number shown below to an Indo-Arabic number (one of our numbers) by interpreting it in two different ways,

(a) assuming the place on the right is the ones place, and

(b) assuming the place on the right is the 1/3600 place.

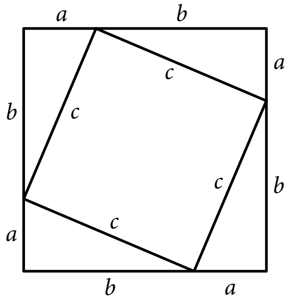


Day 6

How tall is a vertical column that casts a 32-cubit long shadow at the same time as Thales' 3-cubit tall (vertical) staff casts a 4-cubit long shadow?

Day 7

The following diagram shows a square within a square. Compute the area of the large square in two different ways, and show algebraically that the resulting equation implies the Pythagorean Theorem. (A version of this proof appears in the ancient Chinese text, *Arithmetical Classic of the Gnomon and the Circular Paths of Heaven*.)



If a rope tied to the top of a (vertical) flag pole is 3 *chi* longer than the pole itself and if the end of the rope, when pulled taut, just touches the ground 8 *chi* from the base of the pole, what is the length of the rope? (A version of this problem appeared in the Chinese text, *Nine Chapters on the Mathematical Art*.)

Day 8

In which proposition did Euclid prove that the sum of the angles in any triangle is 180 degrees? Explain.

Which proposition guarantees that if a triangle has sides of lengths 5, 12, and 13, then the triangle must be a right triangle, Proposition I.47 or Proposition I.48? Explain.

Day 9

What value for π did the Egyptian scribe Ahmes use (implicitly) in his circle area formula, $A = \left(\frac{8}{9}D\right)^2$, where A is the area and D is the diameter of the circle (in modern notation)?

Day 10

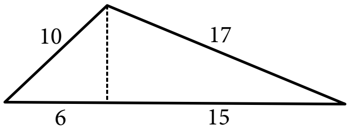
Draw a picture illustrating the algebraic identity $a(b+c) = ab + ac$.

Day 11

Compute the area of the triangle shown at right in two different ways:

(a) using Heron's formula, and

(b) using the usual formula (and the Pythagorean Theorem!).



Day 12

Solve for x in the equation $x^2 + 8x = 20$ by completing the square in three different ways:

1. Arithmetically, by rewriting al-Khwarizmi's written instructions. You should get only <u>one</u> solution.
2. Geometrically, including a picture. You should get only *one* solution.
3. Algebraically. You should get *two* solutions, $x = -10$ and $x = 2$.

Appendix C
Resources

Sources for classroom activities, worksheets, images, and videos:

Apostol, Tom, *Project Mathematics!* videos, California Institute of Technology, including:

- Early History of Mathematics (2000)
- Theorem of Pythagoras (1988)
- Story of π (1989).

Katz, Victor, and Karen Dee Michalowicz (eds.), *Historical Modules for the Teaching and Learning of Mathematics,* MAA, 2005. Activities and worksheets from the module *Negative Numbers* include:

- Chinese Shang (or Stick or Rod) Numerals
- Chinese Counting Board
- Problems from India
- Al-Khwarizmi's Negative Numbers.

Activities and worksheets from the module *Lengths, Areas, and Volumes* include:

- The Greatest Egyptian Pyramid
- Queen Dido (and her isoperimetric problem)
- Eratosthenes' Size of the Earth
- Derivations of Circle Area Formulas.

Life by the Numbers television series, PBS, 1998. See especially the short segment on the influence of perspective drawing from Episode 1.

Lumpkin, Beatrice, *Algebra Activities from Many Cultures*, J. Weston Walch, 1997, especially early number systems of Egypt, Mesopotamia, China, India, and Central America.

MAA Convergence: www.maa.org/press/periodicals/convergence. See especially:

- Caulfield, Michael J., "John Napier: His Life, His Logs, and His Bones," vol. 3 (2006)
- Diamantopoulos, John, and Cynthia Woodburn, "Maya Geometry in the Classroom," vol. 10 (August 2013)
- Molinsky, Michael, "Some Original Sources for Modern Tales of Thales," vol. 12 (November 2015)
- Shell-Gellasch, Amy, and Pedro Freitas, "When a Number System Loses Uniqueness: The Case of the Maya," vol. 9 (June 2012)
- Swetz, Frank, "Mathematical Treasures," to include the following "Anonymous" entries: "Cambodian (Khmer) zero," "Inca quipus," "Ishango Bone," "Mesopotamian accounting tokens," "Old Babylonian area calculation," "Old Babylonian tablet (The Best Known Old Babylonian Tablet)," "Tomb of Menna," and more! See also "Leonardo da Vinci's Geometric Sketches" of Platonic solids.
- Wessman-Enzinger, Nicole, "An Investigation of Subtraction Algorithms from the 18th and 19th Centuries," vol. 11 (January 2014).

Record[e], Robert, *Whetstone of Witte,* London, facsimile of 1557 edition: Da Capo Press, 1969; or TGR Renascent Books, 2010, paperback 2013.

Record[e], Robert, *The Pathway to Knowledge, Containing the First Principles of Geometrie,* London, facsimile of 1551 edition: Walter J. Johnson, Inc., 1974; or TGR Renascent Books, 2009, paperback 2013.

Robins, Gay, and Charles Shute, *The Rhind Mathematical Papyrus: an ancient Egyptian text,* British Museum Press, 1987.

Zaslavsky, Claudia, *Multicultural Math: Hands-on Math Activities from Around the World,* Scholastic Books, 1994. Warm-ups for number systems and arithmetic from various cultures, to include calendars and magic squares.

Zaslavsky, Claudia, *Multicultural Mathematics: Interdisciplinary Cooperative-Learning Activities,* J. Weston Walch, 1993, especially number systems from various cultures, symmetry, magic squares, and networks.

Sources for outside-of-class reading and homework

Burton, David, *The History of Mathematics: An Introduction* (7th ed.), McGraw-Hill, 2010. Selected reading and problems from the first few chapters on mathematics in early civilizations.

Dunham, William, *Journey Through Genius: The Great Theorems of Mathematics,* Wiley, 1990; also available from Penguin Paperbacks, 1991, especially sections on Euclid's *Elements,* Archimedes, Eratosthenes, and the conflict between Cardano and Tartaglia.

Historical Notes: Mathematics Through the Ages, Consortium for Mathematics and Its Applications (COMAP), 1992. Contains 27 short, readable articles, such as "Learning the Ropes: The Origins of Geometry," by Joseph W. Dauben.

Jacobs, Harold R., *Mathematics: A Human Endeavor* (3rd ed.), W. H. Freeman, 1994. See especially the explorations of the Fibonacci sequence and of Euler paths and circuits, which I've used as homework assignments. See also the Mesopotamian numeration puzzle on pages 2–3, which I've used as a classroom activity.

Johnson, David, and Thomas Mowry, *Mathematics: A Practical Odyssey* (8th ed.), Cengage Learning, 2016. See especially the sections on "Egyptian Geometry" and "The Greeks."

Joyce, David E., Euclid's *Elements:* aleph0.clarku.edu/~djoyce/java/elements/elements.html.

Katz, Victor, and Karen Dee Michalowicz (eds.), *Historical Modules for the Teaching and Learning of Mathematics,* MAA, 2005. Reading and homework from the module *Lengths, Areas, and Volumes* includes:

- Origins of Length, Area, and Volume Measurement
- Lengths, Areas, and Volumes in Greece
- Thales' Shadow Measurement
- Euclid's Geometric Algebra
- Pythagorean Theorem
- Practical Pythagorean Theorem
- Al-Khwarizmi's Completing the Square
- Ratio of Circumference to Diameter
- Ancient Circle Area Formulas.

Ore, Oystein, *Number Theory and Its History,* Dover Publications, 1976; original publication: McGraw-Hill, 1948. I especially like the first chapter, "Counting and Recording of Numbers," at least in part because students can be led to see that the chapter itself is a historical document.

Perl, Teri, *Math Equals,* Addison-Wesley, 1978, most often the chapter on Hypatia, but also Maria Gaetana Agnesi, Sophie Germain, and Sonya Kovalevskaya.

Philipp, Randolph, "Multicultural Mathematics and Alternative Algorithms," *Teaching Children Mathematics* 3:3 (Nov. 1996), pp. 128–133.

Swetz, Frank (editor), *From Five Fingers to Infinity,* Open Court, 1994, an out-of-print anthology of articles from MAA journals and other sources, which contains such gems as "Oneness, Twoness, Threeness: How Ancient Accountants Invented Numbers," by Denise Schmandt-Besserat. Other relevant articles from this anthology have been reprinted in Frank Swetz (editor), *The European Mathematical Awakening: A Journey Through the History of Mathematics, 1000–1800,* Dover Publications, 2013.

Tobias, Sheila, *Overcoming Math Anxiety,* W. W. Norton, 1978; updated paperback version, 1995. As noted above, I have adapted a "mathematics autobiography" assignment from this book.

Zaslavsky, Claudia, *Africa Counts: Number and Pattern in African Culture* (3rd ed.), Lawrence Hill Books, 1999; originally published, 1973. See especially Section 1.2, Historical Background, and Section 4, Mathematical Recreations.

Potential textbooks for the course

Avelsgaard, Carol, *History of Mathematics for Liberal Arts Students,* to appear in *MAA Convergence,* 2018 (prerequisite: Intermediate Algebra).

Berlinghoff, William, and Fernando Gouvêa, *Math through the Ages: A Gentle History for Teachers and Others* (Expanded 2nd Edition), MAA and Oxton House, 2015.

Shell-Gellasch, Amy, and J. B. Thoo, *Algebra in Context: From Origins to Application*, Johns Hopkins University Press, 2015.

Additional resources for instructors (and interested students)

Joseph, George, *The Crest of the Peacock: Non-European Roots of Mathematics* (3rd ed.), Princeton, 2011.

Katz, Victor, *A History of Mathematics: An Introduction* (3rd ed.), Addison-Wesley/Pearson, 2009.

Katz, Victor (ed.), *Mathematics of Egypt, Mesopotamia, China, India, and Islam: A Source Book,* Princeton University Press, 2007.

Katz, Victor, *et al.* (eds.), *Sourcebook in the Mathematics of Medieval Europe and North Africa,* Princeton University Press, 2016.

O'Connor, John J., and Edmund F. Robertson, *MacTutor History of Mathematics Archive:* www-history.mcs.st-and.ac.uk/.

Robson, Eleanor, "Words and Pictures: New Light on Plimpton 322," *American Mathematical Monthly* 109 (February 2002), pp. 105–120.

Stedall, Jacqueline, *Mathematics Emerging: A Sourcebook 1540–1900,* Oxford University Press, 2008.

Materials: Toothpicks or strips of fairly sturdy paper of two different colors can be used for counting sticks for the Chinese counting board; "Smarties" or other small round candies can be used for abacus beads and for pebbles with which to form polygons and circles; string for rope; cardstock for puzzle pieces and pizza slices; and, because of the folding, slightly lighter weight paper than cardstock for Platonic solids. It's also helpful to have a supply of scissors, tape, compasses, straightedges, and, if possible, four-function calculators. The instructor should purchase at least one Chinese abacus (7 beads per position: 2 5-beads and 5 1-beads). Save the instructions! They may be your best guide to traditional use of the abacus, from moving beads with the thumb and forefinger to adding (or subtracting) 3 by thinking of it as 5 – 2.

Classroom: A classroom with flat tables conducive to group work and hands-on activities, and with overhead projection of a computer and document camera is strongly recommended.

College Algebra the History Way

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Course Overview

Most institutions of higher learning have a minimum graduation requirement of one college-level mathematics course. The spread of general education and quantitative literacy courses over the past decade has given non-STEM majors a variety of course options to fulfill this requirement. However, some institutions still require an algebra course or higher for graduation. Students who take algebra as a terminal mathematics course in order to fulfill a mathematics requirement often find the course challenging and rarely gain an appreciation of the beauty and usefulness of mathematics. The course described here combines the basics of college algebra with its history as an alternative to algebra or other general-education mathematics courses. It has a prerequisite of introductory algebra or high school algebra I.

There are several manifestations of this course, each dependent on the focus and balance between the mathematical and historical content. One option is to teach this course as an elementary mathematics and algebra course that uses history as the vehicle to present the mathematics. It was designed to fulfill the requirement of intermediate algebra that many colleges have for non-STEM majors, and was why the course and text were initially conceived. However, over time, this course has been taught with varying degrees of rigor and mathematical versus historical content at several different institutions, dependent on student ability and departmental and curricular need. This course can also be taught as a general-education mathematics course for non-STEM majors, with less emphasis on algebra and more on history. As a third option, this course can be used as a course for education majors. With its focus on the basic development of number systems, calculational methods, and algebra, this course is highly effective for education majors. Many of the topics can be used in school mathematics from elementary school through high school. Finally, as an elective for mathematics majors, in which they would learn several topics or methods that they would not have seen in their regular mathematics courses. The investigation of the development of mathematics in different cultures inherent in this course fulfills the multicultural requirements of many colleges.

The combined nature of the course allows students to gain a deeper understanding of mathematics as well as an appreciation of mathematics as an ongoing, creative endeavor. Students are able to leverage their proficiency in writing, languages, the humanities and social sciences, and history to enhance their learning. They find that there are many connections between the topics that they do have an interest in and mathematics. Students are surprised by the interdisciplinary nature of mathematics and find that they can master fairly complicated techniques when placed in a different context. History provides the story that gives life to the mathematics.

Course Design

There are a wealth of historical elements and methods related to the topics covered in a traditional college algebra course. The course described here is for a three or four-credit 100-level general education course. The number systems of the Egyptians, Romans, Babylonians, Chinese, Mayan, Indian, and some Native American tribes are highlighted, culminating in the Indo-Arabic base-10 place-value system. After our base-10 place value system is fully mastered, various methods used throughout history to perform basic operations such as addition, multiplication, division, and root extraction are discussed. These include approximation techniques and checking methods such as casting out nines. Along the way, the history and evolution of mathematical notation is covered. Solving equations up to and including the cubic is covered, in its historical setting, as with all the topics. Equations then lead to exponentiation and logarithms and the impetus for their development (this topic is often omitted in the three-credit version of the course). Finally topics such as Greek number theory and the development of set theory round out the course. The mathematics as well as the history of each concept is explored, but in doing so, the history of each region and cultural is also brought to light. Though the text used does not address geometry and trigonometry, those topics can be included if needed. The topics listed above will make for a very fast pace. Selection of topics will depend on the amount of time available as well as the overall level of student mathematical sophistication and any departmental requirements. Different options are outlined in the appendix.

Most students who will be taking a course of this nature are not always comfortable in a traditional mathematics course or doing mathematical thinking. In order to capture the interest of all students and use what they are interested in and good at, a variety of teaching methods and assignments are available. While covering all of the topics mentioned above, the classroom experience is vital. The two key components are classroom discussions and hands-on practice. Daily discussions allow the students to explore not only the mathematics, but the societal, cultural, and historic events that affect and even shape how mathematics evolves and is transferred (or not) from region to region. Group work, whether it be working problems or using calculation devices such as the abacus or Napier's bones are vital to the learning experience. Details of activities will be given below.

Resources

The text used for this course is:

> *Algebra in Context: Introductory Algebra from Origins to Applications,* by Amy Shell-Gellasch and John Thoo, Johns Hopkins University Press, 2015 [6].

This text was specifically written to fulfill the (terminal) intermediate algebra requirement of many colleges, be they four-year or two-year institutions. The text provides all the mathematics as well as historical material necessary to teach this course. It also provides numerous practice problems, extension problems, and research questions.

Other books such as

> David Reimer's *Count Like an Egyptian* [5] and
>
> Richard Gillings' *Mathematics in the Time of the Pharos* [3]

provide additional historical descriptions, problems, and activities that can be used in the classroom.

> Frank Swetz's *Mathematical Expeditions: Exploring Word Problems across the Ages* [7]

is a wonderful source for additional historical problems, while Bunt's *The Historical Roots of Elementary Mathematics* [1] provides additional material on algebra.

Students often think all of mathematics has always been the way they see it. They have little or no conception that the way we do mathematics and the notations we use are really very recent developments. For that reason I sprinkle the history of notation throughout the course.

Cajori's time-honored *A History of Mathematical Notations* [2] is the perennial choice for this.

A standard algebra text can provide additional practice on fundamentals that you may deem necessary.

Additional materials might include articles on specific topics, videos and websites. In particular there are many websites that present both mathematical applets and historical activities designed for teachers that are appropriate for this course. A very useful resource is the Mathematical Association of America's online journal *Convergence*. This journal provides articles on the history of mathematics and its uses in teaching. For example, the article by Janet Beery and Frank Swetz, *The Best Known Old Babylonian Tablet?* (*Convergence*, volume 9, 2012) can be given to students during the section on Babylonian numbers or during work on taking roots. The article *Logarithms: The Early History of a Familiar Function* by Kathy Clark and Clemency Montelle (*Convergence*, volume 7, 2010) includes useful images of the key players and pages from their work and logarithmic tables. Both these articles allow students to view original materials and help promote discussions in class.

Assignments

The assignments revolve more heavily around problems and calculations than a more traditional history of mathematics course would, due to the emphasis on algebra and the general education quantitative reasoning requirements. Problem sets that require the students to use both the historical and modern methods are assigned daily or weekly from the book. Tests are also based on such problems. However, writing is incorporated in many ways. On homework and tests, students are asked to describe the historical importance of ideas or techniques as well as to compare and contrast various methods of computation. As mentioned above, working problems during class coupled with class discussion is vital.

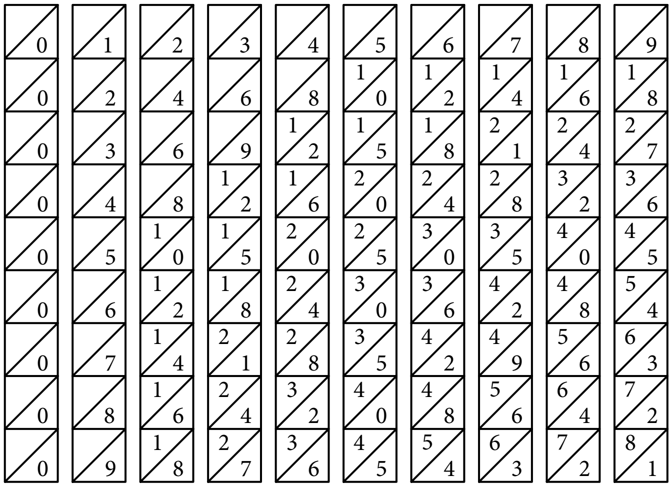
Several short research assignments (1–4 pages) are given about four to six times a semester. In addition to these short research papers, one or two longer written assignments, with a poster presentation, can be used. Part of a day is set aside for students to view and present their posters to their peers. These one or two longer papers are weighted heaver and I expect students to really focus on the production of the paper, as if this was a writing course. (I spend time in class discussing what is expected.) The research suggestions from the book or topics that have come up during the course of the class discussions can be used for any of these research projects.

An example of a one-to-two-page assignment would be the Ishango bone. On the first day of class I present the students with a photo and a diagram of the Ishango bone. They form small groups and discuss what the bone may represent. For homework, students are to research the Ishango bone and write a short paper about it. This activity is a fun way to open the course. It also addresses two themes for the first part of the course: numbering systems and calendars. The earliest number systems were tally marks, which the bone is one of the earliest examples. Much of early mathematics was devised to aid in calendar reckoning. Many believe that the Ishango bone is a lunar record. Calendar reckoning can be a thread through the course, incorporating a longer writing assignment. Each student researches a different calendar used by a culture or religion at some point in history. There have been enough different calendars used throughout history so that most students may have a unique topic. *Mapping Time: The Calendar and its History* [4] is a great source of both the history of many calendars and the mathematics and astronomy behind them. I put this book on reserve at the library for the students during this writing assignment.

Another short writing assignment I enjoy happens right after we learn how to use the abacus. Students must write on the 1946 competition between a US service member using an electric calculator and a Japanese clerk using an abacus. The abacus won hands down! Students are intrigued by this episode, which leads to a nice discussion of electric versus electronic calculators. I bring in an image of the type of hand-crank calculator used so they can see and explore the difference with their modern calculator.

Exams have both a calculation section and a section of short essay questions. In this way, the exam mirrors the course with a combination of calculation and discussion. This allows students who are still struggling with the mathematical portion of class to feel confident about part of the exam. Prior to the exams, I assign four of the short research topics from the book that we have not discussed in class. Students then do a simple internet search on each and come to the exam prepared to answer three of the four questions in three to five sentences.

There are many hands-on activities used in this course. During the section on basic calculations, the students are introduced to various types of calculating devices. They make Chinese counting rods and counting boards on which they learn to do multiplication. They learn how to preform addition, subtraction, multiplication, and even division on various types of abacus. They can make a simple abacus using paper and coins; they can construct one (as an at-home project) using string and a shoe box; or a set of abacus can be provided for the course. During the section on Egyptian numbers, string can be given out and they can become Egyptian rope stretchers, using the 3-4-5 triangle to create a right angle. They are also shown how to make a set of Napier's Bones out of tongue depressors for preforming multiplication using the lattice method.



0	1	2	3	4	5	6	7	8	9
0	2	4	6	8	1/0	1/2	1/4	1/6	1/8
0	3	6	9	1/2	1/5	1/8	2/1	2/4	2/7
0	4	8	1/2	1/6	2/0	2/4	2/8	3/2	3/6
0	5	1/0	1/5	2/0	2/5	3/0	3/5	4/0	4/5
0	6	1/2	1/8	2/4	3/0	3/6	4/2	4/8	5/4
0	7	1/4	2/1	2/8	3/5	4/2	4/9	5/6	6/3
0	8	1/6	2/4	3/2	4/0	4/8	5/6	6/4	7/2
0	9	1/8	2/7	3/6	4/5	5/4	6/3	7/2	8/1

A set of Napier's Bones can be made from large tongue depressors or strips of cardboard.

Finally, students learn how to make a simple slide rule or dividers for doing multiplication during the section on logarithms. Many more activities can be found on the internet. These in-class activities allow students to explore the mathematics as well as the history in depth and to engage with it kinesthetically.

Students are also given a final project to do. I let this be fairly loosely defined so they have the chance to follow their interests and play to their strengths. I have accepted traditional term papers with a short presentation or major book reports. Many students choose to do a biography on someone they have heard of, often a modern figure. For example, one student chose to create an in-depth timeline of the life and career of Katheryn Johnson, an African American mathematician employed at NASA in the 1950s. Often art or music students will do a project that looks at the mathematics behind Bach or M. C. Escher for example. Another favorite topic is for the student to pick a topic, from the book or not, that we did not cover. They need to write up a report on the history and the mathematics that includes worked out examples of how the methods are used. With all of these projects, suitable references and citations are expected. Since this course is a terminal mathematics course for non-STEM students, I am more interested in them doing a final project that they really enjoy and leaves them with a rewarding experiences in mathematics, than having them conform to a traditional notion of a final term paper or final exam. I explain that I want them to really get into something they are interested in, and I have always been pleasantly surprised by how much effort they put into it. To make sure they succeed, I require that each student meet with me well in advance of the due date to make sure their project is appropriate and has at least some mathematical content.

Lessons Learned

This course has been taught at several colleges, from a selective small liberal arts college to a large mostly urban community college. Though the demographics of the students were very different at various institutions, the class proved to be very popular. Whether it is being taught as an algebra course or a general-education mathematics course, the vast majority of students taking it do not describe themselves as mathematically inclined. Quite often they describe themselves in opposite terms. They often enter the course concerned, if not downright scared, given their prior mathematical experiences. However, they soon become comfortable when they realize they can use their knowledge of other topics to help them learn mathematics. Most likely this is the last mathematics course they will ever take. So I view the course as a success if they leave knowing that they mastered some part of mathematics and that they have had an experience with mathematics that was interesting and meaningful. I find that the discussions are very interesting to most of the students. I try to choose in-class examples that really showcase the connections to the mathematics they learned in school and the historical methods. This allows them to discover the modern approach through the historic one. This seems to really catch their interest, as they learn both approaches simultaneously. For these students, approaching mathematics head on often causes them to shut down, whereas the historical approach often does not feel like mathematics. The one issue I have found is that the students are even less likely to get their homework done than in a traditional mathematics course. I assign small daily problem sets. Some days I have a few problems turned in, other days I have them copy down their solution to one of the homework problems in class. I also use homework problems on the tests. This helps encourage the students to try the homework every day.

When presenting each topic, a choice needs to be made as to whether the historical approach or the modern is more intuitive and should be presented first. For a few of the topics it is actually easier to present the modern version and then present the historic, for example, logarithms. Often this has to do with notation or the complicated nature of the calculations. However, most topics can be approached from the historic view first, which then leads to the modern view. It is important to remember that the audience for this course is most likely math-challenged students. Sometimes one or two truly math-challenged students can change the whole dynamic of the course, so stay open to changing topics or assignments as you go. One section of the course may sail through the material and ask for more, while other sections may struggle with the simplest of topics. When teaching this course at a community college near Washington DC in 2016, I had a non-traditional student who was going back to school and very concerned about taking a mathematics course. She enjoyed the course so much she decided to keep the book and not sell if back. She had three teenage children who would be attending the college soon and she wanted all of them to take the course. She noted that she was able to approach the course through reading and writing, which were her comfort zones, and allowed her to have a positive mathematical experience and successfully conquer several mathematical topics, much to her surprise!

By coupling topics in basic and intermediate level college algebra with history, students who would normally just “get through” their mathematics requirement learn that there is a history to mathematics. They learn that it has been an integral part of almost every culture on earth in some way or another. They see the depth and beauty of the subject. And they start to view mathematics as a creative product of humanity, in the same way that music or art is.

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4. Richards, E. G., *Mapping Time: The Calendar and Its History*, Oxford University Press, 2000.
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6. Shell-Gellasch, and Thoo, J. B., *Algebra in Context: Introductory Algebra from Origins to Applications*, Johns Hopkins University Press, 2014.
7. Swetz, Frank, *Mathematical Expeditions: Exploring Word Problems across the Ages*, Johns Hopkins University Press, 2012.

Appendix A

Possible course outlines

Option 1: a lower-level general-education course with only algebra I as prerequisite.

Weeks 1–4: Part I (chapters 1–6). Number systems from other cultures, culminating in the Indo-Arabic positional base-ten system. Short term paper and poster on calendars.

Weeks 5–8: Part II (chapters 9–12). Arithmetic Snapshots. Projects may include counting boards, the abacus, and Napier's bones. Project or paper with presentation. (The section on finding roots can be optional, depending on the level of the students.)

Weeks 9–11: Selections from Part III (chapters 14–17). Topics such as sets or Greek number theory and figurate numbers. Longer problem sets can be given as well as biographies of ancient mathematicians.

Weeks 12-14: Part IV (chapters 19 and 20). Solving equations. Longer problem sets. Biographies of historical mathematicians. Final project or paper with presentation.

Option 2: an algebra-intensive course to fulfill a terminal algebra requirement.

Weeks 1–5: Part II (chapters 9–12). Arithmetic Snapshots. Methods of calculation including root extraction.

Weeks 6–12: Part IV (chapters 19–24). Solving equations through the cubic and logarithms.

Weeks 13–14: Additional topics (possibly chapters 14 and 16). Given the slightly more advanced nature of this version, additional topics not in the book such as geometry or matrices can be covered. Alternatively, a large final project with presentation can be worked on in groups during these last two weeks.

Bringing together Older and Newer Mathematics

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COURSE OVERVIEW

How does one engage a wide variety of students enrolled in a history of mathematics course that has only a mathematics prerequisite of Calculus I? One approach is to emphasize the theme of bringing together older and newer mathematics and to examine how they are intertwined. The basic idea of this theme involves introducing a mathematical concept in a very early form and then following what happens to it over the years, ending as close to the present day as is reasonable. This theme was inspired by the epilogues in William Dunham's book, *Journey through Genius*: *The Great Theorems of Mathematics* [6]. This is an approach that seems to have worked well with the broad audience taking this course which includes undergraduate students majoring in mathematics, computer science, physics, chemistry, biology, earth sciences, business, and economics. Occasionally a few majors in the humanities and social sciences enroll in the course. Many students who take the course are earning education degrees as middle-level mathematics specialists or secondary mathematics education majors. The course is also cross listed as an honors course for the University Honors College and as an elective graduate course for our master's degree in mathematics education program. It should be noted that the number of students who take this course each semester is usually at least thirty.

The history of mathematics course at Millersville University is classified as a perspectives course in the general education program. Its prerequisites are freshman level courses in English composition and speech, one semester of calculus, and junior standing. To be approved as a perspectives course, an application must be made showing that the course has significant interdisciplinary or multicultural components and that the course requirements include significant writing and oral components. Every perspectives course must be a 3-credit upper- level course with no more than one prerequisite course beyond the composition and speech courses. Perspectives courses are intended to require a high level of educational maturity, knowledge, and critical thinking. Every undergraduate student who graduates from Millersville must fulfill the perspectives requirement.

COURSE DESIGN

The course is designed around its objectives, four of which are required of all perspectives courses: (1) To integrate knowledge acquired in previous course work and life, (2) To make independent and responsible value judgments and decisions, (3) To apply analytical and critical thinking abilities to the course content,

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[1] The author would like to acknowledge the support from the Faculty Grants Committee at Millersville University that helped to make this article possible.

and (4) To demonstrate how different areas of knowledge relate and can be used in complementary ways. The other course objectives specifically deal with the learning of history of mathematics. Some of these are (1) To begin to understand and trace the historical development of various mathematical processes, thought, areas, and topics, including causes and effects, (2) To understand the roles that various peoples and cultures and societies have played in the historical development of mathematics including contributions by Indian, Arab, and Chinese mathematicians of the first millennium and a half AD, (3) To compare, contrast, and evaluate differing expositions about the same topic in the history of mathematics, (4) To identify and elaborate on specific situations where "recent" mathematics and "older" mathematics are brought together, (5) To identify and explain how specific mathematical developments changed a given society, and (6) To learn the importance of avoiding the imposition of modern views of mathematics and society on the development and use of earlier mathematics. Every assignment reflects at least one of the course objectives.

On the first day of class, students complete a one page questionnaire. On the top half students write down three questions pertaining to the development and/or history of mathematics whose answers are of interest to them or about which they are curious. On the bottom half, they indicate their major along with any minors or options, whether or not they plan to teach, and the mathematics and statistics courses they have taken beyond the calculus prerequisite. This information is used two ways. At the end of the semester, usually for the final exam, each student is asked to answer one of the questions he or she wrote down on the first day. I choose the question for each student to avoid redundancy with other final exam questions. I also use the questions and the information about majors and mathematics courses to help decide how to organize the course projects and how to determine appropriate topics for specific students.

During almost every class meeting, students are required to turn in a "daily comment". At the beginning of the class, a piece of paper is passed out with the topic for the comment at the top and by the end of class, the students are to have written their comments about the topic. The daily comments help me keep track of how students understand and/or perceive the historical development of mathematics as we progress through the semester. These comments cannot be made up due to absences because they are based on that day's class lesson, so an ancillary result is a form of attendance taking.

All of the course material is presented in the form of PowerPoint slides. A list of these topics is found in Appendix A. Each of these presentations consists of text that I have written based on my extensive reading and research, exact quotes from relevant and cited sources, and pictures and diagrams scanned from print sources. The theme of "bringing together older and newer mathematics" is present in almost every topic. Some examples of how this theme is incorporated are discussed below.

Ancient Egypt: During the first week of class, when we discuss the mathematics of ancient Egypt, the topic of unit fractions naturally comes up. Students are usually very confused about this topic because it seems so alien to them. This gives me the opportunity to bring the topic of unit fractions into more recent times. I introduce two algorithms for decomposing a simplified proper fraction into the sum of distinct unit fractions. One algorithm has been attributed to the Indian mathematician Mahavira (800s AD) and the other to the English mathematician James Joseph Sylvester (1800s). We work through a few problems and invariably students realize that the two methods are mathematically equivalent, although Mahavira's algorithm is given in words whereas Sylvester's is given in algebraic symbolic form. This discussion also provides me with an early opportunity to explain that the algebraic notation we take for granted took an extremely long time to develop.

Pythagorean Theorem: After completing what time allows for the mathematics of ancient Egypt and ancient Mesopotamia, we move into the mathematics of ancient Greece, beginning with Thales and Pythagoras. This leads directly into a discussion of the Pythagorean Theorem and its many proofs across many cultures. We watch the DVD "The Theorem of Pythagoras" [2] which shows a variety of animated proofs of the Pythagorean Theorem. We then connect the old and the new through the article "Loomis Claimed

370 proofs… The Pythagorean Theorem: An Infinite Number of Proofs?" [9], which includes President Garfield's proof published in 1876. We also look at a new and original proof discovered by a high school student and published in the March 1991 issue of *The Mathematics Teacher* [7].

Geometric Constructions: After our discussion of the Pythagorean Theorem and before we begin to look at Euclid and his *Elements*, we spend one class period devoted to performing basic geometric constructions using compasses and unmarked straightedges from my classroom set. These exercises allow students to experience a brief hands-on glimpse into how geometry was done by the ancient Greek mathematicians. It also leads directly into the four classical construction problems of duplicating a cube, trisecting an angle, squaring a circle, and determining which regular polygons are constructible. In this case, we do not bring together the old and new right away. These connections are made later on separate days. For the first two problems we discuss Pierre Wantzel and the polynomial conditions he published in 1837 that characterize constructible numbers. For the fourth problem, I introduce Fermat for the first time in the context of Fermat primes and Gauss for the first time in the context of his result connecting Fermat primes and constructible regular polygons. We then connect Fermat primes to Pascal's Triangle mod 2, as discovered by Gardner in 1977. The third problem, squaring the circle, is not dealt with until later in the course when we discuss transcendental numbers.

Euclid and his *Elements*: We spend two days on Euclid right after the activity on geometric constructions. As we study Euclid, several opportunities exist for bringing together older and newer mathematics. For example, we look at the articles from the December 23, 1996 issue of *U.S. News and World Report* [14] and the January 1997 issue of *The Mathematics Teacher* [10], both about the two high school students who used Geometer's Sketchpad to discover a new way to divide a line segment into n equal parts. Another example happens as we discuss Euclid's proof that the collection of prime numbers is never ending. I introduce Paul Erdős and his idea of THE BOOK containing the most elegant proofs for mathematical theorems, a version of which [1] was published in 1998 to commemorate him. Euclid's proof is the very first entry in this book, illustrating how highly regarded Euclid's proof continues to be in the mathematical community. A third example occurs as we look at the five regular polyhedra to which Book XIII is devoted. A natural connection between older and newer mathematics is the introduction of Euler and his polyhedral formula, Euler is revisited when we discuss Euclid's even perfect number theorem and Euler's converse to that result 2000 years later. Moreover the older idea of perfect numbers and their relationship with the newer idea of Mersenne primes leads to the ongoing modern Great Internet Mersenne Prime Search (GIMPS) as well as the unsolved problem concerning the existence of odd perfect numbers. Also included are several newspaper articles I have collected over the years announcing the discovery of new Mersenne primes.

Archimedes: After Euclid, we study Archimedes, which leads to more opportunities to bring together older and newer mathematics. One of the most interesting examples is the history of the Archimedes Palimpsest and the work done between 1998 and 2011 at the Walters Art Museum in Baltimore. Another example occurs when we discuss Archimedes' quadrature of the parabola and his use of triangles to fill up the region instead of rectangles as we use today. A natural connection is to apply the "newer" Fundamental Theorem of Calculus to the region bounded by the graph of $y = x^2$ on the interval $[-2, 2]$ and the line $y = 4$, showing that the area of the parabolic sector is indeed 4/3 of the area of the triangle with vertices $(-2, 0)$, $(2, 0)$ and $(0, 4)$.

The number π Even though the number π is mentioned early and often during the course, I always devote one day to this number and it can take place almost any time during the semester. As π is ubiquitous across time and cultures, it is the quintessential example of bringing together older and newer mathematics as we trace its history beginning with early approximations through its polygonal and series phases and ending with the computer age. One of the facts we look at is the July 29, 1961 announcement by Dr. Daniel Shanks and Dr. John W. Wrench of their computation of π to over 100,000 decimal places using an IBM computer.

We follow this by reading Dr. Wrench's obituary that appeared in the March 25, 2009 issue of *The Washington Post*.

Definition of Derivative: Towards the middle of the semester, we reach the 1600s during which mathematical ideas began to develop at a very rapid rate, including calculus. Before we discuss Newton and Leibniz, we read a translation of Fermat's work "On a Method for the Evaluation of Maxima and Minima". We dissect his approach and show that it is virtually equivalent to the definition of derivative that we use today except we use the concept of limit explicitly.

Fermat's Last Theorem: Diophantus and his *Arithmetica* are mentioned early in the class when we study the ancient Greeks. Fermat is mentioned many times during the class. Usually on the last day of class I show the NOVA video *The Proof* about Andrew Wiles' quest to prove Fermat's Last Theorem and this is the final example of bringing together older and newer mathematics.

RESOURCES

I use *Journey through Genius* [6] as one of the textbooks for the course because of the comprehensive coverage of the topics that it includes. For the other textbook, I use *Math through the Ages* [3] because it gives students quick access to many different topics in the history of mathematics. However, neither book has in-depth discussions of women in mathematics or the histories of Indian, Middle Eastern, and Chinese mathematics. The PowerPoint lessons I have prepared fill in these gaps.

Videos that are shown during the semester include the Discovery Channel's *What the Ancient Greeks Knew*; the *Project Mathematics! v*ideo on "The Pythagorean Theorem" produced at Cal Tech; the NOVA production, *Infinite Secrets*, about the Archimedes Palimpsest; a clip on Eratosthenes and the Library at Alexandria from Carl Sagan's *Cosmos* which is available online; and the NOVA video, *The Proof*, about Andrew Wiles quest to prove Fermat's Last Theorem.

These videos enhance the student's learning experience by providing course content from a different perspective. I use *What the Ancient Greeks Knew* to introduce Thales and Pythagoras. The Pythagorean Theorem video includes several animated proofs of the theorem. *Infinite Secrets* clearly demonstrates how rare ancient mathematical documents are and how lucky we are to have modern technology to restore them. The Carl Sagan video shows how Eratosthenes calculated the circumference of the earth, and more importantly, it shows a virtual tour of the Library at Alexandria and the tragedy of its destruction. *The Proof* sums up the theme of bringing together older and newer mathematics and it also mentions many mathematicians that we studied during the semester.

Some the readings I use for assignments are described in the next section and a list of articles and books that have been helpful to me is found in Appendix B.

ASSIGNMENTS

Daily Comments

Examples of topics for the daily comments include (1) Please comment on the differences between how we do arithmetic today and how the ancient Egyptians did their arithmetic. (2) Please comment on the current relevance of Euclid's *Elements* by identifying two of Euclid's results that you have used in previous math courses. (3) Please comment on Pierre Wantzel's contribution to the history of mathematics and how it brings together older and newer mathematics. (4) Please comment on the advances in mathematics made by the Indian and Hindu mathematicians of the first millennium AD and a few centuries after. (5) Please comment on the advances in mathematics made by the Arab, Islamic, and Middle Eastern mathematicians of the first millennium AD and a few centuries after. (6) Please comment on any ethical issues encountered during the quest for the cubic. The daily comments are worth between 5 and 20 points each. I expect to read insightful and nontrivial comments from the students that reflect critical thinking. This usually does not happen right away, but with the feedback I provide, most students' comments improve as the semester progresses.

Homework

Homework is assigned throughout the semester. Many of these assignments require students to read an article about a topic in the history of mathematics. During the reading, students encounter answers to questions I have prepared about the article and they write them down. This technique ensures that students will focus on the basic facts in the article. After these questions have been answered, students write a short essay in which they express their own opinions about or reactions to the topic. After this work is collected, students engage in a short class discussion so they can share their thoughts, insights, and questions about what they learned. These assignments help fulfill the required written and oral components for the course. As with the daily comments, I expect to read and hear insightful and nontrivial comments about these readings from the students. Here are descriptions of some of these assignments.

Ancient Egypt: After introducing the mathematics of ancient Egypt, students read the article "Mathematics Used in Egyptian Construction and Bookkeeping" by Beatrice Lumpkin [11]. Students are asked to write answers to questions I prepared about this article. The questions I ask are intended to provide a glimpse into very early practical applications of mathematics and the level of sophistication of this mathematics, as well as an early need for some type of "zero". The student work is graded on correctness and level of insight.

Ancient Mesopotamia: As we discuss the mathematics of ancient Mesopotamia, the topic of Plimpton 322 naturally comes up. Students are required to read and compare the two articles from the *American Mathematical Monthly*: "Sherlock Holmes in Babylon" by R. Creighton Buck [4] and "Words and Pictures: New Light on Plimpton 322" by Eleanor Robson [13]. Students are asked to write short essays comparing the Buck and Robson articles according to criteria that I set. The student work is graded on completeness, insight, and how well they support their opinions. The assignment for this topic specifically fulfills the two course objectives of comparing and contrasting differing expositions about the same topic and the importance of trying to avoid imposing modern views on earlier mathematics. It also provides students with the opportunity to contemplate the relevance of the authors' credentials and the quantity and quality of references used. Two other readings are used as preparation for this homework assignment: excerpts from *The Exact Sciences in Antiquity* by Otto Neugebauer [12] and "A Remarkable Collection of Babylonian Mathematical Texts" by Jöran Friberg [8].

Archimedes Palimpsest: As preparation for this topic, students read Chapter 4 in *Journey through Genius* [6]. We also watch the NOVA production, *Infinite Secrets, The Genius of Archimedes*. The subsequent assignment is to find out what has happened to the Archimedes Palimpsest since the end of the video. Most of the research for this assignment is found online. Students are asked to write answers to specific questions that I pose and to cite the websites where they found the answers. The student work is graded on completeness and correctness.

Early Chinese Mathematics: We discuss the mathematics of ancient China after we have discussed the mathematics of early India and the Middle East. The assignment is to read the introduction and first 22 pages of the book *Fleeting Footsteps* by Lam Lay Yong and Ang Tian Se [15] and to write a reaction paper to the main thesis of their work. What I expect the students to realize from this assignment is that the Hindu-Arabic numeral system that we use every day has a complicated and uncertain history. I also expect the students to defend their opinions on the origin of our number system. The student work is graded on completeness and how well they support their opinions.

The Beginning of Probability Theory: As we discuss the many mathematical triumphs of the 1600s, I assign excerpts from the book *The Unfinished Game* by Keith Devlin [5]. From this reading and the short essays that they write, the students acquire a strong sense of how the development of one mathematical idea can change the outlook and economics of an entire society. The student work is graded on insight and how well they support their opinions.

Calculation based homework: Other homework assignments involve calculations such as implementing the Egyptian doubling method for multiplying, the Arabic lattice method for multiplying, and solving a depressed cubic equation. These are graded on mathematical correctness.

The combined points for daily comments and homework assignments make up 60% of the course grade.

Course Projects

As previously mentioned, each student is required to prepare and participate in a project for the course. Each project has a written component in the form of a handout for the class and an oral component in the form of a class presentation. The entire project is worth 20% of each student's course grade. The student presentations take place during the last two weeks of classes and are assigned around the middle of the semester. Students are required to research their topics using at least four reliable sources and prepare a written outline of the results of their research, including a bibliography of the sources used. Copies of these handouts are passed out to the class when the students present their topics to the class. The topics for and the types of course projects vary from semester to semester, depending on the number of students in the class and its composition. The course projects have been in the form of group projects, individual projects, or a combination of these two formats.

Group Projects: The topics for group projects are always topics that we did not discuss in class during that semester. When I have several students with the same major or concentration, they may be assigned to work in a group to research and present historical connections between their majors and mathematics. This has resulted in projects on the history of statistics, the history of computer science, the history of mathematical topics in physics, and the development of mathematical models in biology beginning with Thomas Malthus and his "An Essay on the Principle of Population". Projects for groups of future middle school teachers include researching the historical development of mathematical symbols, algebraic properties, and/or decimal fractions. Projects for groups of mathematics majors who have taken several upper-level mathematics courses include tracing the histories of linear algebra, abstract algebra, and/or advanced calculus. Another type of group project that I have used involves three or four students per group to whom I assign two or three mathematicians who are connected in some way such as Kovaleskaya and Weierstrass, Noether and Hilbert, or Ramanujan and Hardy and Littlewood. One student in the group acts as an interviewer, asking the other students, who are portraying the mathematicians, questions about their lives and work.

Individual Projects: An example of a very successful course project where each student works individually comes from a 1999 list of the hundred greatest theorems as compiled by Paul and Jack Abad [16]. To each student I assign one of the theorems we did not cover during the course. Each student presents his or her research on the theorem in the following format: statement of the theorem, examples of the theorem, history of the theorem, the student's perspective on the theorem's position on the list, and sources used in the research. All of this is required to fit on one sheet of paper (using both the front and the back) and the students have five minutes of class time to present their theorems. Other types of individual projects are assigned to students based on their majors. For example, a psychology major did her project on the history of mathematics anxiety and an accounting major did his project on Luca Pacioli's development of double-entry bookkeeping.

Honors and Graduate Projects: Students who are taking the course for honors credit or graduate credit are required to complete all of the work for the regular course as well as an extra project. For students in the Honors College, the extra project is usually an in-depth extension of the course project they prepared for the regular course in the form of a term paper. For graduate students, the extra project consists of choosing six topics in the history of mathematics and, for each topic, explaining how they either already incorporate its history into mathematics classes that they teach or how they would incorporate it if they had the space in the curriculum they are required to cover. They also write an essay about their opinions on how value is added or not added to a topic when some of its history and background are included.

Primary sources: A few years ago, a retired faculty member donated his large collection of rare mathematics and science books, some dating back to the 1500s, to the university where they are housed in the library's special collections. I have used this collection as another type of project for honors and graduate students. Students choose a book from this collection to review. They write a summary of the book and take some digital pictures of it. These are then posted on the library's website as a reference for interested parties.

Final Exam

The course ends with a final exam that is the only test in the course. It is worth 20% of each student's course grade and it is in two parts. The first part consists of several questions requiring short essay answers that relate course content to the course objectives. The second part consists of short answer questions about the content learned from the student presentations. The final exam is open book and open notes, but not open computer.

LESSONS LEARNED

The best way to prepare to teach a history of mathematics course is to read, read, and read some more. For me, the most effective way to broaden and enhance my knowledge base has been to read articles published by NCTM, MAA, AMS, and Springer. Some of the older mathematics history articles were written by giants such as Howard Eves, Carl Boyer, and D. J. Struik. When I began to teach the course, I was fortunate to have access to print issues of *The Mathematics Teacher* dating back to 1962 (they are now available on-line) and I have used my own collection of MAA journals. One of the reasons I value articles that were written before 1990 is that they often provide students with a sense of how historical perspectives change, even for scholars. One of the reasons I value recent articles is that they show students that "older" mathematics is still being studied and expanded into "newer" mathematics. In Appendix B I have listed the titles of many articles and books that I have found helpful over the years, either in class or for my own benefit. My own personal library contains many other books that serve as references for my own edification in the history of mathematics.

An important lesson that I have learned while teaching the history of mathematics is stated in my syllabus: "There is absolutely no way that we can address all of the concepts, people, and events that make up and have contributed to the history of mathematics over the ages." This is particularly frustrating for me because every year I learn something new and exciting about the history of mathematics. In order to include these new ideas, I have had to pare down both the number of topics and how much time is spent on the topics that are covered. I have also learned that I cannot always believe what I have read, so it is important to read several sources about each topic.

An important lesson I have learned about the readings I assign to students concerns the questions I ask about those readings. Vague questions elicit vague and superficial answers, so my questions must be very specific, but not so specific that I inadvertently influence their opinions. I want my students to look deeper into the subject matter and connect it to previous knowledge and for this they need specification. I have also found that composing and then evaluating math history assignments is not the same as composing and grading calculus assignments!

Another lesson I have learned is the importance of the daily comments. I began using them in 2007 as an attempt at informal assessment and they have worked well. In the early part of the course, most of the daily comments are superficial and unsophisticated, but by the middle of the semester, virtually all students are writing insightful comments that reflect critical thinking and make connections within and outside the course. The daily comments have also eased the burden on the final exam by providing me with the opportunity to ask about topics and ideas and connections that I used to feel compelled to put on the final exam. Students also tell me that they value the daily comments as preparation for the final exam.

Going into the course, most students have no idea how mathematics developed through the ages. I am very pleased that my students seem to embrace this journey through so many cultures. Most of my students are deeply disturbed that it took so long for women to be accepted as mathematicians and that the accomplishments of early Indian, Middle Eastern, and Chinese mathematicians were held in so little regard by the West for so many years. The lesson learned here is that, for my students, the people who develop the mathematics are just as important as the mathematics they develop.

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APPENDIX A

COURSE OUTLINE

The topics in the course and the general order in which they are covered are listed below. In parentheses next to each topic is the approximate number of class meetings I spend on that topic. Some topics have specific assignments that have already been described. Other topics are only covered during the class lesson. Some topics on this list may be shortened or omitted due to time constraints. Some topics not on this list are covered through the student projects for the course.

1. Origins and Egyptians (2 days)
2. Babylonians (2 days)
3. Thales and Pythagoras (1/2 day)
4. The Pythagorean Theorem (1/2 day)
5. Some Greek mathematicians after Pythagoras and before Euclid (1/2 day)
6. Geometric constructions and Euclid (1 day)
7. Perfect numbers and Mersenne primes and Euler (1/2 day)
8. Pierre Wantzel and two of the four classical Greek constructions (1/2 day)
9. Fermat primes and Gauss's theorem on constructible regular polygons (1/2 day)
10. Archimedes, including the Archimedes Palimpsest (2 days)
11. Some Greek mathematicians after Euclid other than Archimedes (1/2 day)
12. Greek commentators including Hypatia (1/2 day)
13. Prime numbers from Euclid and Eratosthenes to the Prime Number Theorem (1 day)
14. Early Indian/Hindu mathematicians and mathematics (1 day)
15. Early Arab/Islamic mathematicians and mathematics (1 day)
16. Early Chinese mathematicians and mathematics (2 days)
17. The Hindu-Arabic number system and Fibonacci (1 day)
18. Some other European mathematicians (1200 – 1600) (1 day)
19. The Quest for the Cubic and Beyond: prelude, conquest, complex numbers, finale (Ruffini, Abel, and Galois) (3/2 days)
20. Viète, Fermat, Descartes, and The Rise of Algebra in Europe (1/2 day)
21. Newton and Leibniz (1 day)
22. More on Euler (1/2 day)
23. More on Gauss (1/2 day)
24. The history and discovery of non-Euclidean geometry (1 day)
25. Transcendental numbers and (1 day)
26. Some famous women in mathematics (1 day)
27. Graph theory, Euler, and the Four Color Problem (as time permits)
28. Cantor's discovery of transfinite numbers (as time permits)
29. Fermat's Last Theorem (1 day)
30. Class presentations (2 days)

APPENDIX B

This appendix consists of a list of most of the articles and books I have used over the years to help me learn about the history of mathematics. They are listed by publisher and type.

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The Evolution of Mathematics

An Upper-Level Integrative Studies Course

Dick Jardine
Keene State College

Course Overview

The Evolution of Mathematics is a history of mathematics course designed to be accessible to all but first-year students at our public liberal arts college. This blended-learning course emphasizes student engagement in and out of the classroom and includes participation from faculty in other disciplines, differentiating this course from most other history of mathematics courses. One of the unstated but personally important goals of the course is to turn students on to mathematics by engaging them in the history of mathematics, using a lively, student-engaging approach rather than a lecture-based pedagogy.

Some context may help the reader understand the course better. The general education program at Keene State College is intentionally integrative, in fact it is called the Integrative Studies Program (ISP), with courses designed for a broader purpose than specific disciplinary preparation. Upper-level courses in the ISP build on lower-level courses, and the Evolution of Mathematics course is one of the few quantitative offerings at the upper level. The only prerequisites of the course are the ISP foundation courses, Integrative Thinking and Writing (ITW) and Integrative Quantitative Literacy (IQL).

The course was designed to be and is categorized as an interdisciplinary course. At our campus, "interdisciplinary" translates to the requirement that students will think about the subject of the course from a variety of disciplinary perspectives. Additionally, the ISP outcomes emphasize that students are to examine course content from more global and multicultural perspectives. In Evolution of Mathematics, students view the development of mathematics from the historical, educational, multicultural (including parallel developments in both Western and Non-Western cultures), and feminist perspectives in addition to the viewpoints students bring from their disciplinary majors. While many (from a half to two-thirds) of the students in the course are pre-service mathematics teachers at the elementary or secondary levels, some of the best students in the course in recent years have been from the English, Physics, Nursing, and Safety Studies programs. Students in the course learn about the development of mathematics from the ancient Mesopotamian and Egyptian cultures to the present day, using the course text to frame the course (see Appendix A for a typical course schedule, with assignments from the text or from materials posted on the course web page).

Resources

The text used for the course is the extended edition of Berlinghoff and Gouvêa's *Math through the Ages: A Gentle History for Teachers and Others* [1]. I describe how the text is used in later sections. This text is well-liked by students because it is, as advertised, gentle but with the depth appropriate for our students. The exercises in the sketches are mathematically challenging for our students, and many of them are assigned as homework for class preparation. An online discussion forum facilitates student success in accomplishing the homework.

I call upon colleagues from other disciplines to help with the interdisciplinary perspective. At our college, interdisciplinarity means that students in the course will

- Identify the multiple disciplinary perspectives used to explore the development of mathematics
- Synthesize the connections between the multiple perspectives
- Reflect on the position from which they interpret and construct their knowledge of the history of mathematics
- Analyze the impact of cultural assumptions and social values on the development of mathematics from multiple perspectives
- Integrate the insights on the history of mathematics obtained from multiple perspectives.

For example, during the course one of our college's reference librarians and I team-teach a hands-on session in our college library on how to locate resources for the history of mathematics. In another session, a faculty member from our History Department lends insight into historiography and the nature of history, from a historian's perspective. A member of our Women's Studies Department, who happens to have an undergraduate degree in mathematics, presents an engaging session on the history of mathematics from the feminist perspective. I have found that colleagues from other disciplines are always enthusiastic about talking with students about the area of their passion, and the "guest" presentations have received positive reviews from the students.

I use videos frequently in the class, stopping the action to do the math that is presented in the videos. While dated, I continue to use the Open University videos from *MA290 Topics in the History of Mathematics*, no longer taught in their maths programme [8] but available on YouTube. As an example, students watch a portion of the video in class, and I stop the action after watching Graham Flegg, Jeremy Gray, or one of the other presenters describe a particular bit of mathematics, such as one of Euclid's proofs or Roberval's clever method for finding the area under a cycloid. Working in groups, the students then try to duplicate the proof in their own words, with one group called upon to report out to the entire class. We use Marcus du Sautoy's *The Story of Maths* [2] videos in the same way, providing a more visually appealing and effective way of engaging students who have grown up on video. Other videos are also used in the course, to be watched by students in or out of class depending on their accessibility. Some of those include *Top Secret Rosies* [3] and a variety of ever-changing videos available on YouTube.

The text does not have extensive biographies, so students present short (five minute) biographies using information they find at *The MacTutor History of Mathematics* [7] website, or perhaps other web sources, and using one print source, with the *Dictionary of Scientific Biography* [4] strongly encouraged. Each student presents during the class session which addresses the time period in which their mathematician lived and contributed to the development of mathematics.

Course Design

The course is designed to serve students as an upper-level integrative studies (general education) course that reinforces and connects learning that students have already acquired and furthers that learning through the vehicle of a history of mathematics course. The course is writing intensive, as students write a number of essays throughout the course and the major graded activity in the course is the course project (Appendix B), usually a written product. Some students have created podcasts and webpages as acceptable alternative formats.

The course is organized with the Canvas learning management system (LMS). Students use the LMS tools (discussion forums, surveys, quizzes, assignment turn-in, etc.) extensively to interact with each other and the instructor. Students find assignments, rubrics, web-served videos, sample essays, and other resources through the LMS. The course is implemented as a blended-learning course, with student preparation required prior to class, and the LMS facilitates that approach.

Students complete the reading assignments and watch videos and participate in the online discussion forum for that particular lesson *before* coming to class. I post discussion forum prompts that are designed so that students make an initial post and substantive responses to other students' posts. Student participation in the online discussions is graded with a rubric, which can be found in Appendix E. Additionally, solid participation results in bonus points awarded on quizzes that open many class meetings.

Class sometimes starts with a brief, five-minute quiz on the reading. As the semester progresses, the less students appear to be doing the reading, the more frequent the quiz openers. Most class work is done by students in groups of three to four students. Because the class is composed of sufficiently many pre-service teachers and other students with an adequate level of mathematical competence and confidence for the content of this course, less mathematically able students learn the needed mathematics just-in-time from their peers and their instructor as they engage in the in-class group activities as well as the online discussion forum. The class work involves mathematical exercises from the text or prepared by the instructor from other sources. Instructor mini-lectures fill in the gaps of content not addressed in the text, in the student presentations, and to provide the connections: the "big picture." Some instructor lectures are threads that connect different portions of the course, such as the various representations of completing the square over the millennia [5] or the use of approximation (numerical) methods used to solve equations [6]. The student group solutions to the historical mathematics problems are presented in the last third to half of the hour and forty-five minute class that meets twice each week. The plan is for students to do more talking than the instructor, and students seem to enjoy that.

Assignments

While the appendices contain the details, I briefly describe some of the graded requirements of the course here:

- Automathography: This is a one- or two-page autobiographical essay, assigned the first week of the course, in which students review their own personal history with mathematics.
- Graded homework: Some exercises from the text are submitted for a grade, and all assignments are to be written in complete sentences and in paragraph form.
- Original works essays: Students practice the essay writing skills learned in their foundational ITW course while writing mathematical content. The essays are prepared after group and classroom discussion, and students individually submit the essays.
- Historical biographies: Students prepare biographical essays and oral presentations on mathematicians of their choice (as described in Appendix C).
- Book review: This assignment (details are in Appendix D) requires students to read and review, both in written and oral presentations, a "popular" book with significant mathematics history content. The student chooses, but the instructor must approve, the book selection. This assignment is intended to turn students on to the vast array of books on the history of mathematics available to them as lifelong learners. The oral presentations are an opportunity for students to share with their peers, and they enjoy doing just that.
- Reflective summary: This is a "bookend" to the automathography in which the student reflects on the learning that took place in our course over the semester.

- Course project: This is usually an extended writing assignment that is the major effort of the semester. Videos and podcasts have been accepted as well. Students appreciate the structure of the assignment (Appendix B), which provides the scaffolding of multiple submissions to help students succeed in producing a quality final product.

Lessons Learned

I love teaching this course, and the feedback from the students has been positive. Students in recent years not only accept but almost expect the active-learning classroom environment. There will always be a few who prefer the entertaining and informative lecture, but most students find it challenging and enjoyable trying to think like their mathematical predecessors, working with others to do mathematics the way that the originators did rather than the way we have learned to do mathematics. The active-learning classroom enables them to do that.

The reading quizzes have revealed that students either are not reading, that their reading comprehension is minimal, or both. The quizzes always have fewer than five items that are straight from the reading assignment from the text, and example exercises are:

- List the five platonic solids.
- What British mathematician first used the Greek letter π to represent the ratio of the circumference to the diameter of a circle.
- Who was the author of the *Liber Abaci*?

My students generally score less than 50% on these 10-point quizzes. In response to student feedback, I have allowed them to have one page of notes they personally take from the reading available during the quiz. If nothing else, that activity motivates students to learn to take notes while they read the text! Even with the notes, most students do not earn a score at the A level.

One of the lessons learned is the value of group work. Over time I have learned to organize groups that provide built-in tutoring for those students who are not majoring in mathematics. Each group has at least a couple of mathematics students. I select the groups based upon the results of a survey administered at the beginning of the semester in which students identify their strengths and weaknesses in mathematics, writing, and speaking. There will always be some students who prefer to work alone, but most prefer the group learning opportunity. Students who register for the course to satisfy a graduation requirement but have no interest in mathematics usually drop the class within the first couple of weeks.

Over time, the value of rubrics has increased. All the assignments now have associated rubrics given to and discussed with students prior to the assignment due date. Rubrics are the key to grading efficiently, and they provide effective feedback that students are more likely to read. Rubrics are ever-evolving. One idea is to have students participate in the design of the rubric so that they have a sense of ownership, but I have yet to do that as much as I would like.

Another lesson learned is that students give good, sometimes great, oral presentations. I let students know up front that the feedback is constructive and intended to better their grade overall. The written portion of the assignments is due after the oral presentation so that students can incorporate the feedback received in their oral report to improve their written report. Some students initially have significant anxiety about presenting to their peers, but I consistently get positive feedback about the presentations. It is very gratifying to hear students say that they became more confident with public speaking as a result of a mathematics class! The students present early and often, and the quality of the presentations improve as the semester progresses. I find it important to lead by example, as I give a presentation of the type they are to give prior to their assignment. Some professors who are not willing to risk giving students control of their learning may choose to have students present to the instructor prior to the class attendance for quality control. It is my contention that students learn from both good and bad presentations.

Many of our students need help learning to write well, so I have learned to invite tutors from our Center for Writing to provide that assistance. The student tutors come to my classroom to remind students of the research, documentation, and writing skills they learned in their earlier composition course that is a foundation of the Integrative Studies Program. Two workshops at key times in the project report writing process are identified in the Course Schedule (Appendix A). The workshops take valuable class time, but the payoff in the quality of final submissions is worth the tradeoff.

Some of my writing assignments require students to rewrite a primary-source mathematical text into their own words. From my experience, students have a hard time reading primary sources, such as Euclid's *Elements*, because of the vocabulary used by the translators or original authors. For example, students are reluctant to take the initiative to look up words like *gnomon* that Heath uses. Teaching them how to use the internet to do a quick web search to find the mathematical meaning of unfamiliar terms as they do the reading gives them a skill they learn early that pays off later.

I find that utilizing the LMS greatly facilitates the administration of the course. The LMS's continue to evolve, providing great tools such as a feature in the discussion forum in which students have to make an initial post before they can respond/view other posts. Blackboard (Bb) and Canvas both have a very easy to use mathematical equation tool in the various student input windows. Video feedback to students is also available via Bb and Canvas, to include the capability for synchronous interaction with students, either individually or as a group, outside the brick-and-mortar classroom. As indicated in the Course Schedule (Appendix A), I have students frequently go to the course web page to obtain guidance, primary source readings, screencasts with instructions, etc. The contents of the course web page changes daily!

In summary, this is a wonderful course to teach as I selfishly get the gratifying experience of exposing not only mathematics majors but other students at our college to the history of mathematics. Faculty and students from other disciplines bring valuable alternative perspectives that broaden the experience for all of us. It is a course that is fun for both the instructor and the students because students have an appropriate amount of control over what they do in the course, and they learn to take responsibility for their learning.

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Appendix A
Evolution of Mathematics course schedule

COURSE SCHEDULE

Each module is the equivalent of a classroom attendance, twice weekly.

Module	Topic	Reading Assignment
1	Course introduction	None
2	Egypt and Mesopotamia Automathography due	B&G pp. 1–14 Sketches 1, 9
3	Greek mathematics I Book review selection due	B&G pp. 14–24 Sketches 7, 14
4	Greek mathematics II Project proposal due	B&G pp. 14–24 Sketches 12, 15
5	Original readings I: Euclid	Course web page
6	Library presentation and tour	Course web page
7	Original readings II: Euclid	Course web page Sketches 13, 18
8	Original readings III: Archimedes	Course web page
9	Original readings IV: Apollonius and Diophantus, Eratosthenes	Course web page
10	Mathematics of ancient India and Arabia Annotated bibliography due	B&G pp. 24–32 Sketches 3, 10
11	Mathematics of medieval Europe; Original readings V: Fibonacci	B&G pp. 32–34 Course web page
12	Mathematics of the 15th-17th centuries Project outline due	B&G pp. 35–42 Sketches 2, 4, 5, 8, 11, 16
13	Original readings VI: Cardano	Course web page
14	History of complex numbers	Course web page Sketch 17
15	Calculus and applied mathematics I Project draft due; Writing workshop	B&G pp. 42–47 Sketch 6
16	Calculus and applied mathematics II Original readings VII: Euler	B&G pp. 42–47 Course web page
17	Historical book review presentations	Course web page
18	American Mathematics in the age of Jefferson	Course web page

19	American biography presentations	Course web page
20	American biography presentations	Course web page
21	Course project presentations	Course web page
22	Course project presentations	Course web page
23	Mathematics history from a feminist perspective Non-Euclidean and projective geometry	Guest speaker Sketches 19, 20
24	Rigor and professionalism Writing workshop	B&G pp. 47–52
25	Abstraction, computers, new applications, and mathematics today Course project due	B&G pp. 53–60 Sketches 23, 24, 25 Course web page
26	History of probability and statistics	Course web page Sketches 21, 22
27	Original readings VIII: Probability/Statistics; course review	Course web page

Appendix B

Course project description

Evolution of Mathematics Course Project

1. The course project is *the* major effort in our course, is expected to be a significant product, and is worth one-third of the course grade.
2. The project can be done as an individual or group (two persons) effort. The product of a team effort will be expected to be more substantial than that produced by an individual and approved by the professor.
3. The project is due as indicated below, but there are intermediate results due on earlier dates. The final project submission will consist of both a written report and a 5- to 10-minute oral presentation. The scope of the written report depends on the nature of the project undertaken. Web pages, videos, podcasts, etc., are encouraged as part or all of the project submission.
4. Project proposals are due not later than the date indicated below. The proposal will include an overview of what you hope to accomplish in the project and your estimate of the nature of your final submission and presentation. The proposal should be at least a couple of paragraphs but not much more than a page in length.
5. It is important that your project be on a subject of interest to *you*. Suggestions for project topics, based on both my and your preferences, as I understand them, include
   - The role of religion on the development of mathematics
   - A historical and interdisciplinary perspective: Mathematics and Music, or Mathematics and Art
   - Women and mathematics: a feminist perspective on the evolution of mathematics
   - A psychological profile of some famous mathematicians
   - An assessment of the historical mathematical content of children's literature
   - Mathematics and the Maya (or Japan, or China, or Islam, or some other culture)
   - Contributions of a particular mathematician, for example, Alan Turing
   - Astronomical mathematics (spherical geometry/trigonometry) origins
   - Classroom module introducing history on a mathematical topic in the schools
   - Early American mathematicians
   - The mathematics of cryptography and code-breaking
   - The mathematics of the sun-dial
   - A topic from the history of mathematics education
   - A module incorporating the history of mathematics to advance learning of a school mathematics topic
   - Other topics approved by your instructor.

Course project milestones

Task	**Points**	**Due**	**Comments**
Project proposal	20	Module 4	Talk with your professor; Do the research!
Annotated bibliography	30	Module 10	Do the research and reading.
Project outline	50	Module 12	The more detail the better! Include the bibliography (not annotated) and additional works used/cited.
Project draft	50	Module 15	No changes in project subject or scope after this point. Include full citations.
Project presentations	50	Module 21	All submitted on the 6th.
Project submission	200	Module 25	Breathe a sigh of relief!

Appendix C
Biographical sketch assignment (partial listing)

Biographical Sketch Assignments

A graded requirement for our course will be the submission and presentation of a brief essay (biographical sketch) about a person prominent in the history of mathematics. The biographical sketch should contain information about the time period in which the person lived and worked, the major contributions of the person, and, if possible, any interesting (humorous?) anecdotal information. The written essay (sketch) should be at least a couple of paragraphs and preferably not more than one page in length, certainly not more than a couple of pages. The oral presentation should take 5–6 minutes of class time, certainly not more than 5 minutes. *Important*: both the written essay and the classroom presentation must contain and briefly explain a mathematical relationship, figure, or graph that is representative of the mathematician's contribution. Remember that your instructor is available for assistance in this endeavor, and full documentation of resources used is required. At least one web source and one print source is required for full marks.

Module Number	**Topic**	**Student**
3	Thales	
3	Pythagoras	
4	Eudoxus	
5	Euclid	
8	Archimedes	
9	Diophantus	
9	Pappus	
9	Hipparchus	
9	Ptolemy	
9	Hypatia	
9	Apollonius	
9	Eratosthenes	
10	Brahmagupta	

Appendix D

Book review assignment

Evolution of Mathematics Book Review

The purpose of this assignment is to get you started on the way to reading "popular" books involving mathematics, and in particular the history of mathematics. The specific assignment is for you to select a book, read the book, and then report on the book.

Your report will actually have two components: an oral presentation and a written report. The written report need not be more than a couple of pages long, though you may need more space to do the mathematical explanation. You may put that in an enclosure/appendix to your report if you wish. Attached is a checklist that I will use to grade the written report, so use that to ensure your report has the right stuff. The oral presentation will be 5-10 minutes in length, and should be well-rehearsed prior to presentation in class. A PowerPoint, or similar software product, presentation is suggested but not required.

In both the written and the oral reports, emphasis is placed on your reflection on what you learned about the history of mathematics while reading the book. Take care to carefully describe a specific mathematical example presented by the author(s) in a way that demonstrates that you fully understand the topic you chose. As always, drafts are welcomed so you can receive feedback on whether or not you are on the right track.

History Book Review Checklist

1. **Does this essay have the correct content?**

a. Brief overview of the mathematical theme of the book	0	3	6
b. Detailed explanation of an example of the mathematical content	0	3	6
c. Reflection on what was learned about the history of mathematics or the history of mathematics education	0	3	6

2. **Is the essay well-written?**

a. Clearly identifiable introductory and summarizing concluding paragraphs, with positive/negative recommendation	0	2	4
b. Well-organized with coherent sentence and paragraph structure, smoothly integrating the mathematical content	0	2	4
c. Minimal spelling, capitalization, punctuation and/or grammatical errors	0	2	4

General comments:

Book Review Oral Presentation Checklist

Name ______________________________________

	Points		
1. INTRODUCTION			
a. Introduce self	0	1	
b. Provide overview of talk	0	2	
2. PRESENTATION			
a. Brief overview of the book	0	2	
b. Mathematical content of the presentation	0	2	4
c. Reflection on the mathematics history learned	0	2	4
3. **CONCLUSION:** Summarize and recommend or not recommend the book with justification	0	2	
4. ORGANIZATION AND STYLE			
a. Timing (5–10 min)	0	1	
b. Quality of visuals	0	1	
c. Clarity of communication, eye contact	0	1	
d. Apparent preparation	0	2	
BONUS: Creativity, appropriate humo	0	1	2

General comments:

Appendix E

Discussion Board Rubric

Name ______________________________________ Score: ______ /10

Discussion Board Rubric[1]

Expectations: You will write an initial post to each discussion forum thread and you will reply to at least one of the posts of another student. A response needs to be substantive and have more depth than “I agree” or “you are wrong.” Your grade will be determined by your overall involvement in the forum and will be made up of three elements:

Critical Elements	**Distinguished (A)**	**Proficient (B)**	**Emerging (C)**	**Not Evident (F)**
Posting (20%)	You have an initial post and at least three responses.	You have an initial post and at least two responses.	You have an initial post and responded to one post.	You had no initial post and/or failed to respond to anyone.
Exploring and Answering (40%)	Your posts showed you answered the question; you demonstrated that you understood the material.	Your posts showed you answered the question; you did not completely understand the material.	Your posts showed you struggled to answer the question; you show minimal understanding of the material.	Your posts do not answer the question.
Writing (40%)	Your posts were well written with proper citations when necessary and contained no grammatical/spelling mistakes.	Your posts contained a few minor grammatical/spelling mistakes and some citations when needed but could be improved with editing.	Your posts were poorly written and included multiple spelling errors, and/or you failed to cite properly.	Your posts are incomprehensible due to writing flaws and you failed to cite.

General comments:

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[1] This rubric is an alteration of that used at Southern New Hampshire University

Themed History of Mathematics Courses for Humanities Students

Glen Van Brummelen
Quest University

Course Overview

Humanities students are sometimes the forgotten audience in initiatives to bring mathematics to the wider community. Often with issues in their high school mathematics background, they place themselves at the opposite end of the academic spectrum and seldom consider mathematics to play any role in their own pursuits. However, as history repeatedly reveals, mathematics and the humanities have had a rich interaction spanning millennia. Collaborating with colleagues in the humanities to introduce some of these episodes into the curriculum can be a transformative tool, changing deeply held attitudes and fears. Besides, it can be a lot of fun.

In this article we shall explore how one might design a themed course, possibly co-taught with a humanities professor. We consider how one might identify a theme, how one can work with colleagues and students with different perspectives than our own, and how one might prepare for the unique possibilities and challenges that these courses present. Generally, given the audience one cannot expect prerequisites, but this should not unnecessarily restrict one's ambitions; humanities students are often capable of more than we think.

Course Design

Possibilities for topics are endless. One might choose simply to teach the history of mathematics, but options dealing with specific themes might be more manageable and attractive to students. To begin, here are four examples of such courses offered recently at my institution; hopefully they will spark some ideas. Details on each course may be found in an appendix.

- ***Mathematics: A Historical Tour through the Great Civilizations*** (Glen Van Brummelen) — We devote the course to several units covering some of the major mathematical cultures (Egypt, Babylon, Greece, India, China, Islam, and Europe). We discover how mathematics shaped, and was shaped by, the people who practised it; and how it interacts with worldviews and alters ideas. Often the same mathematical topic is revisited several times, with a focus on how the same problem might be conceived similarly or very differently by various mathematical cultures.
- ***Euclid: The Creation of Mathematics*** (Glen Van Brummelen and Jim Cohn (literature)) — The *Elements* is the central textbook for this course; we cover all of Book I, and a sequence of propositions

that lead eventually to inscribing a regular polyhedron in a sphere from Book XIII. Themes include the need for and value of axiomatic reasoning, aspects of the *Elements* that are foreign to us today (horn/infinitesimal angles and ratio theory), the significance of non-Euclidean geometry, and the fundamental shift in mathematics caused by the introduction of Cartesian geometry.

- ***Mathematics and Music*** (Glen Van Brummelen and Laurel Parsons (music)) — This course can go in many directions. We began with the discovery of sound waves and how they give rise to musical phenomena. From there we transitioned into historical notions of tuning and temperament. Historical uses of mathematical structures used to generate music are explored, including serialism, the Fibonacci series, pitch class set theory, and rhythmic practices in global music traditions. Various interludes include the history of the music of the spheres, Hofstadter's *Gödel, Escher, Bach*, fractals, and change ringing.
- ***Information, Certainty, Knowledge*** (Ryan Derby-Talbot) — Many students believe that mathematics is a tool that, if applied diligently and deeply enough, can answer the mysteries of the universe in statements of absolute truth. Mathematics' claim to such certainty, however, is called into serious question when one begins to examine several hard-to-notice subtleties of the subject. Indeed, by recreating the observations made by the exceptionally keen eyes of several mathematicians in the past two centuries, we will encounter a host of irresolvable paradoxes and conundrums in mathematics, lurking in the shadows of ideas such as infinity, geometry, language, and the very nature of logic itself. These observations raise several large philosophical questions about the nature of knowledge and certainty.

Before you begin to plan your course, it is crucial to approach both mathematics and the humanities with a sense of mutual and collaborative appreciation.

Appreciating Mathematics: Mathematics is one of the most powerful intellectual forces that have shaped who we are today. We think the way we do because of Euclid. We have the power of modern technology because of Newton and Leibniz. Although mathematics is an essential tool in science, it represents philosophical method, is a historical presence, and has affected many practices in the fine arts. To become truly educated, humanists and mathematicians need to interact, just as scientists and mathematicians already do. The challenge is to convince humanities majors, usually those with the least taste for (or even fear of) mathematics, to choose to engage.

History is a natural entrance for these students; they are already attuned to the importance of understanding how we came to be who we are, the subtleties of literature and linguistics, the enlightenment of poetry and prose. Much more than in science and mathematics courses, they are accustomed to asking and debating big questions that shape worldviews. A mathematics course for humanities students has the potential to tap into these predispositions to gain an appreciation of the true spirit of the subject, just as liberal arts mathematics courses appeal to students with more practical goals.

The first step in attracting humanities students is to be clear *why* a mathematics course would be valuable to them. It is an enlightening and surprisingly difficult exercise to settle on a single clear response. Formulating and publicizing potential answers helps to shape the direction of the course and engage a skeptical audience. Here are some possibilities, together with historical themes that interact with them:

- Mathematics approaches knowledge in specific ways that have allowed the construction of some of the most powerful and comprehensive systems of thought ever devised (e.g., Euclid's *Elements*, the birth and development of symbolic algebra).
- Mathematics is a creative discipline that gives birth to algorithms that have shaped our world (e.g., ancient Babylonian solutions to quadratic equations, arithmetic processes in various cultures, the development of the notion of a derivative).

- Mathematics develops in response to needs across human experience, including the humanities (e.g., the rise of axiomatic thinking in the context of pre-Socratic Greek philosophy, the birth of logarithms to ease practical calculations for surveyors and astronomers, the invention of perspective geometry in Renaissance art, the use of group theory in musical composition).

Appreciating the Humanities: To be respectful of the interests and perspectives of the audience, the instructor of a course for humanities students should be aware of what makes them tick. One needs to go beyond pure mathematics, engaging also with the richness of historical methods that the students are learning in their other classes, and validating the significance of their own academic passions. As difficult as it sounds, engaging in a genuine and viable interplay between mathematical and humanistic concerns is necessary to make the course relevant, a meaningful experience rather than a short detour through an undesirable program requirement.

But what are the methods of inquiry in the humanities, and how might we educate ourselves to interact plausibly with them? The humanities are not structured as mathematics is; there are widely varying approaches and themes. Consulting with interested humanities colleagues at one's host university is a helpful way to familiarize oneself with the culture; conversations might lead to ideas for course themes and possibly even a teaching partner. Reading crossover books between mathematics and various humanities disciplines *from the perspective of the other discipline* (for instance, perusing reviews in humanities journals) can also enrich one's understanding of the questions that motivate them.

One categorization of methods and interests in the humanities, taken from a description of the foundation program at my home institution, reads as follows:

Culture: Situated knowledge and practice. Examination of a cultural context, theme, or phenomenon with attention to its geographical, temporal, and social context.

- This is a natural area to engage in cross-cultural comparisons between Western and non-Western mathematics. One historiographical debate that might make for an interesting class discussion is the issue of locality versus essence that has arisen in the study of the medieval Islamic mathematical sciences.[1]

Scholarship: A respect for and participation in conversation, arguments and research practices in the humanities. Identifying and addressing questions that are open-ended enough to be interesting but focused enough to be approached.

- One course in this category at my home university discusses the notion of a "canon" of art and literature. One might consider, for instance, whether the fundamental texts of algebra (with their changing aims and language) form a single canon across its development from Babylon and Greece through Islam and modern Europe, and whether a "canon" is a useful or a damaging concept to encapsulate algebra as a discipline.

Texts: Strategies for close reading; understanding of the classification and rules of genres; recognition of a multiplicity of interpretations and richness of texts. Awareness and integration of historical and cultural elements in textual analysis.

- One extended episode that might be worth a unit (or a project for a strong student) is a study of Euclid's *Elements* Book II, leading to the "geometric algebra" debate. Readings of this short text, originally interpreted by historians as a series of (implicitly modern) algebraic statements dressed up as geometry,

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[1] Abdelhamid Sabra's article "Situating Arabic science: locality versus essence" [12] laments the portrayal of Islamic science as a single monolithic entity, not respecting sufficiently the diverse subcultures and developments of ideas that occurred over a span of many centuries.

have since 1975 increasingly emphasized respect for the ancient context, helping to stimulate a trend toward a new historiography of mathematics. Two recent articles in the *Journal for Humanistic Mathematics*[2] take sharply different sides in this controversy. While the mathematical context would need to be approached carefully and perhaps simplified, the historical issues are accessible and ready to be debated.[3]

Choosing a Theme: A mistake made by many first-time teachers is to assume that one must "cover all the ground". Handling five millennia of history and all the major concepts of mathematics in one semester, to an audience often afflicted with math phobia, is a hopeless task. Building a course from one or more themes can be more effective and stimulate greater interest. The list of possible topics is endless. In addition to the courses listed at the beginning of this article, ideas may arise from the following areas:

- *Mathematics and the fine arts:* Dealing with questions related to visual representations, symmetries in art, and interactions between music and mathematical structures, this category lends itself to multiple choices and connects naturally with historical movements.
- *Mathematics and philosophy:* What is the relation between axiomatic reasoning in mathematics and analytic philosophy? How did both arise in ancient Greece, and how are they viewed today?
- *Mathematics and postmodernism:* Are mathematical objects real? To what extent is mathematics a social phenomenon?
- *Mathematics and literature:* Mathematical texts have been shaped by literary genres throughout history. Mathematics has occasionally provided creative spark for novelists and poets (e.g., J. L. Borges).
- *Women and minorities in mathematics:* How has political/social power discriminated for/against certain populations in mathematics? How have these minorities overcome barriers? *Have* they overcome the barriers?

Mathematical Preparedness: Depending on local conditions and how the course is advertised, students may exhibit a variety of quantitative skills ranging from poor to non-existent. In my course the mathematical "wall" is first encountered when we reach calculation with fractions; the next barrier is symbolic algebra. If your university has a quantitative reasoning requirement, it could be made a prerequisite for the course; if a learning commons is available, one-on-one tutoring is invaluable in addressing remedial issues. One can incorporate some remediation within the class by selecting simpler topics such as ancient number systems and arithmetic at the beginning.

Many of the issues humanities students face in mathematics derive from fear, leading to silence and a feeling of powerlessness. I invoke an engaged and questioning spirit in several ways. We spend one hour very early in the course discussing students' prior mathematical experiences (now known affectionately at my university as the "math group therapy session"). Students soon recognize that they are not alone in their anxieties, and they often form connections with others who share them. Simultaneously, almost without realizing it, they have begun to establish the ability to express themselves verbally in the classroom. Finally, a key component to building trust is for the instructor to accept that a remedial issue in basic mathematics does not reflect a lack of intelligence; humanities students who cannot add fractions may yet be every bit as bright as math majors.

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[2] The initial article by Victor Blåsjö [4] complains that the new historiography has formed an insiders' club of historians of mathematics who needlessly eschew modern representations of historical mathematics, while Michael Fried [9] argues that Blåsjö has misunderstood the nature of the criticism of older portrayals of *Elements* Book II.

[3] The current emphasis on the use of original texts in the history of mathematics classroom is a salutary outgrowth of the new historiography.

Classroom Practice: How one structures the classroom experience can vary; the following section provides strategies from the author's experience, and is not meant to be prescriptive. Humanities classes tend to involve seminar-style discussions, which are plausible only with a smaller number of students (say, 25–30 or fewer). With a larger group, it is crucial to find ways of breaking the class into smaller units to allow genuine conversations. If feasible, team-teaching the course with an instructor from the humanities can deepen the experience. Students learn much from the interaction between two instructors, especially if they come from different academic backgrounds. Also, there is a greater chance of expertise being present in the room when an unanticipated question is asked. The advantages of interdisciplinary instruction manifest themselves most strongly when the instructors discuss and debate with each other, so it is advisable for both to be present at all times. To avoid disciplinary specialization, both instructors should be prepared to engage in all aspects of the course, even if this implies entering an unfamiliar area or conversation. Observing how an instructor struggles with foreign patterns of thought and asks linking questions is a wonderful role-modeling moment for the student. In addition, the experience builds student-instructor trust: they are not being asked to take on a challenge that the instructor is unwilling to face herself.

Part of the challenge in teaching this sort of class is to mediate the differing dynamics of a mathematics versus a humanities classroom. Mathematics instruction tends to focus on the instructor as the purveyor of knowledge, as opposed to humanities classes that rely on reflections based on articles or books read in advance of the class. The resolution of this divide will depend strongly on the individual teaching styles of the instructors. If the course is team-taught, there is no reason why the instructors need to alter their natural teaching styles; as long as the course has a coherent narrative, students are not especially jarred (indeed, they are often engaged) by occasional changes in instructional method.

The type of material (mathematical or humanities-oriented) can also vary. Classroom exercises are especially helpful if they involve both mathematics and reflective thought. So, for instance, an activity based on levels of infinity might involve both an application of one-to-one correspondences between different sets (say, even numbers and whole numbers) and a discussion of historical implications (what Galileo intended to say by placing these sets into a correspondence). An activity that lives on only one side of the divide may be tolerable in the short run; a series of activities that never engage one of the two cultures undermines the course's purpose. Similarly, supporting readings should include both mathematical exercises and broader issues. This might require supplementing a standard textbook with readings from other sources.[4]

Resources

A themed course might emerge from one or more well-chosen books that provide a backbone for the narrative. There are dozens of possibilities. Choices for some of the courses and themes listed above include:

- *Music and Mathematics: From Pythagoras to Fractals*, eds. John Fauvel, Raymond Flood, and Robin Wilson (Oxford, 2003). This historically-oriented collection deals with diverse topics and provides many jumping-off points for student projects.
- *What is Mathematics, Really?*, by Reuben Hersh (Oxford, 1997). This iconoclastic historical tour of crises in mathematical thinking leans toward postmodern attitudes and is bound to stir up controversy. Some mathematical instruction and readings from different points of view might be needed to support the text.
- *A Certain Ambiguity*, by Gaurav Suri and Hartosh Singh Bal (Princeton, 2007). This mathematical novel weaves its plot around infinity and what it means to know something, mathematically or otherwise.

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[4] The textbook by Berlinghoff and Gouvêa [3] is especially appealing; the expanded edition contains a number of mathematical exercises, and many of its capsule articles deal with interesting issues. Collections of articles such as Swetz [13] are excellent sources for supplementary readings.

It has been used as a textbook for the Quest University course *Infinity, Certainty, Knowledge*. Consider also David Foster Wallace's *Everything and More: A Compact History of Infinity* (Norton, 2003).

- *The Unimaginable Mathematics of Borges' Library of Babel*, by William Goldbloom Bloch (Oxford, 2011). This book reprints Borges's famous short story and expounds on the mathematical themes embedded in the text. It should be supplemented with sources or reviews that emphasize literary interpretations.
- *Euclid: The Creation of Mathematics*, by Benno Artmann (Springer, 1999). A survey of the main themes of the *Elements*, with connections to other aspects of Greek mathematics and culture.
- *Complexities: Women in Mathematics*, edited by Bettye Anne Case and Anne M. Leggett (Princeton, 2005). A collection of biographical pieces, reflections on institutions and biases, and personal journeys. This is a good basis for dealing with issues, and could be supplemented with mathematical topics related to the research of the authors described in the book.
- *Squaring the Circle: Geometry in Art and Architecture*, by Paul Calter (Key Curriculum Press, 2008). A textbook on a variety of geometric topics, emphasizing historical and biographical connections in art and architecture.
- *Manifold Mirrors: The Crossing Paths of the Arts and Mathematics*, by Felipe Cucker (Cambridge, 2013). Emerging from a liberal arts mathematics course, this beautiful book emphasizes visual art but also extends into music.

Interesting places to look for alternate viewpoints include reviews of the book you have chosen for your main narrative. Along with conventional journal articles, the *Journal of Humanistic Mathematics* is a helpful and easily accessed source for supplementary materials. The usual searches of scholarly databases should help to supplement the main reading list. Be careful to choose articles and other readings that are suitable for your audience's level.

Assignments

In general, assessment should reflect pedagogy. In a genuinely interdisciplinary course, the standard mathematical problems one finds in textbooks should be interspersed with either short answers or essays extending the humanistic themes explored in class. Short response questions I have posed in my course on the mathematics of many cultures include:[5]

- In class we have noticed how different the Greek method of mathematical reasoning is, compared to its predecessor in Egypt. Altschiller-Court ("The Dawn of Demonstrative Geometry", [1]) argues that the birth of deductive geometry might not have been as abrupt as it appears. Do you find his argument convincing? Explain. How well does it square with what we have seen in class?
- The authors of "On Beauty Bare: Euclid's Plane Geometry", Berlinghoff and Gouvêa [2], argue that Euclid's and the Greeks' divorcing of geometry from the real world was a good thing, and that the modern teaching of geometry has lost something important by trying to make it practical. In the authors' view, what is the failing of modern geometry classes? Do you agree that this is a genuine loss? Do you feel that loss in your own educational experience? Explain.
- Critique the main argument (made in Van Brummelen [14], "Mathematical Truth: A Cultural Study") that religious belief affects mathematics through the influence of broad cultural, ultimately religious assumptions. Are you convinced? Why or why not?

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[5] See also Van Brummelen [15], which contains an extended exploration of the trigonometric methods of the medieval Venetian navigator Michael of Rhodes. The exercise ends with a consideration of the difficulties in deciding whether a historical mathematical passage maps directly onto a modern concept — an important issue in settling questions of priority.

- Some mathematicians have reacted negatively to claims of exploitation or suppression against women or racial minorities. They argue that mathematics is inherently separate from culture (2 + 2 = 4 whether you're male or female, European or African), and therefore that social issues do not belong in mathematics. This raises a debate about whether teaching methods for mathematics should differ for girls or children in racial minorities. (a) Do you think that mathematics teaching should be altered for particular cultural or gender groupings? Explain. (b) Do you think that mathematics itself is independent of cultural perspective?

Many of these questions require a form of evaluation different from standard mathematical assignments; often there is no correct answer. Submissions should be graded based on their depth of critical thinking, the extent to which they use evidence (from class or a reading), and clarity of expression. Questions of this sort can also be developed into essays consisting of sustained research and argument, skills that come naturally to humanities students. Longer projects are a common feature of humanities courses, taking the form of essays or artistic statements of various sorts. Projects allow students to personalize the course to their particular interests, although for many students a list of possible projects helps to give students some traction. It should be made clear that, whatever the project topic, students should engage in a substantial way with mathematics. They may need some guidance or some examples on how to accomplish this.

Lessons Learned

Especially in the initial offering of a course, it is difficult to anticipate what sorts of psychological issues might arise. In our course on Euclid's *Elements*, for instance, randomly chosen students were to present propositions at the whiteboard on their own without written aids. This method is used effectively at St. John's College, but the audience at Quest University (in their freshman year) was deeply divided with respect to confidence levels. While about half the class benefited greatly from the intense environment, the rest experienced moments of terror in front of the class and were unable to function. To address the problem, we divided the class into two groups, each instructor taking about ten students. The "weaker" group worked in a more collaborative and supportive manner. While they built their confidence in a gentler setting, by the end of the course they achieved close to the same level of success as the other group. The moral of the story: be aware of how students are reacting, and consider alternate pedagogical strategies if class morale goes downhill.

It is also important to make clear from the very beginning the level of mathematics expected in the course. Some students might expect a humanities-oriented mathematics course to be a narrative *about* mathematics but not *containing* mathematics, while others might expect a significant analytical challenge. It might be worth exhibiting typical assignment questions on the first day, so that students who hoped for something different may make other plans.

Finally, be aware that students coming from the humanities may have an extremely limited understanding of what mathematical reasoning is. For many, mathematics has been a set of algorithms to be memorized and performed, not a lively collection of beautiful arguments. Time should be devoted to helping students understand the nature of mathematical proof. Phrases we use every day almost without thinking about them, such as "for all x" or "if y, then z", need to be unpacked explicitly and in intuitive terms for your audience. Pay close attention at all times to your use of language (both theirs and yours), recognizing that when you say something, students might hear something else.

The challenges in teaching a history of mathematics course for the humanities are many, but the payoff is large. Students who previously felt that mathematics had little or nothing to offer them and their interests can have their perspectives and attitudes transformed. Since some of these students will become writers, teachers, or even public figures, the gift of a restored and healthy relationship with mathematics has the potential to spread far beyond the classroom. It is well worth our attention.

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Appendix

Four Course Descriptions

The courses below have been taught at Quest University, two with a single mathematics instructor, two co-taught with a humanities instructor. For general descriptions of these courses, see the beginning of this article.

Mathematics: A Historical Tour of the Great Civilizations

Topic list

Topics in this course vary from year to year. Other cultures are often covered in projects and short presentations.

- Introduction: The nature of mathematics
- Beginnings in pre-history
- Ancient Egypt (number systems, geometry, proto-algebra)
- Ancient Babylon (computation, geometry, quasi-algebraic problems, mathematical astronomy)
- Ancient Greece (the beginning of deduction, the evolution of Euclid's *Elements*, Archimedes, mathematical astronomy)
- Medieval Islam (inheritance/appropriation of methods and styles of reasoning from Greece and India, their manifestations in arithmetic and algebra; geometry, trigonometry, and astronomy motivated by religious concerns)
- Pre-modern China (sources for the emergence of interest in mathematics, types of justification and proof, geometry and practical problems)
- Europe (rebirth of learning; appropriation from ancient Greece and medieval Islam; symbolic algebra; analytic geometry; calculus)

Assessments: Five written assignments, mostly mathematical but some historical essay questions; student presentation on a historical topic that might convey an important message about mathematics to a wider audience; a large final project (examples: the Pythagorean Theorem in different cultures, different perceptions of the number zero, pre-modern mathematical and astronomical instruments, what does it mean to "know calculus" in different times and cultures?).

Infinity, Certainty, Knowledge

Topic List

- What is knowledge? (introduction to the course)
- What makes us certain of something? (observation and certainty, inductive and deductive reasoning)
- What is the nature of infinity? (cardinality, Cantor's diagonal argument, Cohen and the continuum hypothesis, infinite series)
- How does mathematics correlate with the "real world"? (mathematical models, Ptolemy's model of planetary motion, chaos theory)
- What is the nature of space? (Euclid and axiomatic-deductive reasoning, the rise of non-Euclidean geometry)

- To what extent is mathematical knowledge certain? (discussion of the novel *A Certain Ambiguity*)
- How does language relate to knowledge? (speech acts, Heidegger's notion of "world", blind spots)
- What are the limits of logic? (Gödel and the incompleteness theorems, probabilistic methods and certainty)
- What is knowledge? (revisited) (course discussion and completion)

Assessments: Three mathematical assignments (on infinity, geometry, and logic); three writing assignments on philosophical topics (3-5 pages in length); a project/class presentation (7-10 pages; examples: the axiom of choice, Turing's halting problem, the history of infinity, the discovery of complex numbers and quaternions)

Euclid: The Creation of Mathematics

Topic List

- Definitions, postulates, common notions (the origins of axiomatic thought)
- Book I, first half (hidden postulates and issues; Euclidean constructions in Islamic art)
- I.27 – I.32 (the parallel postulate and non-Euclidean geometry)
- I.33 – I.48 (areas in Greek geometry; the Pythagorean Theorem)
- Parts of Book III (horn angles, infinitesimals, and reactions in Europe; uses in Greek astronomy)
- Parts of Book V (Euclid's ratio theory)
- Parts of Books VI, XI, XIII (solid geometry, the Platonic solids)
- The paradigm shift to Cartesian geometry from the 17th century

Assessments: Four written assignments, geometrical constructions and proofs but also some historical essay questions; in-class theorem proving on a daily basis (prove a particular proposition at the whiteboard); a medium-sized final project (examples: geometry and Plato's divided line, Japanese temple geometry, Max Bill's Euclidean art, axiomatic development of geometry through origami, the Kantian exposition of space).

Mathematics and Music

Topic List

- The mathematics and physics of sound (sound waves, Fourier series, beats, overtones, harmonics)
- Tuning and temperament (Pythagorean, just, equal)
- Mathematics and musical composition
- Pitch class set theory, serialism, and the Fibonacci series
- The history of the music of the spheres
- Music and self-similarity; fractals and the Cantor set in music
- Rhythm and meter
- Permutations and change ringing

Assessments: Three mathematical problem sets, three musical analysis/composition assignments, a final project (examples: Birkhoff's theory of aesthetic measure, 12-tone analysis, Helmholtz's theory of consonance, inversional symmetry in 20th century music).

Mathematics and Drama in Ancient Greece

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Course Overview

This interdisciplinary course investigates the commonalities and the differences in the work of the mathematician and the dramatic artist through the study of the ancient Greeks. We explore the ideas of the ancient Greeks in both disciplines, connections to modern work, and points of overlap between mathematics and drama. The course description articulates these goals more fully.

> *Official Course Description*: We examine the dramatic arts and mathematics from the perspective of the ancient Greeks as well as how they intersect in modern times. The peoples of ancient Greece utilized both disciplines to process and understand their changing world. Ultimately they transformed both drama and mathematics, and created many fundamental elements and practices that continue to influence the modern age. We study the story of ancient Greek developments in drama and mathematics, focusing on the history, the culture, and the detailed questions and techniques that arose in Athens, Epidaurus, Delphi, Samos, and Delos. We examine specific innovations and events that gave rise to practices that remain relevant in the exercise of both disciplines today.

We have taught this course in Greece during a three-week January term, but we also share our ideas for teaching this course during a full term in the United States. We designed this course to be accessible to students majoring in either mathematics or drama, which essentially makes it accessible to any student. We attract roughly one-third mathematics-oriented students, one-third drama-oriented students, and one-third "I want to go to Greece"-oriented students. As might be expected, some students are somewhat drama-phobic, while others are somewhat math-phobic. We intentionally foster a supportive academic community throughout the course, enabling our students to push their boundaries and grow both intellectually and personally. Because we are open to and seek a broad audience, we do not require any prerequisite courses.

Course Design

Inspirations and Aspirations

We were inspired to create this course through the convergence of two factors. First, Hallock is a Professor of Dramatic Arts and McAllister is a Professor of Mathematics. We are good friends and we discuss the ideas, perspectives, and experiences of our respective fields with each other. Through conversations about our disciplines, we came to recognize certain commonalities in the working processes of the mathematician and the dramatic artist. For example, we both affirm the following "process of creation" articulated by the English social psychologist Graham Wallas [46].

1) Preparation: when we investigate the problem from all possible directions.
2) Incubation: when we allow the question to gestate, to percolate, while we do other things or deliberately pursue other activities.
3) Illumination: when the pieces fall together in new and possibly unexpected ways.
4) Verification: when we work out the details and test a new idea by ourselves and/or through peer-review.

Such conversations stirred up our interest in the possibility of team-teaching a course exploring the similarities and the differences between mathematics and the dramatic arts. We decided that the ideas and works of the ancient Greeks would provide a fertile ground for this study.

Second, Centre College's academic calendar includes a three-week January term during which students take a single course. Centre's professors specifically design courses to take advantage of the focused learning environment provided by this short term, and we offer both on-campus and study abroad courses. We independently became interested in teaching a January-term study abroad course and we decided to join forces on this project.

While we originally designed this course for our three-week January term on-site in Greece, Mathematics and Drama in Ancient Greece could be taught during a full term off-site in the United States. Essentially every core topic can be studied and learned in a standard classroom setting. In fact, teaching this course in the United States would remove the constraints of a particular itinerary, providing complete freedom in sequencing the topics. Because we do not require any prerequisites, we are confident that a revised version of this course can be taught in many different settings, including first-year seminars, special topics courses, and honors programs. We provide one model for such a course in Appendix A.

Throughout Mathematics and Drama in Ancient Greece, we aspire to interweave the ideas and perspectives of mathematics and the dramatic arts to create a deeply enriching educational experience for our students. The course focuses on the ideas of the ancient Greeks in their particular cultural, intellectual, and political context. In addition, we discuss the ways that these ideas have informed and influenced the practice of our disciplines through the centuries to modern times. For example, the deductive practice of axiomatic mathematics traces its origins to the ancient Greeks, as do many interesting mathematical results from the 19th and 20th centuries. Similarly, many elements of ancient Greek theatre are present in modern dramatic arts. At regular intervals during the course, we also explore the similarities and differences among the processes central to our fields of study.

Ultimately, our goal is to create a course that embodies the best of the liberal arts. We ask everyone to be a Renaissance person of some sort and to view ideas and experiences from multiple perspectives. Our course draws students from all academic backgrounds, adding depth to the central message of interconnectivity. We find something thrilling and meaningful about students majoring in biology, dramatic arts, English, and mathematics working together to understand the acoustics of a theatre, geometries without parallel lines, and Aristotle's take on both logic and tragedy. In this way, we provide an opportunity for students to see core similarities among their different disciplines as they bring their different skill sets to bear on common questions time and again.

This inherently interdisciplinary course raises the question of whether students could receive credit toward the mathematics major. Centre College's Mathematics Department decided to allow credit for a 200-level elective, with the understanding that a mathematics student's Major Presentation would address a robust topic from a mathematics-oriented portion of the course (more details below). A related question is whether students could receive credit for the mathematics component of our general education curriculum. We chose not to allow this credit, in large part, because the other courses satisfying this general education requirement incorporate significant computational work, while Mathematics and Drama in Ancient Greece does not. While these choices work well at Centre College, we understand how other institutions might reach different conclusions about such a course.

Topics of Study

Roughly speaking, one-third of the course topics are mathematics-related, one-third are drama-related, and the remaining third are more generally about ancient Greece. We also explore points of intersection between mathematics and the dramatic arts. We teach a couple of topics from our respective disciplines and tour guides provide a great deal of information about ancient Greece through multiple on-site tours. However, our students present most of the course content in their Major and Minor Presentations. Details about these presentations are provided in the Assignments section below. Through our individual and corporate brainstorming, we developed the following three lists of potential topics of study.

Mathematics-related topics include: the distinction between inductive and deductive reasoning; Babylonian and Egyptian mathematics; ancient number systems; Thales of Miletus and his geometry; Pythagoras and his mathematics; Euclid; Euclidean geometry; geometric constructions; impossible straightedge and compass constructions (doubling the cube, squaring the circle, trisecting the angle, and the Gauss-Wantzel theorem); alternative tools for geometric constructions; non-Euclidean geometries; Socrates and his method in Plato's *Meno*; Zeno's paradoxes and possible resolutions; Cantor and transfinite cardinalities; Archimedes of Syracuse; the Tunnel of Eupalinos; Aristarchus of Samos and Greek astronomy; Aristotle on logic; Gödel's incompleteness theorems; the architecture and restoration of the Parthenon; the golden ratio; and the acoustics of the Theatre of Epidaurus.

Drama-related topics include: the life and work of important playwrights, including Thespis, Aeschylus, Aristophanes, Sophocles, and Euripides; Aristotle on tragedy; the peripatetic school; the tragic hero in Greek drama; Mycenae and the House of Atreus; the architecture of Greek and Roman theatres, including the Theatre of Dionysus, Odeon of Herodes Atticus, and the Theatre of Epidaurus; the festival of City Dionysia; the tradition of procession; Greek masks; the role of the chorus; and dithyrambs.

General topics include: the Acropolis; the caryatids of the Erectheion; the Agora; the Stoa of Attalos; ancient Greek legal systems; the origins of democracy at the Pnyx; bartering and fair division; the transition from barter to currency; the Greek's holistic approach to wellness at the Aesklepion; Aegean navigation; seafaring, fishing, and trade; the Pythian games, both ancient and revivals; Greek constellations and navigation; the canal of Corinth; and Greek myths.

We also aspire to create an interdisciplinary course that explores the commonalities and the differences in the work of the mathematician and the dramatic artist. As might be expected, identifying these topics has proven challenging, but also quite interesting and deeply engaging. We co-teach these interdisciplinary topics and our preparation process for these blended classes involves very lively, stimulating conversations.

Interdisciplinary topics include: working with written texts; the creative process; the Parthenon; its construction, symmetry, and City Dionysia; acoustics and the architecture of theatres; astronomy, constellations, myths, and story-telling; and the interplay between Apollonian ideals focused on reason and Dionysian ideals focused on emotion.

As an example of how we co-teach an interdisciplinary topic, we detail our class plan for studying the commonalities in how we "work with written texts." We first create groups consisting of three students, each of which includes at least one mathematics-interested student and one drama-interested student. McAllister states the definition of two sets having the same cardinality in terms of one-to-one, onto functions and the students in each group work toward the goal of each person developing an understanding of this definition.

After 10–15 of minutes of collaboration, we ask them to describe the experience of trying to understand this definition. During the 2015 rendition of the course, the students identified the following aspects of their process:

> creating a common baseline of vocabulary and glossary, recording words and ideas, zooming in and out on the words and ideas, repeating the ideas to themselves and out loud to the members of their group, reflecting on and mulling over possible interpretations, stating examples and non-examples, sketching pictures and diagrams, developing metaphors (one group used animal sounds), working through the definition word-by-word, and rewriting the definition in their own words.

Hallock then shares the following, familiar passage from *Hamlet* [39].

> To be, or not to be, that is the question: whether 'tis nobler in the mind to suffer the slings and arrows of outrageous fortune, or to take arms against a sea of troubles, and by opposing, end them. To die, to sleep—no more, and by a sleep to say we end the heart-ache, and the thousand natural shocks that flesh is heir to. 'Tis a consummation devoutly to be wished. To die, to sleep—to sleep—perchance to dream: aye, there's the rub, ….

After one student reads this excerpt to the class, we ask each group (already assigned above) to rewrite the passage in their own words. After 10–15 of minutes of collaboration, we talk through the passage one line at a time with the groups sharing their rendition of each line. After this intensive engagement with the text, our student rereads this excerpt to the entire class and we discuss the difference in how the text is read and heard. We wrap up by revisiting their description of working with the definition of cardinality and exploring the similarities to their work with the excerpt from *Hamlet*. We also compare and contrast the unique interpretation of a mathematical definition (in a given context) with the breadth of possible interpretations of a passage from a play.

Connections with More Modern Mathematics

Throughout the course, we draw connections between the works of the ancient Greeks and those of more modern mathematicians and dramatic artists. Some specific mathematical examples of such connections include the following.

- McAllister continues the discussion about "working with written texts" by exploring the challenge inherent in creating precise mathematical definitions. To illustrate this challenge, McAllister has two students leave the classroom and instructs the remaining students to "define" a supplied image of a "little, green alien." Once their definition is complete, the two students return to the classroom and, working independently, they attempt to redraw the original image based on their peers' newly-crafted definition. As might be expected, the results are decidedly mixed and the students have fun laughing together at what was drawn. We then discuss both sides of the experience: creating the definition and interpreting what is written. McAllister has added this activity to his transition to proofs class to enhance those students' understanding of mathematical definitions.
- As part of our study of Euclidean and non-Euclidean geometries, we discuss the multiplicity of models that can satisfy a given set of axioms. In addition to recalling Euclidean plane geometry, the students explore the sphere model of elliptic geometry by means of beach balls and markers, and the Beltrami disk model of hyperbolic geometry. The real-life applications of elliptic geometry to airplane routes and whale communication, and of hyperbolic geometry to Einstein's theory of special relativity are particularly striking to the students, and counter some students' initial view that non-Euclidean geometries are esoteric and irrelevant.

- We study multiple proofs of the Pythagorean Theorem, including Euclid's original argument as well as algebra-based arguments such as President James Garfield's proof. Among other things, we discuss how various proofs of a given result can provide diverse perspectives and points of view, and so deepen our understanding of mathematical truths.
- In the context of cube-doubling, we discuss existence versus construction arguments and the idea of proving the non-existence of a solution. We also highlight the intellectual flexibility needed to develop diverse solutions of a mathematical question. The students appear intrigued by the Gauss-Wantzel approach to resolving the cube-doubling question as well as the alternative solutions provided by the cissoid of Diocles, the conchoid of Nicomedes, and the paper-folding of Row.
- We study Zeno's paradoxes of motion, Aristotle's distinction between actual and potential infinities, Archimedes' near-development of the calculus, and Cantor's theory of transfinite cardinalities. We discuss the definition of cardinality in terms of one-to-one, onto functions and apply this notion to various infinite countable sets of numbers. We also explore the multiplicity of infinite cardinals and the impossibility of proving or disproving Cantor's continuum hypothesis.
- We discuss Aristotle's logic and its 19th century extension to propositional and predicate logic by Boole and Frege. Building on these ideas, we examine the basics of Gödel's incompleteness theorems, including self-reference, arithmetization of syntax, and the (informally phrased) Gödel sentence "I am not provable."

Making Choices

We incorporate only some of these many interesting topics into each rendition of the course because of time constraints. While we are free to study essentially anything we want, we cannot study everything we want and we have had to choose among these many competing goods. As part of our planning process, we revisit our lists of possible topics, reexamining our past choices and thinking about what collection of topics will work well for the next rendition of the course.

One factor in choosing topics is the personal interests of our diverse group of students, especially because of their significant role in teaching the course content. Generally speaking, we ask mathematics-oriented students to teach mathematics-themed topics and drama-oriented students to teach drama-themed topics, although some students ask to cross-over between disciplines. In addition, we offer students the opportunity to suggest their own topic. The last time we taught this course, a student planning to attend law school asked to present on "Law and Order in Ancient Athens," which proved to be an interesting and timely topic that we were able to incorporate into the course syllabus.

Another Option

While planning and implementing Mathematics and Drama in Ancient Greece, we also discussed developing a course focused around plays with high mathematical content or with mathematical themes interwoven into the story in some essential fashion. While we did not choose to follow up on this idea, we mention it here in case others might be interested. Three such plays are *Arcadia* by Tom Stoppard [45], *Copenhagen* by Michael Frayn [18], and *Proof* by David Auburn [6].

Resources

Our students present most of the course content. More information is provided in the Assignments section, but to offer a couple of details: each student is assigned a major research topic, they conduct some initial research and share an outline of their results with us, after which we provide feedback that may include directing them to additional resources.

For mathematics-related topics, the following books and websites are good, general purpose resources for an initial exploration of these mathematical impossibilities. Both faculty and students are encouraged to begin their study of a particular topic with these sources. They also include references that enable further, more thorough investigations.

- Boyer and Merzbach's *A History of Mathematics* [8]
- Dunham's *Journey Through Genius: Great Theorems of Mathematics* [12]
- Heath's *A History of Greek Mathematics, Volume I* [22] *and Volume II* [23]
- Katz's *A History of Mathematics: An Introduction* [27]
- *Complete Dictionary of Scientific Biography* at www.gale.cengage.com/ndsb
- *Encyclopedia Britannica* at www.britannica.com
- *The MacTutor History of Mathematics Archives* at www-history.mcs.st-and.ac.uk
- *Stanford Encyclopedia of Philosophy* at plato.stanford.edu

In addition to the preceding general sources, additional text resources for particular mathematics-related topics include the following.

- Thales of Miletus, Pythagoras, and their mathematics: Heath [22] is a good source of information about these mathematicians and their work.
- Euclid, Euclidean geometry, and geometric constructions: Heath [14] is an English translation of *Elements* that includes biographical information about Euclid.
- Non-Euclidean geometries: Gray [19] provides a good, general-interest exposition of the development of non-Euclidean geometries. I like the presentation of Euclidean geometry and non-Euclidean geometries given by Greenberg [20]. Davis and Hersh [10] presents the idea of negating portions of Playfair's Axioms as a path to obtaining hyperbolic and elliptic geometry. Antonick [2] describes an application to whale communication.
- Impossible geometric constructions: Jones, et al. [26] and Kazarinoff [28] prove the impossibility of these geometric constructions and Cajori [9] is a good source of information about Wantzel's contributions. Heath [22] details alternative construction tools of the ancient Greek mathematicians and Row [36] presents paper-folding solutions.
- Plato's *Meno*: Grube [33] is an English translation of *Meno*.
- Zeno's Paradoxes: Heath [22] presents Zeno's Paradoxes and Sainsbury [38] discusses multiple paradoxes, including both Zeno's and Russell's paradoxes.
- Infinity: Maor [30], Rucker [37], and Smullyan [40] are good, general-interest books with helpful ideas for teaching infinity, and Dunham [12] is a good resource for understanding Cantor's work with transfinite cardinals.
- Archimedes of Syracuse: Stein [44] is an accessible presentation of Archimedes' work.
- Tunnel of Eupalinos: Apostol [3] discusses the history of this aqueduct.
- Aristarchus of Samos: Heath [24] presents the work of Aristarchus' predecessors as well as his development of a heliocentric hypothesis.
- Aristotle on logic: Smith [4] is an English translation of *Prior Analytics* and Heath [25] provides detailed descriptions of the mathematics discussed in Aristotle's works. Nagel and Newman [31] is an excellent introduction to Gödel's incompleteness theorems and Rucker [37] includes a discussion of Gödel's results.

- The Parthenon: Rhodes [34] is a descriptive overview of the Acropolis and Neils [32] provides a scholarly study of the Parthenon.
- The Golden Ratio: Livio [29] presents a general-interest survey of the history of Phi.
- The Theatre of Epidaurus: Ball [7] presents a nice summary of the Declerq and Dekeyser study [11] of the acoustics at the Theatre of Epidaurus.

Some helpful resources for drama-related topics include the following.

- Aristotle's *Poetics*: Janko [5] is an English translation of *Poetics* and some other works.
- Greek Tragedy: See Easterling's *The Cambridge Companion to Greek Tragedy* [13].
- Greek Myths: Hamilton [21] is a thorough resource.
- Plays of the ancient Greek tragedians: *The Cambridge Translations* are excellent.
  - Aeschylus: de May's translation of *Agamemnon* [1].
  - Sophocles: Franklin's translation of *Antigone* [43], Affleck's translation of *Oedipus Tyrannus* [41], and Dugdale's translation of *Electra* [42].
  - Euripides: Harrison's translation of *Medea* [16], Franklin's translation of *Bacchae* [15], and Shaw's translation of *Hippolytus* [17].

In addition to these print and web resources, the following videos are appropriate for both a January-term course in Greece and a full-term course in the United States.

- *Empires—The Greeks: Crucible of Civilization*, ASIN: B0007KIFUA, PBS, Boston, 2005.
- *Gospel At Colonus*, ASIN: B001B1Q3EO, New Video Group, New York, 2008.
- *Secrets of the Parthenon*, ASIN: B0013XZ6I4, PBS, Boston, 2008, at www.pbs.org/wgbh/nova/ancient/secrets-parthenon.html.

For our January-term study abroad course, the students watch and discuss at least one of these videos as part of our pre-trip preparation process. For a full-term course, these videos can be assigned for homework and discussed during class when the corresponding topics are studied during the term.

Finally, teaching a study abroad course involves many logistical details, a number of which we address in Appendix B.

Assignments

Our students provide a great deal of course content. They complete a number of assignments prior to and during the course to foster intentionality in seeking a rewarding study abroad experience, to create a rich and rigorous academic environment, and to enable both their personal growth and their development as scholars. Assignments identified as being due "prior to leaving for Greece" are submitted electronically to both professors two weeks before our departure. For the 2015 course, the students submitted these assignments by December 22nd, in advance of our departure on January 4th. These assignments include

- Articulation of Goals
- Fact Outline for their Major Research Topic
- A detailed plan for the Engaged Activity component of their Major Presentation
- A script for their Minor Presentation about a deity from Greek mythology

We require this early submission because of limited access to resources and limited time for research while in Greece. The students have time to complete these assignments after the Fall term has ended and we have time to critique their work before leaving for Greece. After reviewing their submissions, we provide feedback and ask students to revise and resubmit their work if necessary. This process helps ensure high quality academic content throughout the course. Our students uniformly express gratitude for this early submission process.

During a full-term course, we would require these same submissions from our students, only we would distribute the submission dates throughout the term based on when students were giving their presentations. For a full-term course, we would still incorporate sufficient lead time to enable the feedback cycle described above to ensure high quality academic content.

Articulation of Goals: Prior to leaving for Greece, each student submits a half-page paper stating and describing at least three academic goals that are directly related to the course content. We encourage focused, detailed statements of goals accompanied by some reflection on each goal. During a full-term course, students could be asked to submit such a paper during the first week of classes.

Major Research Topic: Each student conducts an independent research project on an assigned major topic chosen from among the core ideas of the course. This research is shared with the entire class by means of a Facts Outline and a Major Presentation given at an appropriate time during the course. In addition, each student submits a research paper about their topic. With relatively minor modifications, a version of this Major Research Project with its three components of oral presentation, engaged activity, and research paper would certainly work quite well for a full-term course in the United States.

1) **Fact Outline:** Prior to leaving for Greece, each student creates an outline of 10 to 12 important facts central to their topic. The professors compile these Fact Outlines into a single document and then distribute paper copies to the students upon our arrival in Greece. For a full-term course, this Fact Outline could be compiled throughout the term and a regularly updated version shared with the students electronically, perhaps through a course website or local learning management software.
2) **A Presentation:** For our study abroad course, Major Presentations are required to be 75–90 minutes long. For a full-term course in the United States, they can be appropriately scaled to the length of a standard class period. A Major Presentation provides an overview of the topic, fleshes out the important details from the Fact Outline, leads the class through an engaged activity, and concludes with a question and answer session. These presentations require intensive preparation before leaving for Greece, but are also informed by the students' experiences in Greece. In the Fall term prior to leaving for Greece, we distribute an itinerary and syllabus that include a likely date for each student's Major Presentation, with the standard caveat that these dates may change based on unexpected events. For a full-term course, this schedule would be included with the syllabus or developed during the first week of the term to allow the students to help choose their topic. The two components of each Major Presentation are:
    a) **An Oral Presentation:** These lectures may include physical demonstrations, handouts, or whatever other communication media allows for the greatest effectiveness. For a study abroad course, PowerPoint and other electronic media are not readily available.
    b) **An Engaged Activity:** Each presenter leads the group through an activity designed to provide a practical and/or kinesthetic experience with the goal of helping everyone develop a greater clarity and personal understanding of the topic. Examples of activities include role-playing excerpts of plays, creating physical models and tableaux, sketching non-Euclidean lines and triangles on a balloon or beach ball, creating and using an inclinometer, physically acting out Zeno's Paradox of Achilles and the Tortoise, and a paperfolding construction of the cube root of two. Prior to leaving for Greece, each student submits a plan for their Engaged Activity. During a full-term course, this plan could be submitted for feedback a week prior to their presentation and activity.

3) **A Written Report:** This research paper includes a four-page presentation of content, a one-page reflection on their Major Presentation, an outline of the Engaged Activity, and a bibliography of their resources. An electronic copy of this report is due two days after the Sigma-Delta Session addressing their presentation (as detailed next).

Sigma-Delta Sessions: After four or five Major Presentations, we divide the class into small groups and they identify two sigmas (or positives) and two deltas (or changes) for each presentation. McAllister chose the label "sigma" as a play on the sigma-notation Σ for the addition operation and our use of "+" for positive numbers, while the use of "delta" for change is drawn directly from calculus. The goal of these sessions is to provide detailed peer feedback to help each student reflect on and improve their oral communication skills. This feedback can address academic content, clarity of the ideas, mechanics of presenting, or the speaker's presence and engagement with the class. For our study abroad course, the students share the sigmas and deltas in a public forum, with the stipulation that each presenter will listen attentively, but not verbally reply to the feedback. We would incorporate some version of the Sigma-Delta Sessions into a full-term course. As an example, for his abstract algebra and real analysis courses, McAllister's students give one or two presentations, after which the rest of the class completes an online Sigma-Delta Survey that is reviewed by the professor and then shared with the presenter individually (rather than in a group setting).

Trivia Contests: In concert with each Sigma-Delta Session, we run a trivia contest based on the Fact Outlines from every Major Presentation up to that point in the course. During these trivia contests, the students tend to be ferociously and playfully competitive. These contests strengthen their memory of key facts and ideas from each presentation and provide us an opportunity to highlight the distinction between understanding and remembering. These Trivia Contests arose from the intense, immersive experience of a study abroad course and may not work as well for a full-term course in the United States, although perhaps the reader might think of some ideas.

Minor Research Topic: Each student serves as the local resident expert on an assigned Greek deity, creating a four-minute long informational speech about their deity. These speeches are called for (without prior notice) at appropriate points in the course with most being given during an on-site tour. Most students create a pocket guide for reference during their speech. Prior to leaving for Greece, each student submits a script for their speech. For a full-term course, we would have students give their Minor Presentations at the beginning of a class with at most one on any day of class. In addition, we would have each student give their Minor Presentation prior to their Major Presentation, allowing us to provide some initial, low stakes feedback on their oral communication skills.

Class Engagement: For our study abroad course, we evaluate each student on their level of engagement in the course every six to seven days; for a full-term course, we would evaluate students in this area approximately every four weeks. We rate our students on the seven criteria of promptness, attentiveness, engagement, questions, contributions, group work, and conduct/citizenship. We expect the students to really *show up*: to be attentively present, and to ask questions of each other, of us, and of the tour guides. This feedback process expresses our high academic expectations in regard to their active engagement in all aspects of the course. We find that these evaluations (particularly the first one) play an important role in elevating the intellectual climate of the course.

Votive Offering Event: In ancient Greece, a votive offering was a central element of visits to the Oracle of Delphi and to the Temple of Aesklepios adjacent to the Theatre of Epidaurus. In the spirit of experiencing this aspect of ancient Greek culture, we hold an event in Delphi during which each student and professor makes a "votive offering" about something they are thankful for or experience as sacred in their lives. For example, people have chosen to read essays, letters, and poems; sing songs in their native language; dance;

share pictures of either family and friends; and share sketches they have drawn during their time in Greece. In addition to being an expression of our human experience, this event strengthens the sense of community and connection among the students and faculty. As with the Trivia Contests, this Votive Offering Event arose from the intense, immersive experience of a study abroad course and may not work as well for a full-term course in the United States, although perhaps the reader might think of some ideas.

Fact Test: Each student provides a 10 to 12 point Fact Outline summarizing the key facts and ideas from their Major Presentation. We compile these Fact Outlines into a single document and provide each student a paper copy when we first arrive in Greece for them to study throughout the course. Based on these outlines, we create a Fact Test that is given during one of our last two days in Greece. For a full-term course in the United States, the standard choice of giving two or more tests evenly spaced throughout the term might be more effective for encouraging and assessing student learning.

A Written Reflective Response: At the end of the course, we give each student a paper copy of their "Articulation of Goals" that they submitted prior to leaving for Greece. They handwrite a reflective response to their goals in light of their study abroad experience. This assignment could be readily given at the end of a full-term course in the United States.

Lessons Learned

Some Sigmas

During every pre-trip meeting, we intentionally foster the atmosphere of a community of scholars. We express our expectations for respect, asking questions, challenging ideas not people, responsible intellectual risk-taking, listening to one another, collaboration, and kindness. Our students consistently buy in to these ideals, engaging with and supporting each other's efforts to learn the ideas at hand.

The on-site immersion aspect of the course has a profoundly positive impact on student engagement and learning. We build in free time for personal exploration and the students have some down time during meals and while traveling by foot, bus, and ferry. However, most days are pretty full of planned activities, including some evening events. As might be expected, our students learn a lot from spending so much dedicated time on task. One challenge for a full-term course in the United States would be finding an appropriately modified level of engagement for both students and faculty.

We had some initial concerns about trusting the students with so much of the academic content of the course. We work with them on the selection of a topic, but then have them run with it. The students accept this responsibility seriously and bring high-quality work to their Major Presentations. They are also strikingly creative in developing their Engaged Activities, most of which are genuinely effective in developing student (and faculty) understanding of their topic. We build in some safeguards to help ensure high-quality work by means of pre-course submissions in December, and we provide support and encouragement to students who solicit our input. Also, almost all of the topics addressed in the Major Presentations are familiar to us, allowing us to correct any misrepresentations of ideas or events in real time.

In light of the diversity of abilities and interests among the students, we had some concerns about their success in grappling with very challenging ideas. While not every student reaches the same level of understanding, they all engage and learn important ideas from both mathematics and the dramatic arts. The class as a whole remains engaged throughout, in part from their personal, innate curiosity, but also because of the supportive, encouraging academic community we foster.

We find that the Sigma-Delta Sessions start strong and improve to quite excellent. Our students have good insights into what makes for a great presentation and they articulate high expectations for each other (and for the faculty). While introducing this event, we remind our students that their future personal and work responsibilities will require them to provide direct feedback to others, including both earned praise and constructive criticism. We encourage our students to think of these Sigma-Delta Sessions as an opportunity to develop their skills in providing feedback.

Some Deltas

Our aspirational vision is to create an interdisciplinary course, exploring the similarities and the differences in the perspectives of mathematics and the dramatic arts. We achieved this goal to some extent in our first rendition of the course and we ramped up this element for our second offering. At the same time, this aspect of the course remains very much a work in progress. We seek additional ways to highlight the intersection of mathematics and drama, emphasizing the rich understanding of reality that arises from the commonalities and differences in these two ways of viewing our world. We would welcome ideas from those reading this article.

In the first rendition of this course, all "prior to leaving for Greece" assignments were due on December 30th. With our departure date on January 3rd, we did not have enough time to provide sufficient feedback and the process of assembling the Fact Outlines into a single document ran right up to the last minute. We also had a couple of students submit their work late, further complicating this process. For our second rendition, we moved the due date back to December 22nd, prior to our departure date of January 4th. This worked better, but we are considering further adjustments.

In particular, one adjustment we are considering is shifting the assignment of major research topics to the end of the preceding spring term and having our students submit an annotated bibliography of sources at some point during the summer. Some students use weak or insufficient resources and they are not as successful as we would like at finding better sources in the week prior to our departure. We wonder if shifting some of this preparation work to the summer would both ease some immediate pre-course stresses and lead to higher quality research and presentations.

Thus far, we have not required any common readings or texts. We believe that the course would be stronger if our students share a common, base level of knowledge earlier in the course. The next time we teach this course we plan to pull together a course pack including an outline and readings about ancient Greek history, three or four plays, and some basics of geometry, logic, and paradoxes. For a study abroad course in Greece, we would require the students to read this material prior to their arrival in Greece. For a full-term course in the United States, these readings would be assigned during the first four weeks of the course.

We anticipated some compatibility issues while traveling to and from Greece, but there have been some surprises along the way. For example, Greece uses a two-pin 220V electrical system and we knew to bring the appropriate converters and transformers. On the other hand, we unexpectedly learned that the DVDs we had brought for students to watch were not properly encoded for a European DVD player.

A couple of times during our first offering of the course, we wanted access to certain presentation supplies that were not readily available outside of Athens. Among other things, we now carry extra markers and large writing pads with us.

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Appendix A
Course Outline

A January-term Course in Greece

The day-to-day plan for this course results from an interplay between topics and itinerary. As described above, we first brainstormed about the topics and the people in our respective disciplines contemporary with or subsequently associated with ancient Greece. We then identified corresponding physical locations in Greece and worked to maximize the number of overlaps. While a wide variety of choices could be made, we have decided to spend time in Athens, Delphi, Epidaurus, Samos, Miletus (Turkey), and Delos the two times we have taught this course on-site in Greece. Working with our Athens-based travel agent, we established an appropriate itinerary. While particular details necessarily change each time the course is offered, the following version from January 2012 provides a representative example of an itinerary and identifies the topics we studied in each location.

Day 1–2: Fly from the United States to Athens and settle into our first hotel in Athens.

Day 3: Tour of the Acropolis and the Pnyx. Celebrating democracy, we debated at the Pnyx.

Day 4: Tour of the New Acropolis Museum, Ancient Agora, the Stoa of Attalos, the Theatre of Dionysus, and the Odeon of Herodes Atticus.

Day 5: Presentations about Socrates and his method in Plato's *Meno*, the transition from bartering to currency, caryatids, the architecture of Greek and Roman theatres, City Dionysia, and the Theatre of Dionysus. This day also happened to be the first Sunday of the month and we revisited the Acropolis for free.

Day 6: Bus tour of Athens. Visit the site of Aristotle's Lyceum and the National Gardens. Presentations about the peripatetic school, Aristotle on tragedy, Aristotle's syllogistic logic, and the golden ratio.

Day 7: Presentations about Greek numbering systems and both the life and work of the playwrights Thespis, Aeschylus, Sophocles, and Euripides.

Day 8: Travel to Delphi, where we engaged in some "Pythian" games at Delphi's gymnasium.

Day 9: Tour of the archaeological site and museum of Delphi. Presentations about Greek masks and tragic heroes in Greek drama.

Day 10: Presentations about the tradition of procession in Ancient Greece and the revival of the Pythian games at Delphi. Votive Offering activity in which each student makes a votive offering about something they are thankful for or that they experience as sacred in their lives.

Day 11: Travel to the old city of Epidaurus. Visit the ancient theatre in old Epidaurus.

Day 12: Tour of the Theatre of Epidaurus and the Aesklepion. Presentations about navigation, Aesklepios and the tradition of Greek "wellness" for body-mind-spirit, and trade in the Aegean.

Day 13: Visit Mycenae, return to Athens, travel by overnight ferry to Samos.

Day 14: Arrive in Samos. Presentations about both the life and work of Thales of Miletus, Pythagoras, and Euclid, including straightedge and compass constructions.

Day 15: Tour of Miletus in Turkey.

Day 16: Presentations about non-Euclidean geometries and the Tunnel of Eupalinos. Site visits to the town of Pythagorio, the Tunnel of Eupalinos, and the Temple of Hera.

Day 17–18: Travel by ferry back to Athens and then, the next day, by ferry to Mykonos.

Day 19: Tour of Delos. Presentation about Greek constellations.

Day 20: Presentation about cube doubling, circle squaring, angle trisecting, the impossibility of a straightedge and compass construction based on the Gauss-Wantzel Theorem, and introducing alternative construction tools, including paper-folding.

Day 21: Fact Test and free day on Mykonos.

Day 22: Travel by ferry back to Athens.

Day 23: Fly from Athens back to the United States.

A Full-term Course in the United States

Many different options are available for a full-term rendition of this course in the United States. Without the constraints of an itinerary, the topics can be sequenced in many different orders. Furthermore, how deeply we study various topics can be readily adjusted to match the particular audience of students. These topics can be explored and taught in many different ways, from a lecture-based format to a focus on student presentations and peer-led activities, or some blending of both approaches. The following outline provides one option among many possibilities.

Days 1–2: Comparing and contrasting the Apollonian ideal focused on reason and the Dionysian ideal focused on emotion, and a general overview of the history of the ancient Greeks.

Days 3–7: Basics of the great ancient Greek playwrights and their works: Thespis, Aeschylus and his *Agamemnon*, Sophocles and his *Oedipus Rex*, and Euripides and his *Medea*.

Days 8–12: Basics of the great ancient Greek mathematicians and their works: the mathematics of the Babylonians and Egyptians, Thales of Miletus and the shift to deductive mathematics, Pythagoras and the results of his school, Euclid of Alexandria and *Elements*, and Archimedes of Syracuse.

Days 13–14: Comparing and contrasting how mathematicians and dramatic artists read and work with written texts.

Days 15–18: Interdisciplinary exploration of the Acropolis and its structures: design and practical issues of construction and reconstruction, the art of the Parthenon and the Erechtheion, and the festivals and processions associated with the Acropolis, including City Dionysia.

Days 18–21: Interdisciplinary exploration of the Theatre of Epidaurus: the architecture of Greek and Roman theatres, acoustics, the distinctive acoustics of the Theatre of Epidaurus, the role of theatre in Greek society, and holistic wellness.

Days 22–23: Exploring the creative process, as outlined by George Wallas [46], and the commonality of this process in mathematics and the dramatic arts.

Days 24–26: Aristotle on logic and the development of modern logic, including Gödel's incompleteness theorems.

Days 27–29: Aristotle on tragedy and its role in both ancient Greek and modern theatre.

Days 30–32: Infinity, including Zeno of Elea, Aristotle's distinction between potential and actual infinities, and the development of both calculus and transfinite mathematics.

Days 33–35: An exploration of *catharsis* both as expressed in ancient Greek theatre and in modern storytelling forms, including both live and video performances.

Days 36–39: Non-Euclidean geometry and impossible geometric constructions.

Days 40–42: Interdisciplinary exploration of the heavens: story-telling, myths, constellations, and ancient Greek astronomy.

Appendix B

Logistics for a Study Abroad Course and Full-Term Adaptations

Applications: For our study abroad course, the students submit an application and we approve every student who enrolls. We ask for basic academic information and an essay explaining their interest in the course, including how this experience meshes with their academic and personal plans. The instructions for the application tell students that we expect them to be courageous, curious, responsible, good citizens of the group, patient, and good-spirited. We ask them to explain why one of these six expectations will be more natural for them and why another will be more challenging. Through this process, we seek students who are both intellectually curious and willing to take risks in an academic setting that is simultaneously challenging and supportive.

Pre-Study Abroad Course Meetings: The study abroad programs at Centre College include pre-trip meetings. We hold one meeting shortly after students register for the course during the preceding Spring term and four additional meetings during the Fall term prior to the January-term course in Greece. The Assignments section provides details about the Major and Minor Presentations mentioned in the following meeting descriptions. During a full-term course the content of these meetings would be integrated into class sessions at appropriate points during the term.

1) April Meeting: We provide a big-picture overview of the course, sharing a likely syllabus and itinerary, setting the academic and community expectations, and responding to students' initial questions. As discussed in the Lessons Learned section, we may shift some agenda items from our Fall term meetings into this April meeting to strengthen the course. This meeting essentially corresponds to the beginning of the first day of class during a full term.
2) September Meeting: We provide more specific details about the course, including the syllabus, the budget, the itinerary, and the topics of study. We frame the academic context as an exploration of the processes at the core of both mathematics and drama, and we discuss some basics of Greek culture. For homework, the students watch the PBS video *Empires* before the October meeting. For a full-term course, the students would watch this video early in the term.
3) October Meeting: We provide an update on trip planning. We lead the group through team-building exercises, discuss the homework video, and assign research topics for both their Major and Minor Presentations. For homework, students prepare a draft version of their Minor Presentation and an abstract for their Major Presentation. For a full-term course, these activities and assignments would be included at appropriate points during the term.
4) November Meeting: We provide an update on trip planning and lead more team-building exercises. We randomly select three students to share their draft Minor Presentation (from their homework) and then introduce the Sigma-Delta feedback process described in the Assignments section below. In small groups, the students share the abstract for their Major Presentations and solicit questions, feedback, and ideas from their peers. For a full-term course, the discussion of good presentation practices and the Sigma-Delta feedback process would take place early in the term. In addition, students would be assigned small groups to provide feedback about their Major Presentations at appropriate points during the term.
5) December Meeting: Over a Greek-themed meal, we discuss final details and respond to students' questions in an open question-and-answer session. In small groups, the students brainstorm about the Engaged Activities that are part of their Major Presentations. For a full-term course, the students

would be assigned small groups to discuss Engaged Activities and provide feedback about their plans at appropriate points during the term.

Language as a Non-Issue: We have not experienced any real problems communicating with the people in Greece, even though we do not have any linguistic competence with Greek. English is widely spoken, and many signs and menus have been printed in both Greek and English. The Greeks we have interacted with have been uniformly kind, generous, and welcoming.

Travel Details: Our entire class flies together as a group from the airport closest to Centre College in Lexington, Kentucky to Athens, Greece. We solicit group-pricing bids from three airlines and select the best bid with respect to money as well as the number and length of layovers. We work with our Athens-based travel agency Dolphin Hellas Travel and Tourism to make all other travel arrangements, including hotels (which include breakfast each morning), transportation by bus and ferry, and tour guides. In addition to breakfasts, we include many group dinners, which enable a stronger community and help students manage their personal expenses.

Tour Guides: Greek law requires licensed tour guides at all classical sites and, without a special permit, we cannot independently lecture on-site, although our tour guides provided time for the students' Minor Presentations about deities and brief instructor-led discussions. All of our tour guides have been friendly, articulate, and entertainingly knowledgeable, and our courses were stronger for their contributions. From our perspective, the greatest challenge in offering a full-term course in the United States would be learning and presenting this history and culture of the ancient Greeks.

Classroom Spaces: For our study abroad course, we hold class meetings in our hotels in either the breakfast room or a separate meeting room. None of these spaces are equipped with chalkboards or projectors and, similarly, we have limited access to copying and printing. Faculty and students design their presentations to work within and around these constraints. Some presenters choose to distribute handouts with their PowerPoint slides, while others record information on a large Post-It-Note writing pad that attaches to a wall. For a full-term course in the United States, we would want to teach in a flexible classroom space that could accommodate not only standard lectures, but also working in groups of two to six students at a time. In addition, some Engaged Activities require lots of open space at the front of a room and we would consider access to an outdoor green space a plus.

Enticement to College Mathematics via Primary Historical Sources

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Course Overview

The lower-division honors course *Spirit and Evolution of Mathematics* is based entirely on the philosophy of direct study of primary historical sources [1,2,3,7,9]. The main focus is on learning important and interesting mathematics, but with history as a constant natural companion, so that students learn much history in the context of learning specific mathematics. Each offering of the course is based on one or more sequences of (translated) primary sources, each sequence designed for studying one topic in depth. The intention is for students to experience as closely to firsthand as possible how human understanding of a great mathematical question or problem evolved over many centuries. Primary sources can naturally raise deep questions, and, through often long periods of historical turmoil, lead to deep understanding. Studying mathematics via this process avoids the "cart before the horse" method of most modern mathematics textbooks, which provide answers to questions that haven't been asked.

"Study the masters!" is the course's spirit, to examine great mathematical texts from a breadth of times and cultures, by immersing ourselves in the very words of the masters who first discovered new ideas. We see these ideas develop into modern branches of mathematics by studying selected sequences of primary sources. At the same time we consciously aim at developing an appreciation for and facility with methods of rigorous proof and modern mathematical thinking.

Spirit and Evolution of Mathematics began in 1989 as an honors option for meeting the lower-division general education mathematics requirement at New Mexico State University, and has been offered almost every year since then [4]. The prerequisite is "a good high school mathematics background", in reality the same as for entry into precalculus, and no specific background topics are assumed. However, students are generally expected to have honors eligibility, which in practice means the ability to learn more independently and to contribute actively and innovatively to class discussion in a course with enrollment restricted to 20. At other institutions our materials have also been used for seminar courses at the upper level, or for individual topic study courses.

Students in the course hail from all possible majors, or are undeclared. In addition to general education, the course also attracts students to major in mathematics. Some of our very best mathematics majors have begun in this course, never imagining they would be attracted to major in mathematics, until studying primary sources opened their eyes and excited them to choose mathematics.

Course Design

Our resource for the course is a collection of carefully chosen historical texts, ideally with generous annotation, commentary, and exercises. The goal is to study the original proofs of results in these texts, in the words of the discoverers, in order to understand the most authentic possible picture of the evolution of major branches of mathematics during a span of over two thousand years. It is exciting and illuminating to read original works in which great mathematical ideas were first revealed.

At home and in class we read, discuss, and interpret the theorems and their proofs, with students writing their thoughts and questions about these works, and we discuss how the various sources tie together in the development of important ideas. Students also solve challenging mathematical problem exercises directly related to the primary source texts.

Different instructors use different pedagogies in the classroom. Personally, I use a non-lecture three-part assignment method applied in almost all my teaching, explained in detail in [8], and in a handout for students (see Appendix C on homework guidelines). For each topic, students are first expected to read new material in advance of class, and to write questions about their mathematical reading for me to read before class. Second, students are expected to prepare mathematical work for class based on their reading, usually by attempting solutions of pre-assigned exercises. These two pre-class parts are graded based only on effort and preparation for class, very quick to assess with a plus, check, or minus.

In the classroom we first discuss their reading questions as a group, along with questions I inject. Class discussions are often challenging and fun for all, because the primary sources provide fabulous grist for deep and wide-ranging considerations. Part of the richness of studying primary sources is that today they often raise as many questions as they answer. The majority of classroom time is then spent with students working in informal groups on the previously assigned problems, interspersed with impromptu whole-class discussions or student board presentations suggested by me as common questions and interesting approaches arise.

The third part of each topic's homework assignment consists of completing post-class homework on the topic, primarily a very few challenging exercises not worked on in class. Students are always encouraged to discuss their ideas with others, and then expected to finish and write up their polished post-class homework entirely on their own, in their own words, to hand in for me to read and mark carefully, and possibly to request rewriting for improvement. This post-class homework part ultimately receives a single letter grade for quality, one for each class day.

The course grade is based on a final holistic evaluation of student work: roughly one half on regular assignments (i.e., student writings on the original sources, and related mathematical assignments); roughly one quarter on class participation; and the remaining quarter on a term paper and a brief oral presentation of it, as described below. There are no exams. The course overview handout for students is in Appendix B.

The course has a very flexible timetable (hence there is no course outline appended), often influenced by what explorations happen in the classroom based on student response and activities, according to the classroom methods described above. Whether we engage one topic chapter or two from the textbook during a term varies.

On the first day I introduce and discuss the nature and expectations of the course, we dive into some mathematics, and I ask students to skim the entire five topic chapter materials of the textbook as homework, to provide feedback, as explained earlier, at the beginning of second class period. At the beginning of the second day I select a topic on the spot, based on their feedback, and we begin right away with primary source material from that topic. Succeeding days always have reading/writing in advance, preparatory mathematical work on exercises, in class group and whole class work, and final homework exercises, as described earlier. By mid-term students begin work on individual term paper research, in addition to our regular classroom work, as described earlier. The very end of term is spent on short term paper presentations and discussion.

Resources

When this course was first co-created and taught in 1989, we had little more than a handful of chosen primary sources on a few topics, some of which we had to translate ourselves, often with no annotation, context, or exercises. We assigned some essay readings to supplement the sources, created assignments as we went, and started writing annotation to tie it all together. And indeed this is how anyone can still develop their own materials; I enthusiastically recommend it. Guidance on the pedagogical principles, and on design of materials, can be found in [1, 2, 3, 7, 9]. Today primary sources are much more easily available, and in translation as well, than when we started. The reader may be pleasantly surprised that finding promising and appropriate primary sources for teaching on a given topic is not as hard as may be feared. The bibliography [10] provides a window to many historical sources for teaching. The recent sizeable source book [12] would be a good place to find many good sources on a variety of topics (see the review [11]).

Over the years we settled on and developed five topic sequences of primary sources, translated as necessary, on grand topics in the evolution of mathematics, added extensive annotation, contextual, historical, and mathematical commentary as a guide and overview of the big story, and numerous mathematical exercises for students. We also included copious references to the literature for deeper understanding by both teachers and students. These became the five chapters of the textbook *Mathematical Expeditions: Chronicles by the Explorers* [5]. Each chapter has an extensive Introduction, which tells a large story from the beginning, both mathematically and historically. As it proceeds, the Introduction points the reader to the subsequent chapter sections, which focus in sequence on sources by specific authors. Therefore we have students read the Introduction in tandem with work on individual chapter sections, going back and forth between the main story and the featured primary source sections.

Each one-semester offering of the course typically covers only one or two of the five independent chapters in depth. Students are sometimes asked to start by skimming the book as homework and rating (A-F) and ranking (1–5) the five chapters for interest, before I select which chapter we will study. The point of both rating (absolute) and ranking (relative) is to aim to choose a chapter that most students will find inspiring, and that they will feel was a preferred choice.

The chapter themes and authors of the primary sources we have used are:

- *Geometry: The Parallel Postulate*. The primary sources follow two millennia of the development of non-Euclidean geometry via excerpts from Euclid featuring his parallel postulate, Legendre's early nineteenth-century final attempts to prove the parallel postulate, Lobachevsky's almost simultaneous introduction of the brave new world of planar hyperbolic geometry, and Poincaré's disk model confirming its equal footing with Euclidean geometry.
- *Set Theory: Taming the Infinite*. Bolzano considers explicit mathematical paradoxes of the infinite, Cantor confidently opens the cornucopia of explicit different infinite cardinalities, and Zermelo axiomatizes set theory to give foundations to modern mathematics.
- *Analysis: Calculating Areas and Volumes*. Primary sources follow the development of calculus over more than two millennia. Archimedes calculates areas of parabolic sectors by the method of exhaustion, and reveals his technique of balancing indivisibles for discovering results before proving them by exhaustion. Cavalieri calculates areas of higher parabolas using geometric algebra and indivisibles. Leibniz proves the fundamental theorem of calculus with his infinitesimals, Cauchy rigorizes integration as limits of discrete summations, and Robinson resurrects infinitesimals in the twentieth century.
- *Number Theory: Fermat's Last Theorem*. Euclid classifies all Pythagorean triples, Euler proves Fermat's Last Theorem for exponent four, Germain provides the first general approach, proving many instances of Case I, and Kummer elucidates the beginnings of algebraic number theory.
- *Algebra: The Search for an Elusive Formula*. Primary sources follow the quest for formulaic solutions to polynomial equations. Euclid resolves quadratics with plane geometry, Cardano solves cubics with verbal geometric algebra, Lagrange explores the fading promise for higher degrees, and Galois converts the problem to studying groups of permutations of roots and field extensions.

As illustrations, Appendix A introduces small excerpts from selected primary source material for each theme, along with connected sample exercises for students. The website [6] provides sample sections from each chapter.

Assignments

Regular homework and related classroom work are the heart of the course. Assignments are largely mathematical in nature, based directly on the primary sources, since the course is first and foremost mathematics, set authentically in its history. Exercises often strengthen students' understanding of a primary source, and are sometimes open-ended. To give a diversity of flavors, Appendix A provides sample exercises from our five general themes above, each exercise preceded by a little context and a small excerpt from the relevant primary source.

For the term paper and brief oral presentation (see handout for students in Appendix D), the choice of topic is up to the student, subject to my approval, but should include a meaningful mathematical component (not mostly biographical) that they can genuinely understand and explain to others in a presentation. I do not suggest topics, so students must keep their eyes open for something along the way of interest. I emphasize that this is an opportunity to delve into something personally exciting or innovative. This responsibility is consciously placed on each student, to try to encourage them to take initiative. One obtains a great variety of quality in topics, and while not all are inspiring, some are truly fascinating.

Because a lot of independent judgement is being expected of lower division students in choosing a topic and writing a term paper, one needs to monitor and guide the process closely so that individual students don't end up way off the desired mark, creating disasters for them and the instructor. So students are asked to pick a term paper topic by mid-semester, and I help students in refining ideas for a topic. Each student is first asked to come up with two ideas for a topic, do a preliminary library search to see that adequate research materials are to be found there (required usually to be books, not just internet sources), and write a paragraph describing each topic to me, along with references to what was found in the library. After possible further refinement, often having them show me their resource books, I approve each topic. Students are sometimes required to show me their writing progress along the way, to help them complete an acceptable paper on time. We use the mandatory final exam period, as well as the last few class days as needed, for term paper presentations, with papers due several days before presentations.

Very occasionally a student will not start their term paper in time, despite my best efforts to guide them, and out of desperation submit something plagiarized; a student who requests to change their topic at the last minute is a possible indicator of this. One needs to be on the lookout for this quite rare occurrence, try to prevent it from happening, but be prepared if it does.

Lessons Learned

This is a highly engaging course for students interested in mathematics, no matter what their major, and some students go on unexpectedly to study more mathematics after having their interest sparked. Many students absolutely love the primary sources, and especially those from the humanities take to them very enthusiastically.

It is important that enrolled students share a certain minimum level of ability with the mathematics, since otherwise they will not be able to cope with the challenge of the primary sources. Our current prerequisite is roughly a Mathematics ACT score of 25, and students allowed into the course without this level often do not succeed.

The class discussions are usually highly stimulating for both students and instructor, since grappling with primary sources invariably leads directly to deeper questions than a textbook usually does. But there is often great variation in how students choose to participate, if left to their own devices, so the instructor needs to be constantly aware of this, and act to guide the process. For instance, one needs to try to draw out the quieter students and gently reign in more outspoken ones, both in group work and in whole-class discussion.

Overall the greatest instructor challenge with this course is simply that the students are lower division, often entering freshmen, and the expectations of this course are frequently a big step up from what they have accomplished before. For students without the prerequisites it is usually too big a step.

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Appendix A

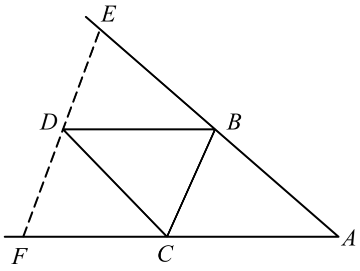
Sample primary source materials and exercises

Here I describe a small primary source excerpt and related exercise for each major theme we have used in teaching the course.

Geometry: The Parallel Postulate

Legendre was the last serious mathematician to attempt to prove Euclid's parallel postulate, while at essentially the same time Gauss, Lobachevsky, and Bolyai were developing the non-Euclidean hyperbolic geometry that negated it. From Legendre's many published attempts, students study an entire proof that the angle sum of a triangle must be two right angles (equivalent to the parallel postulate), and are challenged to find an unsupported assumption. Legendre's text contains the passage

> Let A be the smallest of the angles in triangle ABC, on the opposite side BC make the angle $BCD = ABC$, and the angle $CBD = ACB$; the triangles BCD, ABC will be equal, by having an equal side BC adjacent to two corresponding equal angles [pr. 7]. Through the point D draw any straight line EF which meets the two extended sides of angle A in E and F.



Students are first challenged, from amongst the entire proof, to ferret out that a subtlety in this passage is the key:

Exercise: Before reading our commentary after Legendre's results, find and discuss the flaw in his proof of the parallel postulate.

Set Theory: Taming the Infinite

Students study Cantor's famous diagonal argument in his own words, including the passages

> Namely, if m and w are any two distinct characters, we form a collection M of elements $E = (x_1, x_2, \ldots, x_\nu, \ldots)$ which depends on infinitely many coordinates $x_1, x_2, \ldots, x_\nu, \ldots$, each of which is either m or w. Let M be the set of all elements E.
>
> …
>
> I now claim that such a manifold M does not have the power of the series $1, 2, \ldots, \nu, \ldots$.
>
> …
>
> This proof appears remarkable not only due to its great simplicity, but in particular for the reason that the principle employed in it can be directly extended to the general theorem, that the powers of well-defined point sets have no maximum, or what amounts to the same, that to every given point-set L can be associated another one M which has a higher power than L.

Students are then challenged to extend Cantor's theorem to arbitrary cardinalities, and to construct infinitely many infinite cardinals.

Exercise: Generalize Cantor's argument in the last source to prove that $2^{\mathfrak{m}} > \mathfrak{m}$ for any cardinal number $\mathfrak{m}$.

Exercise: List an infinite sequence of infinite cardinal numbers.

Analysis: Calculating Areas and Volumes

One of Archimedes' triumphs was his determination, without modern calculus, of the area of any segment cut from a parabola. Students study his entire proof by the Greek method of exhaustion. Archimedes states his result as

> For it is here shown that every segment bounded by a straight line and a section of a right-angled cone [a parabola] is four-thirds of the triangle which has the same base and equal height with the segment.

One exercise for students is to connect this with our modern calculational approach.

Exercise: Archimedes proves that the area of a segment of a parabola is four-thirds that of a certain triangle. But this is of limited usefulness unless one can determine the dimensions of the triangle for a given segment, in order to compute its area. Show how this can always be done.

Number Theory: Fermat's Last Theorem

Sophie Germain was the first to provide general results towards proving Fermat's Last Theorem. Students study her unpublished manuscripts, including the main result she has been known for:

> If p is an odd prime, and if there exists an auxiliary prime θ with the properties that p is not a pth power modulo θ, and the equation $r' \equiv r+1$ modulo θ cannot be satisfied for any two pth power residues, then Case I of Fermat's Last Theorem is true for p.

With Germain's Theorem, students can start proving special cases of Fermat's Last Theorem.

Exercise: Use Germain's Theorem to show that there are no Case I solutions to the Fermat equation for exponent 7.

Algebra: The Search for an Elusive Formula

The search for the elusive roots of polynomials had many twists and turns. One was a poem divulged to Cardano by Tartaglia for solving a certain type of cubic equation:

When the cube and its things near
Add to a new number, discrete,
Determine two new numbers different
By that one; this feat
Will be kept as a rule
Their product always equal, the same,
To the cube of a third
Of the number of things named.
Then, generally speaking,
The remaining amount
Of the cube roots subtracted
Will be your desired count.

An exercise for students is to decipher how this poem yields a modern looking formula.

Exercise: Turn Tartaglia's poem into the formula in the Introduction.

Appendix B

Handout for students on course overview

SPIRIT AND EVOLUTION OF MATHEMATICS

Prerequisites: A good high school mathematics background: for instance, a Mathematics ACT score of 25 or better, or meet placement requirements for entry into precalculus, or consent of instructor.

Course Text: *Mathematical Expeditions: Chronicles by the Explorers*, by R. Laubenbacher and D. Pengelley (at the bookstore).

Study the masters! is the spirit of this course, in which we will study great mathematics from a breadth of times and cultures, by immersing ourselves in the very words of the masters who first discovered new ideas. We will see these ideas develop through time into modern branches of mathematics, by studying selected sequences of primary historical sources. Themes may include the emergence of non-Euclidean geometry, the concept of the infinite, the calculus, number theory, and the quest to solve algebraic equations. At the same time we shall aim at developing an appreciation for and facility with methods of rigorous proof and mathematical thinking.

Our primary objects of study will be a collection of historical mathematical texts with annotation and commentary. The goal is to study the original proofs of the theorems in these texts, in the words of the discoverers of the mathematics, in order to understand the most authentic possible picture of the evolution of major branches of mathematics during a span of over two thousand years. It is exciting and illuminating to read original works in which great mathematical ideas were first revealed. At home and in class we will read, discuss and interpret these theorems and their proofs, with you writing your thoughts and questions about these works, and we will discuss how the various sources tie together in the development of important ideas. I expect our class discussions to be both challenging and tremendous fun. Other written assignments will consist of proving related results, and solving other mathematical problems related to the texts.

You will have the opportunity to work both individually and collaboratively with others. I encourage you to discuss your ideas with others, and then I expect you to write up your homework entirely on your own, in your own words. As we examine the development of mathematical ideas in the original texts, we will also discuss their historical context and biographies of their creators.

I always expect to receive your homework when due, unless there are extenuating circumstances you discuss with me; speak with me if your homework must be late for a special reason. You should always hand in your homework when due even if it is incomplete. Late homework will receive only partial or no credit. On-time homework may sometimes, at my suggestion, be re-worked after I critique it, to bring it to perfection, due at the next class period after being returned by me. My goal is to help you perfect your work to your and my satisfaction.

I will expect active class participation from everyone, since this is one of the most valuable and enjoyable features possible in a small Honors class. Thus in particular I expect everyone to attend regularly. Please also be here so we can all start class on time, since late arrivals are particularly disruptive for a small class based primarily on discussion. Our class environment should be devoted entirely to our joint endeavors. In particular, I expect that we should never, ever be interrupted by a cell phone, even a buzz or vibration. Please take the responsibility to make absolutely sure that any cell phones remain completely turned off at all times in the classroom.

I will expect you to write a term paper and give a brief oral presentation on it. The choice of topic will be up to you, with my approval, but should include a meaningful mathematical component (in particular, it should *not* be mostly biographical). So keep your eyes open for something along the way of interest, since this is an opportunity to delve into something particularly exciting to you. I will discuss the timetable for the development of your paper in class, and will ask you to pick a topic by mid-semester. I will help you in refining your ideas for a topic. Please familiarize yourself with the university's policy regarding plagiarism and academic misconduct at www.nmsu.edu/~vpsa/SCOC/misconduct.html. I will discuss this further when you select a paper topic.

Your course grade will be based on a final holistic evaluation of your work as a whole: roughly one half on regular assignments (i.e., your writings on the original sources, and related mathematical assignments); roughly one quarter on class participation; and the remaining quarter on your term paper. There will be no in-class exams. However, we will use the mandatory final exam period for class activity, probably term paper presentations; please plan to be there. Please note that in Honors College courses, if you are taking the course on an S/U basis, a B– is required to produce an S.

Regarding what I write on your papers: My goal is to help you learn. If I write a note asking you a question, write back and bring my attention to it, or talk to me! If I circle something and put a question mark, I could not decipher or understand it. If I circle something with no comment, I am alerting you to check on a question of spelling, grammar, or meaning that needs correcting. If something needs further work, I will often expect you to redo it to perfect it, and give credit when it is corrected.

Help: Here are places to go for help.

a) I strongly encourage group work on homework, with individual write-ups. The only **rule** is that if you are to hand in your own individual assignment, then after you've talked all you want with sources of help, go home and write up your own assignment, *by yourself*, to hand in. Your paper should never read like anyone else's, since it should always be your own thoughts in your own words.

b) Also, I'm available, and want to help. I should be your primary source of help after those above. Try to prepare specific questions about things you are having trouble with. "I can't do this problem" is not very helpful to you or me, whereas "I started to do this, and then this happened, and I got stuck" gives us much more to work with. I can help you a lot in class, and am also available during my office hours, or by appointment.

Appendix C

Handout for students on homework assignment guidelines

Keep this sheet

Guidelines for all regular homework assignments

Please put your name (and any nickname your prefer) on the first page, *staple* your pages together, and *do not* fold them. Use both sides of the paper if you wish, to save paper. Please *do not* write in light pencil. Please write clearly. Thank you.

Parts A, B, C of each homework are equally important.

Part A: Advance preparation. Hand this in at the beginning of class, one class period before our class discussion and work on new reading. Reading responses (a), questions (b), reflection (c), and time spent (d):

You do *not* need a new page for each part (a), (b), (c), (d).

a) Read assigned material. Reread as needed for complete understanding. Then write clear *responses* to assigned questions about the reading.

b) Write down some of your own explicit *questions* about your reading, ready to bring up in class. This may involve new or old concepts which are confusing to you, and connections to other ideas. You should also consider writing down what was well explained and interesting, what was confusing, what you had to reread but eventually understood.

c) Reflection: Write two or three sentences *reflecting* on the process of your work; this should only take a few minutes. Write about how things went with any assignment or reading done for class, and other course work. This should reflect both your ongoing personal feelings about the course as a whole and your interaction with the material at hand.

d) Write how much *time* you worked on part A.

Part B: Warmup exercise preparation to present in class. This is due during class when we begin to discuss new material. Work individually, and then with others in your group outside class time, on a few assigned easy warmup exercises on the new material we will discuss, based on your advance reading in Part A. Write up the solutions to these individually, to hand in in class. I will ask individuals and groups to present some of these to the class, to get us started discussing new material. Be sure to hand these in before leaving class.

Also always write how much time you worked on part B, and with whom.

Part C: Main exercises. These will be assigned after class discussion and work on new material. They will normally be due next period. Work individually and with others in your group on these. Also come to see me during office hours or at other appointment times about these. I am happy to help you. Then go home and write up your final solutions completely by yourself, without comparing with other people. The paper you hand in should be entirely your own writing, not the same as anyone else's.

Appendix D

Handout for students on term paper guidelines

Guidelines for term paper and presentation

One quarter of your work for the course is a well written term paper on a mathematical topic of your choice, along with a brief class presentation.

- I will ask you to take the initiative in finding a term paper topic. First I will ask you to come up with two ideas for topics, do a preliminary library check that adequate research materials are likely to be found there, and write one paragraph describing each topic to me, along with references to what you found in the library. Use your imagination and interests in selecting topics!
- The principal requirement for a topic is that it should be primarily about mathematics in some form. If it is about a mathematician, then it should focus at least half on that person's mathematical work, and not more than half on biography. The other main requirement is that you should be able to discuss the mathematics in your paper with some genuine understanding of it. Writing a paper that lists mathematical results you have absolutely no understanding of is not fruitful.
- I will give you feedback on your tentative topics, and may ask you to seek further source material in the library to show me, to make sure a topic is appropriate. Then I will approve your selection of a topic.

Please read lib.nmsu.edu/plagiarism.

- ◽ I expect your paper to be well written. It should include a complete list of references. If you wish to quote directly from a source somewhere in your paper, instead of writing something in your own words, you should indicate the quotation clearly. Any consistent format for the paper and references is acceptable.
- ◽ Your paper should be about 10 pages long.
- ◽ You may handwrite your paper clearly. If you type it, please use 1½ or double spacing.
- ◽ I may suggest that you show me a rough draft or outline before the final version of your paper.
- ◽ When you give me your paper, keep a copy to prepare for your presentation to the class.

IV

Online/Blended Courses

Mathematics History in Three Acts
A Graduate Education Course

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Course Overview

The course, "*Using History in the Teaching of Mathematics*," is designed for graduate students who are current secondary mathematics teachers or who are preparing to teach mathematics as part of an alternative certification program. Although the course that is described here was taught in a blended learning environment —that is, containing both face-to-face and online elements—the course was previously taught as either a fully online or completely face-to-face course. Thus, the course described here is a modification of previous versions, designed to meet the needs of a particular student population, with a focus on teaching.

The goal of the course is to engage participants in the study of the history of topics that they may be expected to teach in the content areas of number, algebra, geometry, pre-calculus, and calculus in order to consider (and, hopefully use!) a historical perspective in teaching mathematics. However, this explicit goal implicitly requires students to gain expertise in identifying, building upon, and creating appropriate resources for the purpose of integrating a historical perspective in teaching mathematics. Thus, there are three key objectives that I seek to achieve during the course: (1) to investigate essential mathematical ideas found in grades 6–12 mathematics that evolved over time; (2) to study and discuss the historical and cultural influences acting upon, and also as a result of, the mathematics being developed; and (3) to develop the pedagogical knowledge needed to integrate a historical perspective in the teaching of school mathematics.

Course Design

The course described here has been taught at Florida State University (FSU) and meets face-to-face, once per week for two hours (for 12 weeks in a summer session), along with a one-hour per week online session. Regardless of the students' teaching experience, none had experienced history of mathematics as an undergraduate student.[1] Although there are no expectations that students in the course have prior experience with the history of mathematics, it is expected that they have prior adequate mathematical preparation. For example, those currently teaching mathematics in secondary school have already completed an approved mathematics education program, which included at least 21 semester hours in mathematics course work beyond the calculus sequence. Students not yet certified to teach secondary mathematics are required to have a baccalaureate degree in mathematics or a related field.

Although the course is designed as a full summer semester course (typically 12 to 13 weeks), my department encouraged alternative course offerings in the summer and as a result, I agreed to teach it as a hybrid

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[1] However, the course described here is still appropriate for students who have previously taken a history of mathematics course.

course; that is, the course would be delivered as a combination of face-to-face and online sessions. The online sessions, as well as the overall course organization, were delivered via Blackboard, which was the Learning Management System (LMS) used by FSU prior to 2017. For the most part, students respond favorably to the course design since it enables them to examine a variety of content topics while transitioning from historical to pedagogical foci. Using this design structure (see Appendix A), students are able to focus first on the mathematics and history and then on history and pedagogy.

The course is divided into three phases. In the first phase, weeks 1 through 5, we meet as a class for two hours and students are required to contribute online to student and teacher posts and prompts, for the equivalent of one hour per week, according to the directions they receive for the week. For example, students are often asked to post solutions to mathematical tasks, and then discuss or ask questions of their peers (or respond to my prompts and questions). In preparation for class each week, students complete a reading assignment, which usually includes the assigned Berlinghoff and Gouvêa [1] sketches (see Appendix A), as well as an article related to an exploration we work on together in class. For example, for week 5 students are asked to skim the chapter "Archimedes' Method" from Laubenbacher and Pengelley [2] so that they gain a brief exposure to the proofs and tasks we will complete in class. Then, during class we work on the chapter in order to give students a feel for the pacing of using such an intervention (a more advanced historical module – depending upon an original source—than those in Katz and Michalowicz [3]). Using such a source serves as a precursor to weeks 9 and 10 (which take place completely online), when students focus on the use of original sources in the mathematics classroom. As the table in Appendix A indicates, a focus on mathematical content is a dominant part of the course (eight of the 13 weeks). It is important to note that although mathematics is of key importance—particularly the mathematical content that students are most likely to teach in secondary mathematics classrooms—it is also my goal to provide students experience with historical materials and resources. Pedagogical considerations occur organically during the first two-thirds of the course but are elevated to the primary focus in the final third of the course.

For the two transition weeks (weeks 6 and 7), the class begins to meet online and I also hold face-to-face sessions in the classroom for two hours each week. This provides a transition period for students to become accustomed to the increased intensity of online assignments and interactions while still being able to meet in person as needed. Appendix B displays the content for the first week of online content (week 6).

Finally, weeks 8 through 13 take place completely online, while students focus on pedagogical techniques (and their rationale) for incorporating history of mathematics in teaching. The pinnacle of the students' most diligent work with regard to thinking about the role of history in the secondary mathematics classroom takes place in week 9. As an example from the last implementation of the course, the discussion board assignment for week 9 drew 191 responses from just seven class participants. My rationale for this online component of the course is to encourage and promote collaboration and sharing of perspectives among the students in the course. In-class discussions—for either two or three hours per week – are not going to push students in the way that I need them to think about, reflect, and discuss (though in writing!) the techniques, methods, and perspectives of scholars in the field. And, with a small number of students, even minor differences can impact an in-class discussion. For example—again, from the last implementation—one student rarely spoke during the face-to-face component of the course because of her command of spoken English; however, this was not an issue in her written posts. Another advantage to the online component is that all discussion threads continue to "live" online; that is, participants can return to previous posts and use them verbatim, without the danger of losing precious content or misrepresenting another's perspective.

Resources

Two key texts provide the mathematical and pedagogical foci for the course. In the first eight weeks of the course, students read and use the brief "sketches" found in *Math through the Ages: Expanded Edition* [1] to review the mathematics content and to gain an initial and brief glimpse into the historical development of the mathematical ideas of focus for the week.

The second text for the course—actually, a multimedia CD—is the *Historical Modules for the Teaching and Learning of Mathematics* [3] resource, which contains over 1300 pages of historical information, sample lessons (many of which can be used as stand-alone lessons or tasks), and classroom resources. The *Historical Modules* are used for two key purposes during the course. First, I want students to familiarize themselves with the sort of resources that they can potentially produce for their own classroom. There are two different course assignments that rely upon the *Historical Modules* (see Assignments section: the Lesson Critique assignment and the Historical Lesson assignment). Second, I want the *Historical Modules* to serve as a resource from which the students can learn about the history of mathematics, gain insight to historical problems and methods, and to view school mathematics topics from a different perspective.

Finally, I use a variety of supplementary resources when I teach any history of mathematics course. For this particular course (taught as a blended face-to-face and online experience) and for these particular students (a combination of those without teaching experience and current teachers), I use additional resources (see Appendix C). Additionally, students are encouraged to use the resources for their assignments, particularly for the capstone project.[2]

Assignments

Students complete five primary assignments in addition to being assessed for their participation during the course. Brief descriptions are given here; students are given detailed descriptions for the five primary assignments. An example of such a description (for the "Response Papers" assignment) is given in Appendix D.

Response Papers (3 total; comprised a total of 15% of semester grade)

On three occasions during the semester, I require students to locate a journal article and complete a two-page (double-spaced) response to the article. The detailed description of this task includes a sample set of prompts from which students could develop the response, as well as a brief tutorial for accessing appropriate journals and articles through JSTOR.

Historical Modules Lesson Critique (completed with a partner or two; comprised 15% of semester grade)

Students are assigned a lesson (or series of related lessons) from the resource, *Historical Modules for the Teaching and Learning of Mathematics* [3], to review and critique with one or two peers. All critiques are then posted on Blackboard for class members to access and utilize for future use.

Historical Resources Review Log (on-going; 20% of semester grade)

Using history of mathematics in teaching is much easier when teachers have access to well-constructed, appropriate materials. For this assignment, students locate and explore various materials that are available. I strongly believe in the idea that teachers must gain experience in (1) becoming familiar with curriculum (in this case, content informed by a historical perspective), (2) critiquing the content, structure, and usefulness of curricular materials, and (3) developing appropriate classroom tasks based upon effective ideas from multiple resources. To this end, I encourage students to allot time to explore materials outside of the other formal course assignments and discussions. Then, students compile a log of what they review during the semester. Students maintain the log as a Word document so that I can easily insert comments, questions,

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[2] The texts listed in the reference list were placed on course reserve in the Dirac Science Library, Florida State University. Also, items in Appendix C marked with an asterisk (*) are used explicitly in the course. Course participants use the unmarked items as references, as needed.

and suggestions when I review their individual work. I provide students with a log template and I review logs twice during the semester.

Midterm Examination (10% of semester grade)

The midterm examination includes a combination of historical problems or application of historical techniques, as well as short-response items.

Historical Lesson (capstone project; 20% of semester grade)

Students create a historical lesson for a mathematical concept or topic that they select. The lesson must include elements such as "history as anecdote" and "history as problems" and students must utilize a minimum number of resources in the construction of the lesson.

Participation (20% of semester grade)

The participation score is based upon students' efforts to engage with and complete assigned readings, homework problems, small group discussions, whole class discussions, and course tasks. Much of the evaluation in this category is based on the effort put forth; however, contributions or attitudes that detract from the positive culture of the online class environment may result in a reduction of the participation score.

Lessons Learned

I have taught a variety of history of mathematics courses over many years, yet this particular instance is unique in that many of the students—as graduate students—are seeking certification in order to teach. Additionally, the course is designed for students with adequate mathematical preparation, since the graduate program in which the students are enrolled requires that they possess an undergraduate degree in mathematics or a closely related field. A strong mathematical background is essential for the course I designed because I do not want students to struggle with the mathematical content of a given mathematical task. Instead, I want them to focus on the *historical* mathematical content. However, it has been my experience that even though a program is designed for a specifically prepared type of student, it does not always result in each admitted student fitting the profile. Thus, I designed a brief content assessment (given on the first day of class) that provides me with information about each person's mathematical content knowledge. I then use the information to plan for future assignments and construction of groups for during the course.

I also believe it is essential to have well-designed tasks for the online component of the course and to develop good discussion board skills with students. Since I have taught numerous online courses this is typically not a problem for me. If this is new territory for readers, however, be aware that if you design really good discussion questions, you need to keep up with the discussion board responses from students. In the course implementation previously mentioned, I missed one evening of responses and the next day there were 102 new posts within two different discussion board topics for a class of only seven students!

Finally, the composition of the course allows the students and me to maintain two emphases throughout the semester. First, there is a focus on mathematical content using historical methods and problems, which is emphasized during the first eight weeks of the course. Second, an emphasis on a study of the ways in which history of mathematics can be used in secondary mathematics classrooms works well with the students. And, the hybrid delivery of the course enables the two emphases to develop organically throughout the course. That is, the natural questions of "how" and "why" to use history emerge while studying the historical development of secondary school topics, and these are formally addressed later in the course. Then, because of the concentrated study of the development of mathematical ideas earlier in the course, the mathematics remains at the forefront when questioning pedagogical issues during the final weeks of the semester.

References

1. William P. Berlinghoff and Fernando Q. Gouvêa, *Math through the Ages: Expanded Edition*, Oxton House and Mathematical Association of America, Farmington, ME and Washington, DC, 2004.
2. Reinhard Laubenbacher and David Pengelley, *Mathematical Expeditions: Chronicles by the Explorers*, Springer-Verlag, New York, 1998.
3. Victor J. Katz and Karen D. Michalowicz, *Historical Modules for the Teaching and Learning of Mathematics,* Mathematical Association of America, Washington, DC, 2005.

Appendix A

Weekly structure of "Using History in the Teaching of Mathematics."

Week	Focus
1	Introductory Tasks including participant introductions, diagnostic assessment, review of syllabus, investigation of sample materials, and preparation for "Library Assignment" (B & G*: pp. 1–60) *Weeks 1–5: Face-to-face sessions*
2	Content Foci: Cultural and historical development of mathematics, number systems, development of the complex number system (B & G: Sketches 1–6)
3	Content Focus: Solving polynomial equations (B & G: Sketches 8–11, 17)
4	Content Focus: Solving polynomial equations, continued (B & G: Sketches 8–11, 17)
5	Content Focus: Geometry (B & G: Sketches 7, 12, 14, 15)
6	Content Focus: Geometry, continued (B & G: Sketches 7, 12, 14, 15) *Transition to online course delivery begins; Weeks 6-13 online sessions*
7	Content Focus: Advanced mathematics (e.g., trigonometry, analytic geometry) (B & G: Sketches 13, 16, 18, 21, 22, 25)
8	Content Focus: Calculus (B & G: pp. 43–53 again; supplemental reading)
9	Pedagogical Focus: Using original sources in teaching mathematics
10	Pedagogical Focus: Using original sources, continued
11	Pedagogical Focus: Emphasizing historical problems
12	Pedagogical Focus: Cultural- and Ethno-mathematics
13	Pedagogical Focus: Biographies and anecdotes

* "B & G" refers to the pages or sketches in Berlinghoff and Gouvêa (2004). [1]

Appendix B

Inclusion of online focus: Week 6 (example)

Task 1	**Task 2**
a. Complete the activity, "Euclid's Algorithm for the Greatest Common Divisor," which you can access in the Week 6 folder on Blackboard. Here, *complete* means do as much as you can—and where you have difficulty, make a note about what the difficulty is and why you're experiencing it. Some of what you note here will be used in **Task 2a**.	a. Add your "2 cents" to the discussion thread entitled "Euclid's Algorithm." You must respond to at least one of the three prompts: (i) What did you have difficulty with? (See **Task 1a** above for how explicit to get!) (ii) What did you learn from the sequence of readings and questions in the activity? (iii) Discuss the format and progression of the activity. If you were to adapt something like this for middle grades or high school, what elements could you or would you use? Why?
b. For Sketches 14 and 15, complete the following exercises: Sketch 14: Projects 1 and 2. Also, produce a proof of Euclid Book II, Proposition 5. (This means your work is to produce a proof, not just a copy of what Euclid did.) Sketch 15: Questions 3, 4, 5, 6	b. Please post your assigned project/question work (thread entitled "Assigned Problems"): Student 1: Sketch 15, Question 4 Student 2: Sketch 15, Question 6 Student 3: Sketch 15, Question 3 Student 4: Sketch 15, Question 5 (You obviously don't have to post the actual object!) Student 5: Sketch 14, Project 1 Student 6: Sketch 14, Project 2 Student 7: Moderator
c. There is obviously a lot of geometry to cover in Euclid's *Elements* alone, and, historically speaking, his work only represents a tiny proportion of the historical content related to geometry. Explore one of the people **or** topics given below (in book, article, web form; your choice), and see item 2c for how to share what you found with the class. **Geometers** Pappus Heron (or, Hero) Ptolemy Descartes Lobachevsky **Topics** development of the formula for the area of a circle Euclid's 5th Postulate (or, the Parallel Postulate) principles of Descartes' geometry (*La Geometrie*) the theorems of Menelaus and Ceva the classical centers of a triangle	c. Share what you found for **Task 1c** on the discussion thread entitled "Geometry Exploration." Please label your thread with the name of the person or topic and briefly respond to: (i) One or two classroom-worthy things that you learned about the history of the topic or the person; and (ii) Was one of the resources you accessed worth sharing with others? If so, provide the URL or bibliographic information.

Appendix C

Supplementary materials: Articles, websites, and texts.

(Items marked with an asterisk (*) are used explicitly in the course; unmarked items used as references, as needed.)

Note: Citations are given in American Psychological Association (APA; 6th Edition) format, which is the citation manual used in the course.

Articles, papers, and book chapters

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*Robson, E. (2002). Words and pictures: New light on Plimpton 322. *American Mathematical Monthly, 109*(2), 105–120.

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*Wisdom, J. O. (1953). Berkeley's criticism of the infinitesimal. *The British Journal for the Philosophy of Science, 4*(13), 22–25.

*Wilson, P. S., & Chauvot, J. B. (2000). Who? How? What? A strategy for using history to teach mathematics. *Mathematics Teacher, 93*(8), 642–645.

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Appendix D

Sample Detailed Assignment Description: "Response Papers" Assignment[3]

On three occasions during the semester, you are required to locate a journal article and complete a two-page (double-spaced; size 12 font; 1-inch margins) "response" to the article. Please use the appropriate assignment link in the **Assignments** folder in the online course site to submit your work.

Directions

The articles that you select, read, and respond to must be focused either on *the history of a mathematical topic* (the article can focus on mathematicians, mathematics, or both) or on *some aspect of teaching about the history of a mathematical topic*, including research studies on teaching using history. If you so choose, all three articles can possess the same focus (if you would really like to specialize over the course of the semester) or you can select a different topic of focus for each response paper. It is essential that each article selected is an actual feature article (and not a book review or website), and the article must be at least five journal pages in length (not including references).

Regardless of the focus of the article, the two-page response paper should address the following prompts (minimally). You might want to read through these before searching for and selecting your articles. Your response paper should be written in narrative form.

Prompts

(1) In what ways has your understanding of the mathematical topic changed or what knowledge do you have of the mathematician(s) as a result of reading the article you selected?

(2) What aspects of the topic or person(s) you selected were of particular interest to you?

(3) In what ways did the article encourage you in considering the use of the history of mathematics (related to the topic of the article or otherwise) in teaching?

(4) What questions did the article raise for you?

Each Response Paper should end with a proper citation of the article you selected. A sample template for the APA format for journal articles is:

Author1LastName, F. M., Author2LastName, F. M., & Author3LastName, F. M. (YEAR). Title of article with only first letter capitalized: And any Proper Names also capitalized. *Fascinating Journal Name, Volume Number*(Issue Number), xxx–yyy.

An example of an actual citation is:

Proia, L. M., & Menghini, M. (1984). Conic sections in the sky and on the earth. *Educational Studies in Mathematics, 15*(2), 191–210.

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[3] Students receive a detailed description of each primary course assignment at the beginning of the course. For space considerations, I include one of the shortest descriptions as an example.

Assessment Rubric

Score (possible points)	Category
	Completeness
1	The response paper is complete and submitted on time and as directed (e.g., via the assignment link, by the deadline, adhering to format, page restrictions (for the article and the response paper), and citation format)
	Topic Selection
1	The article selected is appropriate
	Content of Response Paper
3	The response paper captures the overarching idea of the article selected and each of the discussion prompts are addressed
2	The response paper only addresses *some* of the discussion prompts required **or** in addressing the discussion prompts, the overall idea of the article's content is concealed
1	The response paper addresses only one or two of the discussion prompts **and** the overall idea of the article's content is concealed
	Depth of Responses
2	The student's response includes thoughtful comments and supporting reasons for the ideas discussed. There is evidence of impact on learning or considerations for use of the history of mathematics in teaching.
1	The student's response is mostly descriptive with limited support for statements made. Lack of evidence of impact (i.e., the article and response paper exercise completed to fulfill course requirements without consideration of personal learning)
	Total points earned

Moving the History of Mathematics into the 21st Century

An Online Course that Includes Projects and Exercises

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Course Overview

This contribution describes an online history of mathematics course taught primarily to mathematics education majors at University of Central Florida (UCF), the second largest university in the country. The course is a required course for mathematics education majors who typically enroll in the course during their senior year. While the course was historically offered each fall semester as a traditional lecture course, strong motivations to offer online mathematics courses resulted in the course being converted to an online format. The course conversion provided the opportunity for a change in focus of the course to align it with the background and program of study of the enrolled mathematics education students. While the previous offerings of the course did not require the completion of any mathematical exercises, the converted course requires the students to complete mathematical exercises using applicable historical methods. Lesson plan projects were also added to the course in an effort to increase student value of the course material and provide learning experiences for the students which required them to make connections across curricula.

This course is designed to develop an appreciation of the contributions made by various cultures and individuals to the growth and development of mathematical ideas. To this end, throughout the course students explore the nature and character of the changes in mathematics over the centuries and investigate the role of mathematics in the development of humankind. They learn about ancient mathematics, medieval mathematics, and both early and late modern mathematics with particular attention paid to the development of topics such as geometry, algebra, and number theory.

Course Design

Looking at the development of mathematics over the centuries is an important part of all future mathematics educators' knowledge base. One of the primary goals of learning the history of mathematics is to effectively employ it in teaching the subject [1].

With this in mind, the course content selections are based not only on the enrolled student's prerequisite knowledge, but also on the applicability of the content to the sixth- through twelfth-grade mathematics classroom. As many of the enrolled students have not previously completed their required matrix and linear algebra course or the required introductory logic and proof course, special attention is paid to ensure that the selected readings from the textbook, *A History of Mathematics: An Introduction*, 3rd edition, by Victor J. Katz [2] contain content applicable for a student with knowledge of high school geometry, college level algebra, trigonometry, precalculus, and the first two semesters of calculus with analytical geometry. Furthermore, the courses typically taught in sixth- through twelfth-grade, including algebra, trigonometry, geometry, statistics, probability, and calculus, are also considered when selecting content for the course.

The course is divided into an introduction module, eleven content modules, and three test and project modules (see appendix A). Each week the student completes one of the content modules or one of the test and project modules. The modules are contained in the learning management system (LMS). The modules are opened on Monday morning with a due date on Sunday night which allows students with a busy schedule during the week to focus on their history of mathematics assignments on the weekends if necessary.

The introduction module contains the syllabus, protocols for online discussions, and course rubrics. It is available to enrolled students prior to the start of the semester and remains open for the entire semester so that students can reference it as necessary.

Each of the eleven content modules include an instructor-written introduction, a listing of assigned readings from the textbook as well as supplemental readings and videos, mathematical exercises assigned from the textbook, an online quiz, and a discussion post prompt. In the textbook readings, a description of the historical method used to solve a particular type of question is often provided instead of a worked out example of a similar question. As such, the assigned exercises typically require an understanding of the associated readings in order for the student to successfully complete the exercises using the historical method specified. While the assigned exercises are not routinely collected, each reading quiz has at least one question related to the assigned exercises and each of the tests given during the semester include three exercises selected from the exercise assignments that are to be completed using the specified historical method.

During each of the three testing and project weeks, no new content is introduced to the student. Instead, the student uses the time to complete their test and project. The testing and project modules contain general test information, project requirements, and applicable hints or reminders. The test information includes a listing of the covered content and the number and types of questions included on the test. The project requirements, including options for learning objective(s) to be included, and associated rubrics are also provided in the test and project module.

Resources

Resources used in the course include the learning management system (LMS) used to deliver the online components of the course, a required textbook, supplemental readings, videos, and websites.

The LMS used for the course is Canvas, which provides a variety of communication methods, discussion board capability, and online quiz assignments. Students enrolled in the course receive video and audio messages from their instructor regarding feedback on their course submissions and progress. These messages are made using the comment options within Canvas. The instructor communicates course information via written documents, embedded videos, and announcements in text, audio, or video formats.

The textbook, *A History of Mathematics: An Introduction*, 3rd edition by Victor J. Katz [2], includes exercises that can be modified to be prompts for essay questions, discussion prompts, or topics for a lesson plan project. The quality of the exercises was one of the motivating factors for selecting the textbook.

While the textbook coverage is complete, the history of mathematics student often benefits from considering a different perspective or point of view. As such, it was common for the students to share complementary resources they would find online with each other via the discussion board in addition to any supplementary readings, videos, and websites identified by the instructor (see References for a partial listing of supplementary readings, videos, and websites).

Assignments

Reading Quiz

A weekly reading quiz built using Canvas' quiz assignment feature is assigned in each content module. The format of the quiz is multiple choice, fill in the blank, and short answer. Each quiz includes at least one question based on the assigned mathematical exercises. The students are permitted a single attempt at the quiz. The questions are designed to assess the student's knowledge, application, and comprehension of the readings.

Discussion Post

A weekly discussion post is also assigned for each module. Some of the posts are public posts that required peer reviews or comments while others were in the form of a private post to the instructor. A discussion post rubric is provided (see appendix B). The discussion prompt is provided for each content module. The majority of the discussion prompts are created to require the student to use not only their comprehension of the readings, but to also analyze the current reading and/or synthesize the current reading in relation to the previous readings. An occasional prompt might require the student to complete some additional research beyond the assigned text and supplemental readings.

Tests

Three tests and a final exam are given in the course. The division of the test material is based on content modules that are grouped by time period. The first test covers ancient mathematics, the second covers medieval mathematics, and the third covers early modern (fifteenth- through seventeenth-century) and late modern (eighteenth-century to present) mathematics. Each test is comprised of fill in the blank, matching, true/false, and short answer questions as well as three math exercises, and an essay.

Typically, an online course does not have required class meeting times, and consequently, a common time is not reserved in the student's schedules for testing. A Doodle poll[1] is created to determine an appropriate group of testing times for each scheduled test date. Typically, two to three testing times are sufficient to ensure that all students are able to test without a class meeting conflict.

Projects for Mathematics Education Majors

During the semester, students complete three lesson plan projects related to the associated test content. For each project, the students are provided two or three learning objectives to select from for the focus of their lesson plan project. They are instructed to think of a substitute teacher, with a mathematics background, teaching their class for the day. The submitted project is to provide all the resources necessary for the substitute teacher to successfully conduct the class in their absence.

While a typical lesson plan would not include all the resources required for the project, the student is reminded that their submission should be a project that includes a lesson plan and all resources necessary to conduct the class, not just a single-page lesson plan. Typical project resources include a completed lesson plan for the class, the historical background for the lesson, a sheet of applicable terms and definitions, detailed lecture notes including worked out examples, and a method of evaluation such as a test, quiz, or worksheet, as well as an answer key with grading rubric for the included assessment. A lesson plan template is provided but students are free to use their own template (see appendix C).

Lesson Plan Peer Review

The purpose of the peer review is to provide an opportunity for students to evaluate and provide input on lesson(s) for appropriateness in meeting the developmental and learning needs of students in the specified grade level.

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[1] Doodle poll: Doodle.com offers a group event "poll" that doesn't require registration to the professional booking service.

Immediately following the lesson plan assignment deadline, the LMS randomly assigns each student three of their classmate's lesson plans to peer review. The peer review includes two parts. The first part of the review is to assign up to 40 points for the lesson plan based on the provided rubric (see appendix D). The second part of the review is to provide meaningful feedback regarding the lesson plan. The feedback is required to be typed, double-spaced, and 1–1 ½ pages per lesson plan reviewed. Peer reviews are submitted to the LMS within four days of the assignment deadline.

Students are encouraged to consider critical thinking, diversity, and human development and learning when completing the peer reviews. Specifically, they are asked to comment on whether the plan they reviewed

- Used appropriate techniques that promote and enhance critical, creative, and evaluative thinking capabilities of students.
- Used teaching and learning strategies that reflect each student's culture, learning styles, special needs, and socioeconomic background.
- Used an understanding of learning and human development to provide a positive learning environment that supports the intellectual, personal, and social development of all students.

Additionally, the peer reviews are required to address three questions:

1. Are there learning tasks appropriate for the cognitive developmental stage of students in the specified grade level?
    a) Provide one example from the lesson plan that supports your conclusion. When giving examples, please provide identifying information (paragraphs, headers, etc.) on which the examples appear.
    b) Provide one example of how the lesson plan could be improved to be more congruent with the theme of your question. Make sure that the example here is sufficiently described so that someone could implement that improvement on their own just from reading your description. When giving examples, please provide identifying information (paragraphs, headers, etc.) on which the examples appear.
2. Are there activities that require students to engage in complex problem solving or critical thinking? *Please provide information to identify where the activities appear in the lesson plan.*
    a) Provide one example from the lesson plan that supports your conclusion.
    b) Provide one example of how the lesson plan could be improved to be more congruent with the theme of your question.
3. Does the lesson plan satisfy the requirement(s) of the assignment? Provide evidence supporting your evaluation.

The peer reviews each student completes are graded by the instructor based on the peer review evaluation rubric. (See appendix E.) Proper conventions such as punctuation, capitalization, grammar, and spelling are expected. The student earns up to 10 points based on the quality of the peer reviews they complete. While the students often comment that the lesson plan peer review is their first attempt at grading, the quality of the feedback they provide to their classmates is typically very good. Students have also commented that the reviews provided them the opportunity to expand their ideas of different ways to include the history of mathematics in the mathematics classroom.

The overall project grade consists of the average of lesson plan grades by the peer reviewers (40 of the 50 possible points) and the grade assigned by the instructor for the completed peer reviews (10 of the 50 possible points). The professor reserves the right to adjust the lesson plan grade and this was clearly stated in the syllabus.

Research Papers for Non-Mathematics Education Majors

The course is a required course for mathematics education majors, however mathematics majors are not able to use the course to satisfy requirements in their program of study. As such, rarely does a non-mathematics education major enroll in the course. In the event that an enrolled student is not a mathematics education major, he or she can submit a five- to seven-page research paper instead of a lesson plan project. The student proposed research topic requires instructor approval. Instructor approval is typically given as long as the student's proposed topic pertains to the associated test content and provides the opportunity for exploration beyond the current depth of the course coverage. The research papers are submitted to the LMS via a Canvas assignment that has the option selected to enable turnitin[2] submissions. As there is typically at most one enrolled student who is not a mathematics education major, peer reviews on the research papers are not completed.

Lessons Learned

The initial online offering of the course taught by a colleague was subdivided into twenty content modules and the instructor of that course shared with the author that this was too ambitious of a goal [3]. The author decided that one content module a week would be used resulting in eleven content modules for the semester. The weekly module structure appears to be a good fit for students who are typically completing an internship during the day and attending the lectures for their other two courses during the evenings on Monday through Thursday.

The initial offering of the course provided up to six different testing times for each test, resulting in testing sessions where only one or two students completed the test during the session. By using the Doodle Poll, it was determined that two to three sessions were sufficient to avoid conflicts with other classed scheduled and this change was implemented during the second offering of the converted course. While the decrease in offered testing times might be inconvenient for the student, it provides a more efficient use of instructor time and classroom space.

Although it is challenging to include mathematical exercises in an online version of the course, it is this author's opinion that the decision to include them is a sound one. The students find these exercises the most challenging part of the course as the textbook does not provide complete examples for the students to follow when attempting the assigned exercises. During discussions with students, they mentioned that these types of exercises improve their critical thinking skills. Several students have compared having assigned exercises without the benefit of completed examples to their mission in the classroom to have students discover the method or answer instead of them being directly told the method or answer.

During the initial offering of the converted course, only some of the content modules required a discussion post. As online discussion posts increase student interactions in the course as well as with each other, the content modules were modified to require a discussion post in each of the content modules. Student interactions increased as a result of this change.

While there were certainly concerns regarding the conversion of the course to an online format, student response has generally been positive. The most common student comment on student perception surveys is that the course is more demanding than they anticipated. From a job satisfaction point of view, the author finds the course challenging and time consuming but extremely rewarding.

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[2] Turnitin: Tool that provides the instructor the ability to easily check for potential plagiarism, check spelling and grammar usage, use drag and drop comments, and leave voice comments. Instructors receive an originality report and similarity index, based on how much matching text was found. The similarity index can be used to locate potential sources of plagiarism, or text that may have been incorrectly cited.

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Appendix A:
Course Outline

The following content modules include the chapter titles from the textbook that is the primary resource for the course.

Course Introduction

Week 1 or prior to the start of the semester if possible:

- Syllabus
- Protocols for online discussions
- Rubrics for discussion post, peer reviews, and projects

Ancient Mathematics

Week 1

- Content Module: Egypt and Mesopotamia and The Beginnings of Mathematics in Greece
- Reading Assignments
    - Chapter 1 Katz textbook (pages 1–27)
    - Chapter 2 Katz textbook (pages 32–46)

Week 2

- Content Module: Euclid [4], [5]
- Reading Assignments
    - Chapter 3 Katz textbook (pages 50–89)
    - Paul Daus, "Why and How We should Correct the Mistakes in Euclid," *Mathematics Teacher* 53 (1960), 576–581
    - View the Khan Academy video Euclid as the Father of Geometry

Week 3

- Content Module: Archimedes and Apollonius; Mathematical Methods in Hellenistic Times, and The Final Chapters of Greek Mathematics
- Reading Assignments
    - Chapter 4 Katz textbook (pages 94–126)
    - Section 5.3 Katz textbook (pages 157–167)
    - Chapter 6 Katz textbook (pages 172–190)

Week 4

- Test 1 and Project 1: Ancient Mathematics

Medieval Mathematics

Week 5

- Content Module: Ancient and Medieval China and Ancient and Medieval India
- Reading Assignments
    - Chapter 7 Katz textbook (pages 195–225)

    - Chapter 8 Katz textbook (pages 230-259)

Week 6

- Content Module: Mathematics of Islam [6]
- Reading Assignments
    - Chapter 9 Katz textbook (pages 265-317)
    - J. Hoyrup, "The Formation of Islamic Mathematics: Sources and Conditions," *Science in Context 1*, pp. 281-329, 1987.

Week 7

- Content Module: Mathematics in Medieval Europe and Mathematics Around the World Including Mathematics in America, Africa, and the Pacific
- Reading Assignments
    - Chapter 10 Katz textbook (pages 324-358)
    - Chapter 11 Katz textbook (pages 364-378)

Week 8

- Test 2 and Project 2: Medieval Mathematics

Early and Late Modern Mathematics

Week 9

- Content Module: Algebra in the Renaissance
- Reading Assignments
    - Chapter 12 Katz textbook (pages 383–417)

Week 10

- Content Module: Mathematical Methods in the Renaissance [7], [8]
- Reading Assignments
    - Chapter 13 Katz textbook (pages 423–461)
    - E. Marcotte, "Eric's Slide Rule Site," www.sliderule.ca/intro.htm.
    - The Oughtred Society, "Slide Rule History," www.oughtred.org/history.shtml.

Week 11

- Content Module: Algebra, Geometry, and Probability in the Seventeenth Century [9]
- Reading Assignments
    - Chapter 14 Katz textbook (pages 467–500)
    - S. Singh, "Homer's Last Theorem," 2013-2014. [Online]. Available: http://www.slate.com/blogs/browbeat/2013/10/29/the_simpsons_and_fermat_s_last_theorem_wizard_of_evergreen_terrace_has_brilliant.html.

Week 12

- Content Module: The Beginnings of Calculus and Newton and Leibniz
- Reading Assignments
    - Chapter 15 Katz textbook (pages 507-538)
    - Chapter 16 Katz textbook (pages 543-578)

Week 13

- Content Module: Selections from "Late Modern" Mathematics and Women in Mathematics [10], [11]
- Reading Assignments
    - Chapter 19 Katz textbook (pages 665–682)
    - Section 21.4.1-21.4.2 Katz textbook (pages 740–744)
    - Section 24.4 Katz textbook (pages 858–862)
    - Selected sections chapter 25 Katz textbook
        - 25.2 Women in Mathematics sidebar (page 899),
        - 25.3 New Ideas in Algebra introduction only (pages 890–891)
        - 25.4.1 Ronald Fisher and Hypothesis Testing (pages 903–904)
        - 25.5–25.5.3 Computers and Applications introduction (pages 907–915)
        - 25.6.3 Proof of the Four Color Theorem (pages 922–924)
        - Selections of the Katz textbook that highlight women in mathematics: Sophie Germain (page 715), Maria Agnesi (pages 616–617), Sofia Kovalevskaya (page 787), Ada Byron Lovelace (page 911), Emmy Noether (page 898), and Grace Chisholm Young (page 883)
        - Appendix A.2: (pages 935–939)

Week 14

- Test 3 and Project 3: Early and Late Modern Mathematics

Week 15

- Final Exam

Appendix B

Discussion Post Rubric

Proficient	9–10 points	Response goes beyond simply addressing the basic factual content of the question to incorporate multiple sources and/or stimulates and provides analysis of the question.	Evidences a very high level of comprehension of the readings and/or the historical context of the question.
Satisfactory	7–9 points	Response deals correctly with the basic factual content but fails to adequately provide analysis of the question.	Evidences a high level of comprehension of the readings and/or the historical context of the question.
Superficial	6–7 points	Response correctly but superficially addresses the material of the question.	Evidences a superficial level of comprehension of the readings and/or the historical context of the question.
Unsatisfactory	5–6 points	Response does not adequately address the content of the question.	Evidences a low level of comprehension of the readings and/or the historical context of the question.
Deficient	0–5 points	Unresponsive to the question.	Insubstantial evidence of comprehension of the readings and/or historical context.

Appendix C

Lesson Plan Template

NAME:	SUBJECT:
LESSON TITLE:	GRADE LEVEL:
COMMON CORE STATE STANDARDS: select at least one Standard and at least one Benchmark.	0–4 points assigned per lesson plan rubric
RATIONALE: explain why students should be learning this content	0–4 points assigned per lesson plan rubric
GOALS OBJECTIVE: this is the long-term objective	0–2 points assigned per lesson plan rubric
CONTENT: outline what you are teaching and include terms/definitions	0–8 points assigned per lesson plan rubric
PROCEDURES: be sure to include ESOL strategies	0–8 points assigned per lesson plan rubric
CLOSURE/SUMMARY: explain how you will end the lesson	0–2 points assigned per lesson plan rubric
RESOURCES: list any materials needed for the lesson, e.g., graphic organizer	0–2 points assigned per lesson plan rubric
EVALUATION: how you will assess-include condition, performance, and criteria	0–8 points assigned per lesson plan rubric
LOOK AHEAD: Describe how this lesson would fit into the "next" lesson plan or topic in the unit	0–2 points assigned per lesson plan rubric

Appendix D

Lesson Plan Rubric

Proficient	4 points	Response goes above and beyond addressing the lesson plan component requirements	Evidences a high level of comprehension of the historical context of the assignment and/or above satisfactory effort put into completing the lesson plan component
Satisfactory	3 points	Response addresses the lesson plan component requirements	Evidences a satisfactory level of comprehension of the historical context of the assignment and/or satisfactory effort put into completing the lesson plan component
Superficial	2 points	Response superficially addresses the lesson plan component requirements	Evidences a superficial level of comprehension of the historical context of the assignment and/or superficial effort put into completing the lesson plan component
Unsatisfactory	1 point	Response does not adequately address the lesson plan component requirements	Evidences a low level of comprehension of the historical context of the assignment and/or unsatisfactory effort put into completing the lesson plan component
Deficient	0 points	Unresponsive to the lesson plan component requirements	Insubstantial evidence of comprehension of the historical context of the assignment and/or effort put into completing the lesson plan component

Appendix E

Peer Review Evaluation Rubric

Proficient	9–10 points	Completes three peer evaluations	Evidences a very high level of effort put into the peer evaluation with meaningful comments and suggestions. Addresses the three questions completely with a very high level of detail. Meets the page length requirement. Evidence of proper conventions such as punctuation, capitalization, grammar, and spelling used throughout the review.
Satisfactory	7–9 points	Completes three peer evaluations	Evidences a high level of effort put into the peer evaluation with meaningful comments and suggestions. Addresses the three questions completely with a high level of detail. Meets the page length requirement. Evidence of proper conventions such as punctuation, capitalization, grammar, and spelling used throughout the review.
Superficial	6–7 points	Completes two or three peer evaluations	Evidences a satisfactory level of effort put into the peer evaluation with superficial comments and suggestions. Addresses the three questions completely with a superficial level of detail. Meets the page length requirement. Evidence of proper conventions such as punctuation, capitalization, grammar, and spelling used throughout the review.
Unsatisfactory	5–6 points	Completes two peer evaluations	Evidences an unsatisfactory level of effort put into the peer evaluation. One of the following is present: Does not address the three questions completely with an appropriate level of detail, does not meet the page length requirement, or evidence of improper conventions such as punctuation, capitalization, grammar, and spelling used in the review.
Deficient	0–5 points	Completes fewer than two peer evaluations	Evidences a deficient level of effort put into the peer evaluation. Two or more of the following are present: Does not address the three questions completely with an appropriate level of detail, does not meet the page length requirement, or evidence of improper conventions such as punctuation, capitalization, grammar, and spelling used in the review.

An Online History of Mathematics Course Directed Towards the Development of Western Intellectual History

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Course Overview

In the spring of 2012, the Department of Mathematics at the University of Central Florida found itself under the obligation to offer a number of online courses. A determination was made at that time to offer our History of Mathematics course in the Fall Semester of 2012 as one of these courses. To see a very different version of this course offered in the Fall Semester of 2013 see the article by T. Muhs in this volume.

History of Mathematics is a required course for Mathematics Education majors at the University of Central Florida (UCF). The course enrollment in the history of mathematics course consists almost entirely of students majoring in Mathematics Education. Due to scheduling reasons, the course is usually taken during the same semester as the course in Modern Geometries and a three-hour pre-internship Classroom experience. This made it particularly advantageous to offer this course in an online format.

As must be expected for any contemporary online course, a major design factor in designing this course is the requirement that student access is possible across a large variety of platforms including cellphone access. The twin factors of the bandwidth limitations imposed by the requirement of cellphone access and the need for transcripts for any video component eliminated any possibility of relying on recorded lectures as a foundation for the course.

The prerequisites for the history of mathematics course are completion of Calculus with Analytic Geometry II or consent of the course instructor. This is regarded as essential background to ensure that the student will have some understanding of integration.

Course Design

Teaching an online course requires a very different approach to the process of course development and preparation than is required for a face-to-face lecture. As part of the preparation for teaching this course as a online course, the author was required to take a required course offered by the university: Interactive Distributed Learning for Technology-Mediated Course Delivery. This course addressed in some detail issues that have been detected in offering web-based courses. Although there was nothing specific in the training that was offered for online mathematics courses, the course did offer a serious examination of the issues [1] in online pedagogy and consulting with persons who have taught online courses before is highly recommended to anyone who is interested in development of an online course.

The nature of an online course requires that several key decisions be made early in the course design process. The first key decision was to determine the course instructor's expectations of intellectual content of the course.

The course description of the course is: A chronological study of the evolution of mathematical thought from primitive counting through modern ideas of the 20th century. The initial concept was to develop a comprehensive course examining the broad history of mathematics in multiple cultural and sociological settings. Examining the limitations of the course design parameters it soon became evident that original expectations of the instructor of being able to explore the development of mathematics in non-western/middle eastern traditions could not be realized. Online courses are intrinsically different from a face-to-face course; there is no opportunity to use the devices of a lecture to subdivide the lecture or to measure the success of the presentation from the participation of the audience. Instead the modular structure that is required mandates the presentation of the material in a coarser manner than might be realized in a lecture. These conflicting design parameters, requiring the development of ideas from the primitive to the twentieth century and the limitations of using a modular course design required that choices be made in focusing the course. There are many possible solutions to this dilemma. In this case the choice was made to concentrate on the development of mathematics within western intellectual history and focus the course on the development of algebra and analysis from Babylon to the mid-twentieth century.

This was embodied in the following statement to the students in the course syllabus.

> *This course provides an introduction to the development of mathematics through history and culture. The course will focus on the development of algebra and analysis from the beginnings to the early 20th century. Other topics including the development of mathematics in the United States, women in mathematics, historiographical issues in mathematics, and the role of mathematics in pedagogy will also be addressed.*

The second key decision was the division of the course into modules. In an online course the module rather than the class is the fundamental unit. To realize the instructor's expectations of the course as indicated above, it was subdivided into twenty modules with an expectation that this would correspond to approximately two hour-long classroom sessions per module. In addition to the modules a midterm examination and a final examination were required.

There were a number of issues that impacted the design of the modules. The most determinative issue was bandwidth. One of the design parameters was that students would be using their smart phones as a principal means of accessing the course. This meant that videos needed to be held to less than 5 minutes in length. In addition, accessibility issues required a transcript for all videos. The combination of these issues eliminated the recorded lecture format as the primary means for the conveying the course material to the students.

If lectures are not the focus of the course, there is a gap in the course. What replaces the lecture as the principle means of information delivery? Our answer is to replace the lectures of the course by extensive student reading of primary and secondary sources to address course content. Each module has a considerable reading assignment that would constitute the central component of the module. The limited video instruction directs the student reading by placing it in a context, summarizing the reading, and leading students to critical questions to be addressed in the module. They would then communicate on the discussion forum with other students regarding the material and then be assessed on their understanding. All of the videos used in the course were produced in-house specifically for the course. All of the videos were fully scripted and the scripts were used to provide subtitles to ensure that the course would meet university requirements for accessibility.

A factor that did not affect the course design was the software that was used to formulate the course. This course used Canvas but the course was designed so as to be implementable on other software.

Module Design

Since as noted above the modules were based on two hours of lecture for a three credit course. The pace of the course was set to cover three modules every two weeks. Each of the modules (with the exception of Module 01 which is an introduction to the course) had the following components:

- Introduction: Introduces the topic of the module
- Objectives: States the goals for each module
- Readings: Directs the students to the materials from the textbook and other literature on the topic of the module
- Assessments: Provides quizzes and short response essays to assess and direct student understanding

This was realized in the following structure:

Preamble:

Every module has a number identifier and a title. This is followed by a short paragraph that introduces the topic of the individual module.

For example, the preamble and objectives for Module 06, Medieval Mathematics, was

> *The mathematics of the medieval time period was closely linked with classical ideas. We focus on two particular aspects of the period that advanced the development of algebra. One is the mathematical school that developed in the Islamic civilization that extended between the Indian subcontinent and the Iberian peninsula. The second is the commercial school that developed on the Italian peninsula.*

- *Identify major contributions to the development of Algebra in the medieval period*
- *Understand the development of algebraic concepts*
- *Analyze the consequences of the development of algebraic concepts*

Assignment:

Readings for the module were given. The readings were introduced by a video. This is the critical part of the module. Students were expected to have completed all of the assigned reading and to have considered their meaning before proceeding. The reading assignments for several modules can be found in the appendix to this article.

Assessment:

A reading quiz confirms that the readings have been completed. Each module had a 10 point reading quiz. The reading quizzes were ten-question multiple-choice assessments to ensure that the students had in fact completed the readings. The questions addressed factual information that was contained in the readings.

Assignment:

A video introduced the discussion of the readings. This video directly asked the students the five questions of the short response quiz that followed and directed students to consider the questions in the light of the course readings. The questions themselves were open-ended and were meant to first inspire discussion and then to allow individual students to provide their own answer. Questions from Module 06 included: “Is the work of Al-Khawarizmi algebra?” and “What were the roles of the *Liber mahameleth* and the *Liber abaci*? What influence did these books have?”

Assignment/Assessment:

Students were required to participate in the discussion board for the module. Up to 20 points could be earned through the activity on the discussion board.

Assessment:

A 20 point short-response quiz indicates the assimilation of the goals of the module. The questions for this assessment were exactly the questions that were addressed in the video after the readings and were presumably discussed in the module discussion assignment.

Resources

As can be seen from the sample module readings provided in the appendix, many of the readings for the course were available online. Nevertheless, it was determined early in the process of course design that a variety of texts would be needed to provide additional reading and guidance for the students. Ultimately students were required to purchase six texts for the class [2–7]

The texts were used in different ways. Some texts were used for only parts of the course. Text [5] was used only for modules 5, 6 and 7. Text [6] was only used for modules 7, 8, and 9. Dawson's biography of Gödel [2] was only used in module 18. Text [3] served as a major resource for the development of the history of analysis in modules 10, 11, 12, 15, and 16.

The major texts for the course were Stedall's source book [6] which provided students with source reading in Modules 6 through 17 and Katz's text [4] which was valued not only for its coverage but also its historiographical approach to the material. It was used in modules 3 through 17. It was also a valuable source of readings for the course from the copious references that it contains.

While some might consider the employment of a comprehensive textbook overkill for such a course, it was appreciated that a uniform source for information would provide students a structural framework for the course. It also provided an enhanced context for the readings that increased student appreciation of the content of the readings.

Lessons Learned

The course would have to be considered a qualified success due to a number of difficulties. The problems were not those that the reader might immediately think when regarding the course. Neither the large amount of reading nor the expense of so many textbooks occasioned any complaint from students. A number of students believed that this was a course that awakened a real understanding of mathematics as an important element in the development of Western Intellectual History. Indeed the course content and format were not a difficulty. The problems arose from the unexpectedly large time commitment that the course imposed on the instructor, difficulties in administration of examinations, and technical difficulties in linking references to a variety of platforms with differing permissions for accessing material.

The first problem that arose was that less than one-third of the course had been prepared by the instructor before the beginning of the semester. As the preparation of each module and particularly the creation of scripts for the videos took a considerable amount of time, this left the instructor constantly in a position of making certain that the material for the next module was available in time for the module to be viewed by the students. For an online course, all of the material for the course needs to be developed and tested well before the beginning of the semester.

The second problem was the difficulties in examination administration. The students not being physically on campus for the course meant that the examinations must be administered in a wide variety of remote locations by a number of very different exam administrators. In the future the administration of the examinations and the required response time and method of examination administration will have to be more closely controlled.

Third, problems developed over remote access to materials through our library. Links that would work on a platform in one location would not be accessible when students would access the material on another platform. The platforms that students used were widely varied: iPhones and Android phones, laptops using wifi, high bandwidth connections and low bandwidth dial-up connections, and connections both inside and outside the campus firewall. Eventually the problems that were caused by the varying manner in which the student's differing platforms addressed the course content were resolved. However, prior to that resolution the variance in how each student's platform addressed the course content occasioned considerable frustration to both instructor and students. The importance of testing the material on a wide variety of platforms cannot be underestimated.

There is one change that should be made in the course to improve it. The instructor was advised early in the development process that rather than making the module the basis for two hour-long class lectures, that each module should correspond to a real week to promote the establishment of a course routine with assignments due at the same time and day of the week for each module in the course. This model was rejected to provide a greater flexibility in the introduction of topics. Based on the experience in online teaching obtained in this course, I would judge this as a serious mistake and recommend adoption of the one module per week model even at the cost of omitting major topics of mathematical development.

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Appendix A
Module Objectives

Module 01: Class organization

Introduction

This module provides an introduction to the web course version of MHF4404.

Module Objectives

- Acquaint students with the pedagogical, logistical, and technical aspects of the course
- Introduce the scope and content of the course
- Introduce the course instructor

Module 02: Pedagogy and the History of Mathematics

Introduction

This module examines the role of the history of mathematics in pedagogical praxis.

Module Objectives

- Identify the reasons for the study of the history of mathematics
- Understand the role of the history of mathematics in pedagogical praxis

Module 03: Historiography of Mathematics

Introduction

This module addresses the historiography of the history of mathematics and introduces the student to problems in the historiography of mathematics.

Objectives

- Identify issues in the historiography of the history of mathematics
- Understand the influence of the historians perspective on the history of mathematics
- Analyze a historiographical problem in the history of mathematics

Module 04: Mathematics, Philosophy

Introduction

This module examines the connection of philosophy to mathematics.

Objectives

- Identify issues relating mathematics and philosophy
- Understand questions and problems relating mathematics and philosophy
- Analyze the role of mathematics as part of the intellectual history of humankind

Module 05: Algebra: Ancient Mathematics in Mesopotamia and Greece

Introduction

This module focuses on the mathematical traditions of ancient Greece and Mesopotamia. In Mesopotamia we focus our attentions on a clay shard called Plimpton 322 and the debate over its meaning. In Greece we look at the question of the solution of the quadratic equation.

Objectives

- Identify issues regarding the development of mathematics in ancient Mesopotamia and Greece
- Understand the development of mathematical technique in ancient Mesopotamia and Greece
- Analyze mathematical concepts within the context of the developer

Module 06: Algebra: Medieval Mathematics

Introduction

The mathematics of the medieval time period was closely linked with classical ideas. We focus on two particular foci of the period that advanced the development of algebra. One is the mathematical school that developed in the Islamic civilization that extended between the Indian subcontinent and the Iberian peninsula. The second is the commercial school that developed on the Italian peninsula.

Objectives

- Identify major contributions to the development of algebra in the medieval period
- Understand the development of algebraic concepts
- Analyze the consequences of the development of algebraic concepts

Module 07: Algebra: Renaissance Mathematics

Introduction

The development of algebra in the renaissance leads to algorithms for the solution of cubic and biquadratic equations (before those concepts exist). It also brings a change in understanding of the concept of number.

Objectives

- Identify major contributions to the development of Algebra in the renaissance period
- Understand the development of algebraic concepts during the renaissance period
- Analyze the consequences of the development of algebraic concepts

Module 08: Algebra: The seventeenth century

Introduction

This module addresses the development of algebra in the seventeenth century.

Objectives

- Identify major contributions to the development of algebra in the seventeenth century
- Understand the development of algebraic concepts during the seventeenth century
- Analyze the consequences of the development of algebraic concepts

Module 09: Algebra: The eighteenth century

Introduction

This module addresses the development of algebra in the eighteenth century.

Objectives

- Identify major contributions to the development of algebra in the eighteenth century
- Understand the development of algebraic concepts during the eighteenth century
- Analyze the consequences of the development of algebraic concepts

Module 10: Analysis: The proto-analytic era

Introduction

The differential and integral calculus grew from the steady development of ideas from predecessors of Newton and Leibnitz. The module illustrates that development.

Objectives

- Identify major contributions to the development of analysis in the period before the invention of the calculus
- Understand the development of analytic concepts in the period before the invention of the Calculus
- Analyze the consequences of the development of these analytic concepts

Module 11: Analysis: Newton, Leibnitz, Berkeley

Introduction

The differential and integral calculus are an integral part of our understanding of higher mathematics. The path to this understanding was not assured and the development of the calculus over the last 500 years is a story of the development of modern mathematics. In this module we consider Newton and Leibnitz and their competing theories of calculus.

Objectives

- Identify major contributions to the development of calculus by Newton and Leibnitz
- Understand the development of analytic concepts by Newton and Leibnitz
- Analyze the consequences of the development of analytic concepts

Module 12: Analysis: The eighteenth century

Introduction

This module considers the development of the techniques of the differential and integral calculus through the eighteenth century.

Objectives

- Identify major contributions to the development of analysis in the eighteenth century
- Understand the development of analytic concepts in the eighteenth century
- Analyze the consequences of the development of analytic concepts

Module 13: Euler, Gauss, and Abel

Introduction

Euler, Gauss, and Abel were dominant mathematicians of the latter half of the eighteenth century and the first half of the nineteenth century. We investigate their influence on mathematics

Objectives

- Identify major contributions to the development of Mathematics by Euler, Gauss, and Abel.
- Understand the mathematics of Euler, Gauss, and Abel in the context of the late eighteenth/early nineteenth century
- Be prepared to analyze the consequences of the work of Euler, Gauss, and Abel in later mathematics

Module 14: Algebra: Galois, Dedekind

Introduction

In very different ways, Galois and Dedekind strongly influenced the development of algebra in the nineteenth century. Galois brought the concept of symmetry to algebra. Dedekind brought to algebra the concept of abstraction.

Objectives

- Identify the influence of Galois and Dedekind on mathematics
- Understand the influence of Galois theory and the question of what is a number
- Analyze the application of Galois theory and the consequences to the study of number

Module 15: Analysis: Cauchy, Bolzano, and Weierstrass

Introduction

Analysis continued to develop throughout the nineteenth century. This module has its focus on the requirement for rigor and abstraction in analysis with particular concentration on the example of uniform continuity.

Objectives

- Identify the development of rigor and abstraction in nineteenth century analysis,
- Understand the need for the development of rigor in the analysis of the nineteenth century
- Analyze the abstraction that was made possible by the advancement of rigor

Module 16: Analysis: Cantor and the Foundational Challenges

Introduction

Cantor's introduction of set theory revealed issues with the foundation of mathematics.

Objectives

- Identify the crisis in the foundations of mathematics that occurred in the latter half of the nineteenth century and the early part of the twentieth century.
- Understand the issues underlying the crisis in the foundations of mathematics

Module 17: Klein, Hilbert, E. Noether

Introduction

Algebra and geometry recombine in the light of advances made by the Klein, Hilbert and E. Noether.

Objectives

- Identify the crisis in the foundations of mathematics that occurred in the latter half of the nineteenth century and the early part of the twentieth century.
- Understand the issues underlying the crisis in the foundations of mathematics

Module 18: Gödel

Introduction

This module introduces the work of the principal contributor to mathematical logic for at least the first half or the twentieth century and a towering figure in the intellectual history of humankind: Kurt Gödel.

Objectives

- Identify the importance of Gödel 's work with particular attention to the Incompleteness Theorem, and the Continuum Hypothesis
- Identify the cultural context of Gödel
- Understand the issues underlying the crisis in the foundations of mathematics

Module 19: Mathematics in America

Introduction

This module presents the growth of academic mathematics in America during the very late nineteenth century. For conciseness we concentrate on one (admittedly) atypical department: the Department of Mathematics at the University of Chicago.

Objectives

- Identify the growth of American mathematics
- Analyze the particularly American aspects of mathematical history

Module 20: Women in Mathematics

Introduction

This module examines the role of women in mathematics.

Objectives

- Identify the role of women in mathematics
- Analyze the struggle for gender equality in mathematics

Appendix B
Module Reading Assignments

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V

Specialty Mathematics Courses

Geometry Through History

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Course Overview

Geometry Through History aims to strengthen the geometry skills of current and future teachers of secondary mathematics, while providing a historical understanding of the main topics from the secondary curriculum. It is offered as a summer course for graduate students in secondary mathematics education, but well prepared undergraduates have been successful students. The prerequisites include a mastery of calculus through computing area and volume, some familiarity with high school geometry, and comfort with proof.

Geometry Through History mixes preparation for secondary classroom teaching with exercise in proving and problem solving, all in a historical context. It has worked very well, and it would not be difficult to vary the proportions devoted to the several goals of the course. It is worth noting that the new Common Core State Standards place more emphasis on proof in secondary geometry, more than current high school teachers saw in their own secondary experience. In this sense, Geometry Through History helps prepare teachers for Common Core mathematics.

Course Design

The organizing principles are three: to develop geometry in a logical order, to apply original methods whenever pedagogically feasible, and to set the logically ordered material in historical context. Several assumptions lie behind this course. First, that secondary geometry still carries marks of its history. Why do we have both ASA and AAS congruence theorems? Why does geometry, unlike the other secondary mathematics courses, have axioms and propositions? Second, that removing modern trappings may let us see more clearly the heart of a topic, a problem, an issue. Why is it interesting that the area and the perimeter of a circle involve the same constant, π, and how do we explain the connection? Third, that mathematics is a great human achievement, spanning ages and civilizations.

Geometry Through History consists of seven sections, described below, starting with Euclid. The foundational material is found in the first two sections, on Euclid and on Dilations; these sections together took about 50% of class time, much of that in problem solving assignments that stretched through the entire course. The remaining time was nearly equally divided among the remaining five sections, with a slightly larger piece for non-Euclidean Geometry.

Euclid

The first section of the course is based on Books 1, 3, and 6 of Euclid's *Elements*, of about 300 BC. This works best with an edited translation of the original; I recommend the online [10], while [11] has detailed commentary and [6] has a student-friendly treatment of Book 1. We emphasize triangle congruence, straightedge/compass construction, parallels and Euclid's Postulate 5, the Pythagorean Theorem, the properties of circles, and triangle similarity. The students work through Euclid's postulate-proposition-proof development of the subject, while comparing Euclid to a typical secondary development of the subject. We start with Euclid's Book 1 Prop 4, SAS triangle congruence, proved as Euclid did in an informal "transformation" approach, after which the other congruence theorems are proved by synthetic methods.

Foundational issues are kept in mind, with the distinction of *absolute* or *neutral geometry*—which we treat as geometry based on Euclid without Postulate 5—and *Euclidean geometry*—that which results from adding Euclid's Postulate 5 to our list of assumed statements. Euclid's Book 1 Prop 16 is pivotal in absolute geometry, that an exterior angle of a triangle is greater in measure than either opposite interior angle. Prop 16 is the basis for the AAS congruence theorem, in Prop 26, and for Prop 27, that when lines are cut by a transversal so alternate interior angles are congruent, then the lines are parallel. Beginning with Prop 29, we have a sequence of equivalent propositions of Euclidean geometry, depending on Euclid's Postulate 5. Proving equivalence is a major theme of the course.

With Euclid's Book 3, on circles, we trace the development of theorems relating arc measure to the angles formed by chords and secants and tangents, culminating in Book 3 Prop 17, Euclid's construction of a tangent to a circle from a given outside point.

With Book 6, triangle similarity is developed. In contrast to the typical modern development, the Side-Splitter Theorem is proved first, based on area considerations. (Side-Splitter Theorem: A line parallel to one side of a triangle cuts the other sides in proportion.) Then the triangle similarity theorems follow.

Similarity by Dilations

Following on Euclid's treatment of triangle similarity in Book 6, a modern approach, based on dilations in the coordinate plane, is introduced. One finds this material developed at the secondary level in [20, Chapter 12]. Important applications include construction of a common tangent to two given circles. We see in this section a facet of the history of mathematics: change in the school mathematics curriculum.

Liu Hui and π

This introduction to ancient Chinese mathematics is intended to show some lovely mathematics, developing the circle area formula, a development that is quickly comprehensible to the students. At the same time, we see a cultural context that contrasts to that of ancient Greece, resulting in contrasting presentations of mathematics.

This exploration is limited to Liu Hui's remarkable commentary, from 263 AD, on the *Jiuzhang Suanshu* —translated as *The Nine Chapters on the Mathematical Art*. The section in Geometry Through History is based on Chapter 1, Problem 31. The *Nine Chapters* gives the correct relation of area to circumference: "Multiplying half the circumference by the radius yields the area of the circle," but gives 3 as the ratio of circumference to diameter [13, p. 88]. Liu Hui approximated π by computing the area of polygons that inscribe and circumscribe a circle. Along the way he justifies the rule of Problem 31. [See Appendix B for more details.]

Archimedes and Cavalieri's Principle: Area and Volume

We apply the *method of exhaustion* to find areas. Two nice examples from Archimedes are studied, the area of the parabolic segment and the area of a turn of a spiral. Nice explication of Archimedes' arguments is found in [19]; there one can also find Archimedes' derivations of sphere volume and surface area formulas, although they are harder than those noted below. The primary source, *On Spirals* [3, pp. 151–188], is helpful.

Basic material is taken from [20], where Cavalieri's Principle is introduced. Cavalieri's Principle is called on to show that the volume of a pyramid is one-third of the volume of a prism with the same base and equal height. In [20], further, we find an important application of Cavalieri's Principle: finding the volume of the sphere. [See Appendix B for more details.]

Early Trigonometry and How to Make a Trig Table

We begin with the *chord* of Greek trigonometry, which goes back to, at least, Hipparchus (c. 190–120 BC). Work by Hipparchus does not survive, so we turn to Ptolemy (c. 100–178 AD), who told us how he built a chord table for a circle of radius 60. In Geometry Through History, we follow Asgar Aaboe's chapter "Ptolemy's construction of a trigonometric table" [1, pp. 101–130]. However, the instructor's decision was to derive some elementary properties for chords, but then for more intricate formulas, including trigonometric values for sums and differences, we look at the sine and cosine formulas.

In class, we find *crd*(36°) by applying triangle similarity to the 36-72-72 isosceles triangle. *crd*(60°) is 60, from which we find *crd*(30°) by the half-angle formula. This lets us find values for $6 = 36 - 30$ degrees, then 3 degrees, and 1.5 degrees, and .75 degrees.

Then we move to India, where the *sine* was introduced, and then follow the transmission to Islamic mathematicians. In the *Exhaustive Treatise on Shadows*, by al-Biruni (973–1055), all six of the modern trigonometric functions appeared, without their modern names [12, pp. 306–308].

Now, to build a table in intervals of a half degree, as Ptolemy did, we need to know *crd*(1°) or sin(1°), to the precision of our table. This sets us onto the next issue in the history of trigonometry, the quest for an accurate value of sin(1°). We note that Ptolemy used a geometric argument to approximate *crd*(1°), but more precise and more interesting is that of the Islamic scholar Abu al-Wafa (940–997). The Indian and Islamic mathematics is taken from Katz's general history [12, p. 309].

Spherical Trigonometry in Medieval Islam

After an overview of the mathematics of medieval Islam, we focus on spherical trigonometry. Our main source is Berggren's *Episodes in the Mathematics of Medieval Islam* [5]. The students learn the concepts of spherical trigonometry, and then trace the path, with proof, attributed by Berggren to Abu al-Wafa, to the spherical Law of Sines. With just the Law of Sines, we compute the direction to Mecca for various cities when their longitude and latitude are given. This was an important computation in Islamic lands.

Some Non-Euclidean Geometry

For this section, I chose one of my favorite subjects, projective geometry. Projective geometry has its roots in the study of the conic sections, beginning with ancient Greek work, especially the *Conics* of Apollonius of Perga (ca. 250–175 BC). [See Appendix B for more details.]

Two reasonable alternatives to projective geometry, to illustrate a non-Euclidean geometry, are spherical trigonometry and hyperbolic geometry. For spherical trigonometry, [21] is a fine resource; for hyperbolic geometry, Donald Crowe's [7] includes a very readable treatment of modest length.

Resources

A review book for secondary geometry, Lawrence Leff's, *Let's Review: Geometry* [15], was a required text. It covers the typical secondary curriculum of the last half century, largely in the sequence set out by Euclid. The bulk of course material was in a package of some 100 pages, purchased by students at the beginning of the course. This package was made up of readings, worksheets, and problems. Included were the instructor's list of the relevant propositions of Euclid, the development of Liu Hui's work behind his approximation to

π, arguments of Archimedes, and the material for projective geometry. It included readings lifted from the variety of sources noted above.

To my knowledge, no book is available that fits a course with this goal, for this audience. I recommend the long chapter by Donald Crowe, "Some Exotic Geometries," in *Excursions into Mathematics* [7, pp. 211–314], as a marvelous introduction to the spirit of Euclid's geometry, its shortcomings, and the hyperbolic and projective geometry that emerged both from and against Euclid's geometry.

The instructor consulted secondary school materials to find connections of course topics to the secondary curriculum. The secondary text *Geometry* [20] was invaluable, as it gives a transformation approach to geometry and includes material on volume and surface area. The instructor needs a good general history, such as Katz's [12], and Euclid's *Elements*—I recommend David Joyce's site [10]; Heath's classic [11] supplies detail and background for the instructor. Heavy use was made of *The Nine Chapters* [13], of Aaboe's [1], and of Berggren's [5] on medieval Islam.

Van Brummelen's [21] is a readable introduction spherical trigonometry. I. M. Yaglom's books [22] on geometric transformations are background for dilations and projections, including many interesting problems. *What is Mathematics*? [8] has an introductory survey of projective geometry.

In the classroom, the instructor made several demonstrations with the free geometry program *Geogebra*. Expanded use of dynamic geometry programs is certainly possible. *Geogebra* has commands appropriate to projective geometry, including drawing a conic on five given points. For supplemental information on mathematics history, students are encouraged to start at the MacTutor history of mathematics web site, www-history.mcs.st-and.ac.uk.

Assignments

Course assignments are largely standard exercises, both proof and computational types, many from [15]. These exercises are more about geometry and less about history, reflecting the goals of the course. Students are asked to complete proofs from Euclid or related to Euclid. There are construction exercises, several based on dilations, and there are "straightedge only" constructions involving central collineations. Several of these exercises are found in Appendix B.

Each student makes a problem presentation to the class. And the students are asked to write a short biographical report on one of the figures encountered in the course: Euclid, Archimedes, Liu Hui, Ptolemy, al-Biruni, Abu al-Wafa, Apollonius, La Hire, or Poncelet. The final assignment, in 2014, was to read J. B. S. Haldane's "On being the right size" and to answer a few questions about it.

Tests cover history, including historically important computations and proofs, along with questions testing geometry skills.

Lessons Learned

Offered several times over the last decade, Geometry Through History fills a need for future and active teachers of secondary mathematics. They come into the course needing strengthening of problem solving skills and needing more comfort with theorem-proof geometry, and they have little awareness of the history of geometry. Work with mathematics from China and medieval Islam is a welcome broadening of cultural awareness. The greatest challenge in planning and presentation is with the section on non-Euclidean geometry. Despite the urge to cover a lot, the instructor needs to focus on a limited body of mathematics—and its history—that will be engaging and meaningful to secondary teachers.

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Appendix A

Course Outline

Geometry Through History has been offered as a three-week summer course, where each class ran about 2 hours 45 minutes. The last class was for a couple of student problem presentations and the final exam, so there were, in effect, 14 class meetings. So the 14 days described below would correspond to the 14 weeks in a typical semester. Proof, computational problems, and straightedge-compass problems were assigned daily.

Day 1. The historical background and context, and the logical structure, of Euclid's *Elements*. We pay attention to definitions and the postulates, and the first propositions of Book 1, including SAS and SSS triangle congruence, through Prop 15, on vertical angles. Comparison is made to the corresponding material in Leff's modern secondary text. Homework is reading from Euclid and Leff, and several problems from Leff.

Day 2. Straightedge-compass construction in Euclid and secondary geometry, with problems. Euclid's theorems without the Parallel Postulate, including Book 1 Prop 16–20, 27, and the ASA and AAS congruence theorems. Exercises in proof, especially proof-by-contradiction.

Day 3. Lines cut by a transversal, what can be proved without the Parallel Postulate, and what requires it. Statements equivalent to the Parallel Postulate, including that the angle sum of a triangle is two right angles. Parallelogram properties.

Day 4. Pythagorean Theorem. Circles, as in Euclid's Book 3, and the angle measures associated with chords, tangents, and secants. Computational and proof problems from Leff's secondary text. Quiz. Homework: Read from Crowe's "Some Exotic Geometries" on spherical (no parallels) and hyperbolic (many parallels) geometry and absolute geometry (Euclid without the Parallel Postulate).

Day 5. Comparing absolute, hyperbolic, spherical, and Euclidean geometry. Problems based on the first week's material. Liu Hui's commentary on the area of a circle, his formula for the area of an inscribed regular polygon and its application to find tight upper and lower bounds on π.

Day 6. More on Liu Hui. Triangle similarity as developed by Euclid in Book 6. Introduction to dilations and similar figures as those related by a dilation. Straightedge-compass constructions justified by dilation properties. Quiz.

Day 7. More problems based on dilations. Cavalieri's Principle and its application to volume of cones and pyramids. Hour test 1.

Day 8. Derivation of the volume of the sphere by Cavalieri's Principle, and surface area of the cone and sphere. The method of "exhaustion" to find area, including two examples from Archimedes and some modern area-under-a-curve computations.

Day 9. Legendre's proof that in Absolute Geometry the angle sum of a triangle cannot exceed two right angles. Early trigonometry, with attention to Ptolemy's construction of a table of chords. Simple theorems about chords are proved, but more emphasis is on the corresponding modern trigonometric identities.

Day 10. More plane trigonometry. Historical notes on trigonometry in ancient India and medieval Islamic culture. Biographical information on al-Biruni. Spherical trigonometry in Islam, with notes on mathematics in Islamic culture, including the spherical Law of Sines. Computation of the direction to Mecca from various cities when their latitude and longitude are given. Quiz.

Day 11. More trigonometry in Islam, including al-Biruni's computation of the height of a mountain. The *Conics* of Apollonius, with introduction to *diameter* and *ordinate* concepts and equations.

Day 12. More on Apollonius. Pole and polar properties of the conic sections, developed as by Philippe de la Hire in 1673, with related problems. Hour test 2.

Day 13. Theorems of projective geometry: Menelaus's Theorem, Desargues' Theorem, Pascal's Theorem. Introduction to central collineations and the point-by-point ruler construction of conics as images of a circle under a central collineation.

Day 14. Review. More projective geometry, including an example from Poncelet. Homogeneous coordinates for projective geometry, as in *What is Mathematics?*

The final assignment, dues several days after the last class, was to complete exercises on the ratios of corresponding lengths, surface area, and volume for similar solids, then to read Haldane's "On being the right size," and answer questions.

Appendix B

Supplement to Course Design

Euclid

Euclid's Book 1, Propositions 16 and 27

Because these are theorems of absolute geometry, their proofs must avoid any dependence on the Parallel Postulate, which is Euclid's Postulate 5. In the proofs below, use only the following statements from Euclid's Book 1.

Definition 23. *Parallel lines* are those that never meet, even when extended.
Common Notion 5. The whole is greater than its part.
Postulate 1. Two points can be joined by a line [segment].
Postulate 2. A line [segment] can be extended.
Postulate 3. A circle can be drawn with a given center and radius.
Prop 4. SAS triangle congruence [and CPCTC].
Prop 10. To bisect a line [segment].
Prop 13. Angles that form a linear pair are supplementary.
Prop 15. Vertical angles are congruent.

Prop 16. *An exterior angle of a triangle is greater [in measure] than either opposite interior angle.*

Given: ΔABC with $\overline{BC}$ extended to D.

Prove: $m\angle ACD > m\angle A$. See Figure 1 below.

Plan: We start by marking the midpoint, M, of $\overline{AC}$, then joining B to M and extending this segment to F so $BM = MF$. Then draw $\overline{CF}$. We find a pair of congruent triangles.

Justify the above steps using Common Notions, Postulates, and earlier propositions. Then finish the proof.

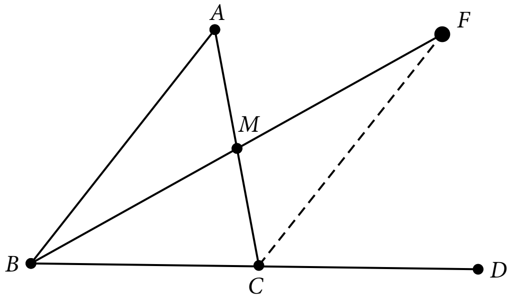

Figure 1. Euclid Book 1 Prop 16

Prop 27. *If two lines are cut by another in congruent alternate interior angles, then the two lines are parallel.*

Given: Lines l, m, cut by line t, with $\angle 1 \cong \angle 2$.

Prove: l and m are parallel. See Figure 2 Left.

Plan: Suppose l and m meet at a point X creating ΔGHX. Why is this impossible?

Construction based on Euclid's *Elements*

Mack proposes this construction of a perpendicular to a line n from outside point P. We select points Y and Z on n so the perpendicular from P should meet n between Y and Z. Circles are drawn on P, with centers at Y and at Z, meeting at Q (in addition to P). See Figure 2 Right.

Mack claims that $\overline{PQ}$ is the perpendicular to n.

Prove that Mack is correct.

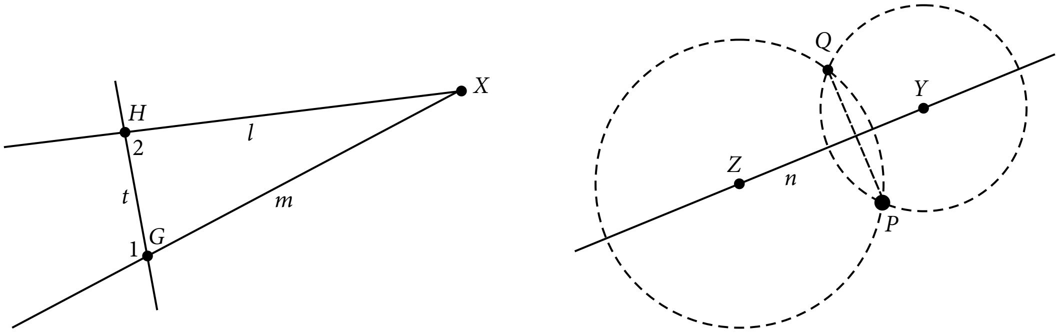

Figure 2. Left: Euclid Book 1 Prop 27 Right: Construction Exercise

Similarity by Dilations

Development of dilations in the coordinate plane can be found in [20, Chapter 12]. A *dilation* S_k, with *center* at the origin, maps the point (x, y) to (kx, ky), where k is not zero. Then two plane figures are *similar* if the image of one under a dilation is congruent to the other. One proves by coordinate methods that distances are multiplied by $|k|$, that a line is mapped to a parallel line, and that angles are preserved. The first application is proving the triangle similarity theorems: If, for example, ΔABC and ΔDEF agree in angles A and D, and B and E, then a dilation of ΔABC can produce an image congruent to ΔDEF by ASA. The second application is in straightedge/compass constructions.

Figure 3 (next page) is an introduction to dilations; at the bottom, the students find centers for dilations in preparation for straightedge construction problems. Those problems include constructing the internal and external common tangents to two given circles; the first step is to construct centers of the dilations that map one circle to the other.

Liu Hui and π

Figure 4 is included in materials given to students, illustrating Liu Hui's derivation of the area of inscribed $2n$-gon when the sides of the inscribed n-gon are known.

The side, s_{2n}, of the $2n$-gon is related to s_n by $s_{2n}^2 = 2 - \sqrt{4 - s_n^2}$, found by the Pythagorean (*Gou-Gu*) Theorem. The circle has radius 1. The upper bound for the area of the circle is the area of the related polygon in which each of the last kites, with diagonal $\overline{BE}$, is replaced by the 5-sided figures *ABCDE*.

The inscribed area A_{192} is 3.141024. This decimal notation is anachronistic. The ancient Chinese had a multiplicative system, meaning that the value of the digits was indicated not by their position but by a word label. Liu Hui used length units; this would be like presenting .034 as "3 centimeters 4 millimeters."

Archimedes and Cavalieri's Principle: Area and Volume

Cavalieri's Principle is applied in finding the volume of the sphere, in an argument found at [20, p. 505]. We have the cylinder of height $2r$, whose base is a circle of radius r, from which are removed the pair of cones with apex at the center of the cylinder and whose bases are the bases of that cylinder. And resting on the same base plane is a sphere of radius r. When sliced by a plane parallel to that base plane, we have equal-area slices, so the volumes are equal. Easier than the derivation of Archimedes, it is found, essentially, in a book of 1604, by Luca Valerio. See [4, p. 105]. And once we know the volume of the sphere, the surface can be found, as given in [20, p. 510], by approximating the volume of the sphere by the sum of volumes of pyramids whose

Dilation Worksheet

1. Draw the image of the balloon figure under the dilation with *center* Q and *scale factor* 2.
2. Find the *center* of the dilation that maps $ABCD$ to $A'B'C'D'$.
3. Find the two *centers* of dilations that map *circle* E to *circle* E'. [Hint: Draw a radius in *circle* C and then construct its two possible images.]

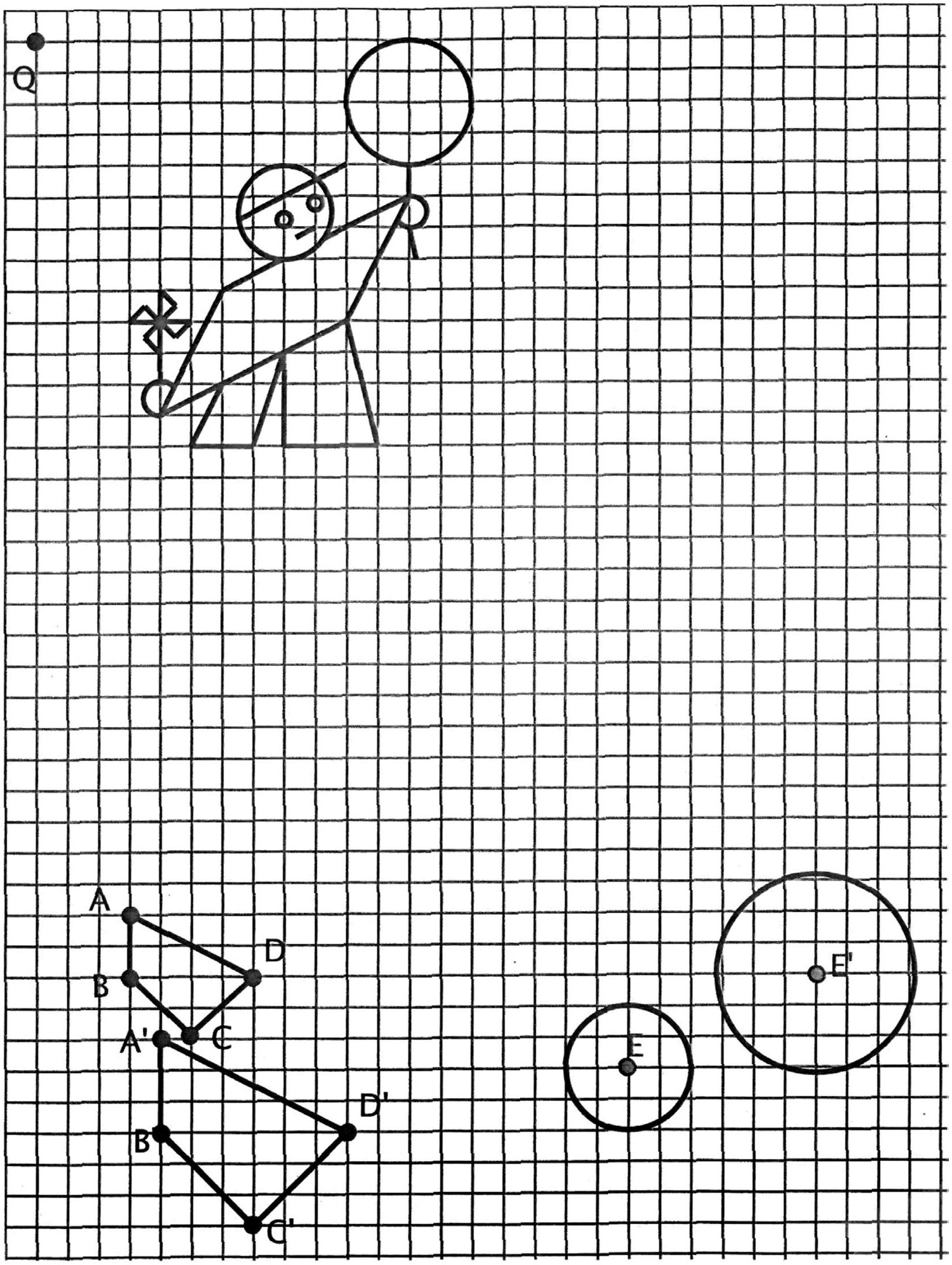

Figure 3. Dilation Exercise

Liu's Theorem

Let r be the radius of the circle, s_k the side of the inscribed regular k-gon, and A_k the area of the inscribed regular k-gon. Then

$$A_{2n} = {}^1/_2 \cdot n \cdot s_n \cdot r.$$

Why? The regular $2n$-gon is formed of n kites, with diagonals r and s_n. The kite has half the area of the r-by-s_n rectangle. Then multiply by n to obtain A_{2n}.

Note that the *Gougu Theorem*:

$$gu^2 + gou^2 = xian^2$$

for a right triangle, was discovered in China around the 1st or 2nd century BC.

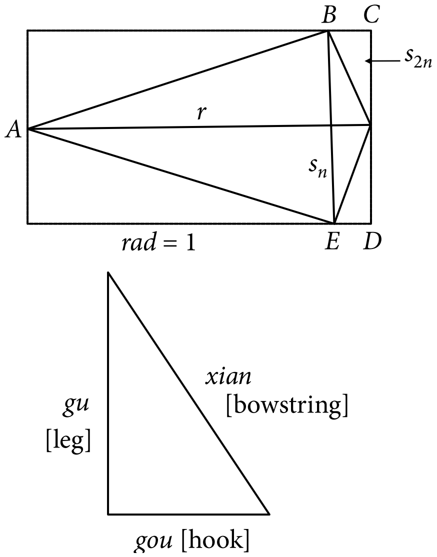

Figure 4. Area of the inscribed $2n$-gon as found by Liu Hui.

bases together approximate the surface of the sphere and whose heights are the radius of the sphere.

Some Non-Euclidean Geometry

The section on projective geometry begins with Apollonius. The conic section of Apollonius is just what the name indicates, the intersection of a cone with a slicing plane. In *Geometry Through History*, the section on projective geometry begins, briefly, with the concept of a diameter and the corresponding ordinates of a conic section, and with the derivation of the abscissa-ordinate equation for the ellipse. The ellipse $EKDM$ in Figure 5 Left has equation $MN^2 = \text{constant} \cdot EN \cdot ND$. Our modern equation in rectangular coordinates follows. Students are asked to perform the corresponding derivation for the parabola. This material is nicely presented in [12, pp. 115–118].

However, a conic section can be seen as the projection from the vertex, A, of the base circle, with diameter BC in Figure 5 Left, onto the slicing plane. The properties of conic sections are explored as circle properties preserved under projection.

This new approach grew with the Renaissance study of perspective, and experienced its first flowering in seventeenth century France, in the work of Girard Desargues (1591–1661), Blaise Pascal (1623–1662), and Philippe de la Hire (1640–1718).

At this point, the instructor has several choices of direction. I chose to work with central collineations, as found in [16, pp. 1–17], where the corresponding material is found nicely developed in a 1673 work of Philippe de la Hire[14]—in French. We need to add what La Hire did not, a line at infinity, where all "parallel" lines meet, to create the real projective plane. A *central collineation* is a one-to-one mapping of the projective plane onto itself under which lines are mapped to lines, and where there is a line, ℓ, of fixed points. It is to be proved that there is a point S, the *center*, where all lines on S are mapped to themselves. The image of a circle under a central collineation is a conic section and all conic sections are formed this way. Images under a central collineation can be found by a straightedge construction, as in Figure 5 Right.

An alternative to the central collineation approach can be found, for example, in [8, Chapter IV].

Additional topics in this part of *Geometry Through History* included Menelaus's Theorem, Pascal's Theorem [When a hexagon is inscribed in a conic, then opposite sides meet in collinear points], and the solution

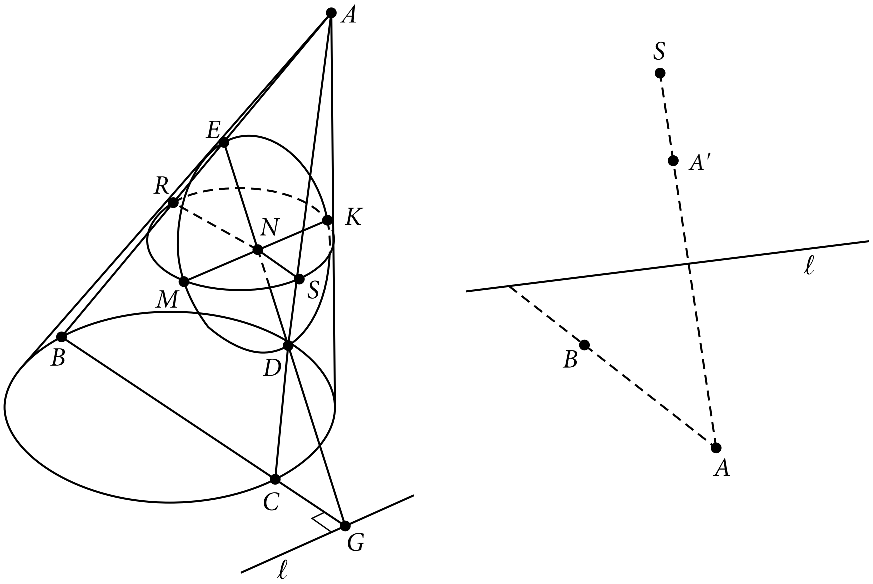

Figure 5. Left: Ellipse *EKDM*. Right: Construct B' for the central collineation with line of fixed points ℓ and center S.

by Jean-Victor Poncelet (1788–1867), in an 1809 paper [17], of the problem to construct a circle tangent to three given circles. Justification for Poncelet's construction is found in propositions of his 1813 *Notebooks* [18], written as a prisoner of war in Saratov, Russia.

History of Calculus

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University of San Diego

Course Overview

"Of making many books there is no end," says the Preacher ("and", he continues, pessimistically, "much study is a weariness of the flesh") (Ecclesiastes 12:12). In the history of mathematics, we can certainly confirm the "making of many books". The history of mathematics is as extensive as the history of humanity. Rather than try to survey this vast subject in a single semester, I have preferred in recent years to choose a particular theme for the history of mathematics course each time I have taught it. The theme for the course I am describing in this chapter is the history of calculus. I have taught this particular course twice.

The invention of calculus was a critical development in the history of mathematics. Modern mathematics, certainly, would be unimaginable without it. Isaac Newton and Gottfried Leibniz are conventionally credited with this invention, but of course they, like all mathematicians, built on the work of their predecessors. My course begins in the ancient world, with the work of Archimedes. The story then continues in western Europe in the 17th century, with early calculus-like techniques in the work of Cavalieri, with the integration of positive integral and fractional powers by Fermat, Pascal, Wallis, and others, and with early methods for finding tangents. Two topics of special interest are Napier's invention of logarithms and Wallis's derivation of Wallis's product. Finally, we get to the work of Newton and Leibniz, respectively. For lack of time in the semester, I regretfully had to omit the work of Euler, Cauchy, Riemann, and Weierstrass.

The History of Mathematics course at the University of San Diego is a junior or senior level course with a stated prerequisite of third-semester calculus. The course satisfies an upper-division writing requirement (instituted on the basis of the slogan "Writing Across the Curriculum"). It also satisfies a requirement for students working toward a single-subject teaching credential in mathematics. At USD, these students are mathematics majors. Thus, most of the students in the course are upper-division mathematics majors, a sizable fraction of whom intend to be secondary school teachers. Occasionally an odd physics major may turn up.

Course Design

The course meets three times weekly, for 55 minutes. The class format is mainly a lecture from me, with discussion by students. Typically, I try to help the students get oriented with respect to the material in the textbook —for example, by pointing out which are the main themes, or by showing the connections between concepts discussed in different chapters. Sometimes, of course, I work through mathematical derivations, but I do not attempt to read the textbook to the students. They are expected to do that themselves. There are regular homework assignments. As a basic textbook for the course, I used C. H. Edwards's book *The Historical Development of the Calculus* [2].

I always begin my course by devoting a few days to questions of historiography. Here I talk about "Whig history" (I usually bring in a copy of Butterfield's book, *The Whig Interpretation of History* [1]), and about the distinction between the *internalist* and the *externalist* approaches to the history of mathematics. For the first writing assignment, I give the students André Weil's article "History of mathematics: why and how" [7], (originally a talk given at the International Congress of Mathematicians in Helsinki, 1978); and "Mathematics: an historian's perspective" by Joseph Dauben [4], a reply to Weil. The assignment is to discuss the issues raised in these articles. I can report that the students almost unanimously dislike Weil and register a strong agreement with Dauben. These are not my own views, so obviously I have not been coaching the students on how to respond.

Edwards's chapter on John Wallis's "Arithmetic of the Infinite" is (in my opinion) particularly interesting. This chapter describes how Wallis derived his infinite product for π:

$$\frac{\pi}{2} = \frac{2\cdot 2\cdot 4\cdot 4\cdots}{1\cdot 3\cdot 3\cdot 5\cdots}.$$

Wallis gave a remarkable interpolation argument, the validity of which is far from clear. Edwards remarks that Wallis's method can be understood more clearly if we describe it in terms of Euler's gamma function. In fact, Euler's discovery of the gamma function was partly inspired, as he tells us himself [3, Section 2], by Wallis's work. This being so, it is a good excuse, I think, for including in the course an excursus on the gamma function, a topic that is seldom included in the undergraduate curriculum these days. In more recent courses, I have used Euler's own paper on the gamma function, "On transcendental progressions, that is, those whose general terms cannot be given algebraically" (Eneström number 19) as a primary reading source, in my own English translation.

In fact, one of the things that I like to do in the history of mathematics course, besides introducing the students to a historical way of thinking about mathematics, is to show the students some beautiful mathematical ideas that they most likely would not learn about in their other mathematics courses.

Resources

The basic text is Edwards's *Historical Development of the Calculus*. This is a very well-written book which gives a good description of the work of each of the mathematicians discussed, with references to the primary sources.

Edwards's book contains much more material than can be discussed in a single semester. (At USD, our semester is short —only 13 weeks.) Thus, I had to make a selection. Edwards's first three chapters deal with mathematics in the ancient world, especially the work of Archimedes. I discussed part of this material, in particular the "method of exhaustion", but omitted most of it.

The meat of the course was devoted to Edwards's Chapter 4 (early calculus techniques), Chapter 5 (construction of tangents), Chapter 6 (Napier's logarithms), Chapter 7 (Wallis's product), Chapter 8 (Newton), and Chapter 9 (Leibniz). For lack of time, I omitted the rest of the book, which discussed later developments.

When I taught the history of calculus course, I had just begun to experiment with the use of primary sources in teaching. For this course, I used Dirk Struik's *Source Book in Mathematics: 1200–1800* [6], which the students were required to purchase. Many of the writing assigments asked the students to read one of the excerpts from Struik's book and to write about it. In particular, I used selections from Galileo's *Discourses Concerning Two New Sciences*, Torricelli's discovery of an infinitely long solid with finite volume, Jacob Bernoulli's proof of the divergence of the harmonic series, and George Berkeley's criticisms of the calculus, from his *Analyst: A Discourse Addressed to an Infidel Mathematician*. Another source was Pascal's essay "On geometrical demonstrations", available in the series *Great Books of the Western World* [5]. (The *Great Books* volume was on reserve in the library.) There was also a writing assignment on the work of the Egyptian mathematician Alhazen, based on the treatment in Edwards's book.

Assignments

Edwards's book contains a large number of interesting mathematical exercises, many of which I was able to assign. All of his exercises involve solving mathematical problems, however. There are no essay or discussion questions that ask students to do historical analysis. Consequently, I provided these assignments myself.

Since the course is supposed to satisfy an upper-division writing requirement, I gave a writing assignment almost weekly. The writing assignments were typically not very long—perhaps two or three pages. I prefer several short assignments to one or two long ones, on the principle that one should walk before one runs. I do not require the students to turn in rough drafts and revisions, but I do comment very thoroughly on each paper that is turned in. One of the assignments is for the students to write a short biography of a mathematician of their choice. This paper is a little longer—perhaps 4 or 5 pages. It is usually assigned sometime during the second half of the semester. Also, one of the "writing" assignments is actually an oral presentation. For the history of calculus course, the students were allowed to choose for the oral presentation some topic connected with the history of calculus. In the history of calculus course, I had only about 10 students, and thus I was able to have the students give 15-minute presentations. In later history of mathematics courses, I tended to have closer to 20 students, and consequently had to reduce the length of the presentations to 10 minutes. These presentations come toward the end of the semester.

Many of the writing assigments asked the students to take a particular "role"; for example, they had to write a letter to Galileo, and they had to explain Torricelli's result to a calculus student. A final writing assignment asked the students to sum up what we had learned in the course.

The exams for the course include essay questions asking for historical analysis (for example, who really invented calculus?), and more purely mathematical questions (for example, use Descartes' method to find the tangent to the curve $y=\sqrt{x}$). I also include a set of short identification questions. I give the students a list of names of some of the mathematicians we have encountered in the course, and ask them to give the nationality, approximate chronological period, and some notable mathematical accomplishment for each mathematician in the list. It seems to be necessary to insist that students know these simple facts. It is impossible, I think, to understand history in the absence of a basic chronological framework. I have had students tell me, for example, that Isaac Newton was an Italian mathematician from the 14th century.

Lessons learned

I found that students generally responded positively to this course. This is perhaps because it is different from their other mathematics courses, but I think the students also find it very interesting to see that mathematics was created by real people who were trying to solve problems that were important to them. It is a revelation that the calculus of the mathematicians who invented calculus looked quite different, in many ways, from the calculus in textbooks today, because, in our modern textbooks, the context of discovery has been scrubbed out. The students see that the pioneers of calculus had to struggle to evaluate $\int x^n dx$, an integral that, from our present perspective, is completely elementary.

Students often, on encountering a proof or a derivation in a textbook, ask, "How could anyone have thought of that?" In this course, we begin to see the answer. Nobody could have thought of some of the treatments in the textbooks, because in fact nobody did. They were not discovered by any individual person, but are the product of a long process in which the original discoveries have been studied, rewritten, reorganized, massaged, and homogenized until they have reached the form in which we have them today. To be sure, many of the mathematicians whose work we study were very brilliant men, smarter than we ourselves are. Yet when we look at their original work in its own context, it is at least possible for us to have some notion of how they got the ideas that they did.

In the history of calculus course, I think that I achieved a good balance between writing and doing mathematical exercises. In some later history of mathematics courses, I think that I tended to over-emphasize writing. Because I required the students to spend so much time writing, I reduced or eliminated the mathematical exercises. In retrospect, I think that this was a mistake. Many of the mathematical topics discussed in the course are new to students; and indeed, as I said, I think that this is one of the desirable features of the course. However, in order for the students to be able to develop an understanding of this new mathematics, it is necessary for them to do exercises, just as in any mathematics course. What I found was that the students were not developing a good grasp of the mathematical concepts. In my most recent offerings of the history of mathematics course, I have reduced fairly drastically the amount of writing, in fact by about half, in order to be able to assign a greater number of mathematical exercises for the students to do. In my judgment, this gives a more satisfactory balance.

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Course Outline

Introduction and Historiography of Mathematics (3 class periods)
Archimedes (1 period; Edwards, Chapter 2)
Medieval science (1 period; Chapter 3)
Early work on integration (5 periods; Chapter 4)
Early work on tangents (4 periods; Chapter 5)
Work of Napier (4 periods; Chapter 6)
Euler’s gamma function, Stirling’s formula, and Wallis’s product (6 periods; Chapter 7)
Work of Newton (8 periods; Chapter 8)
Work of Leibniz (5 periods; Chapter 9)
Oral presentations by students (3 periods)
Exams (2 periods)

Isaac Newton's Principles of Mechanics

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Course Overview

"Nature and Nature's Laws lay hid in Night:" wrote Alexander Pope, "GOD said, *Let Newton be!* and all was Light" [3, p. 330]. Thus did the English poet describe Newton's legacy. Newton's *Principia Mathematica* of 1687 had opened up a new way of seeing the world. Pope's epitaph shows how much Newton's severely mathematical book had affected the imaginations of the educated members of society. In this chapter, I describe a history of mathematics course that was based primarily on Newton's essay *De Motu* ("On Motion"), which was a first sketch for what later became his *Principia*.

The History of Mathematics course at the University of San Diego is a junior or senior level course. Its stated prerequisite is third-semester calculus. The course satisfies an upper-division writing requirement. Also, it satisfies a requirement for students working toward a single-subject teaching credential. These single-subject students at USD are mathematics majors. As a result, most of the students in the History of Mathematics course are upper-division mathematics majors, many of whom intend to be secondary school teachers. A few physics majors have occasionally taken the course.

Course Design

The course meets three times each week, for 55 minutes. The class follows mostly a lecture format, but, as explained below, I also regularly call on students to give presentations. There are regular homework assignments.

My original idea for this course was to use Dana Densmore's excellent book *Newton's Principia: The Central Argument* [2] as the basic text. This book consists of a selection of passages from the *Principia*, in a new translation by William Donahue, with explanation and commentary by Dana Densmore.

As I worked with this material, however, it seemed to me that it would be more satisfactory, for my class, to go back to an earlier essay of Newton, the *De Motu*. This essay contains Newton's first formulation of the principles of motion and their application to the motion of the planets around the sun. It was this work that Newton later revised and considerably expanded to form the *Principia*.

The *De Motu* is a much shorter work than the *Principia*; it occupies only about 30 pages in Whiteside's edition, which contains both Newton's Latin text and Whiteside's own English translation. For my purposes, it had the advantage of being a self-contained work, and short enough to cover essentially the whole essay in my course, so that it was not necessary to work with a series of extracts from a larger work.

About the first two weeks of the course were devoted to setting the historical stage for Newton's work. We discussed ancient astronomy, then the work of Tycho and Kepler, then that memorable discussion among

Halley, Hooke, and Wren, in January, 1684, which proved to be the catalyst for Newton's work on the *Principia* (see [6, pp. 402–403]).

After this preparation, we went through the *De Motu*, "line by line", omitting only the final two problems, numbers 6 and 7, on motion in a resisting medium. After finishing the *De Motu*, we turned back and went through a few sections of Densmore's book, to get an idea of how Newton had expanded the *De Motu* to produce the *Principia*.

The course was thus almost entirely based on the reading of the primary source, namely, the *De Motu*. This text required quite a bit of explaining, to be sure, to help the students understand what it was saying. Also, during most class periods, I asked individual students to go to the board to present some of Newton's arguments. These presentations were assigned ahead of time, so that the student could come and consult with me if he or she was having difficulty understanding what Newton said.

Resources

The main text for the course was *Newton's Principia: The Central Argument* by Dana Densmore [2], which I asked the students to purchase. Newton's essay *De Motu* is available in the sixth volume of D. T. Whiteside's *Mathematical Papers of Isaac Newton* [4]. I provided the students with copies of this essay. Aside from the historiographic essays by André Weil and Joseph Dauben, mentioned below, and from some handouts that I prepared myself, these were the only texts that I used in the course.

Assignments

One difficulty for us was that Newton assumes, on the part of the reader, a good familiarity with classical geometry, the work of Euclid and Apollonius. College students nowadays frequently know little about the geometry of Euclid, let alone that of Apollonius. Newton, of course, proposed to deduce from his law of gravitation the consequence that the planets move around the sun in elliptical orbits. His arguments, which were given in strict geometrical form, rely heavily on Apollonius's treatment of the ellipse. In particular, Newton made use of Apollonius's concept of *conjugate diameters* of an ellipse. In order to enable the students to cope with this material, I developed a series of exercises on the geometry of the ellipse, which are included as an appendix to this chapter, in the hope that they may prove to be of use to other instructors who might want to try this approach. (I should remark here that these exercises were not intended to present the work of Apollonius in its own historical setting, but simply to help the students understand Newton's arguments.)

These geometrical exercises were of course challenging to the students. Some of the students in the class were very talented (one is now on the faculty of another university in southern California), and were able to cope with the exercises quite well. Other students, as is true in any class, had more difficulty. I should mention that I gave this particular course in fall, 1998. Perhaps the students in today's classes would have had more difficulty.

In addition to the mathematical exercises on the ellipse, there were several writing assigments. Some of these were the same as assignments I have used in other versions of the history of mathematics course. One of them asked students to respond to the essays "History of mathematics: why and how", by André Weil [5], and "Mathematics: an historian's perspective", by Joseph Dauben [1]. Another was to write a brief biography of a mathematician of the student's choice (not necessarily related to Newton's *Principia*). For the course on Newton, one assignment asked the student to go beyond Newton's formula $a = v^2/R$ for the acceleration a of a body moving in a circular orbit of radius R at speed v, to write down a derivation of the more general formula

$$\mathbf{a} = \frac{d^2s}{dt^2}\mathbf{t} + \kappa\left(\frac{ds}{dt}\right)^2 \mathbf{n},$$

where s is the distance (or arc-length) travelled by the body, $\mathbf{t}$ is a unit vector in the direction of the tangent to the curve of motion, $\mathbf{n}$ is a unit vector perpendicular to $\mathbf{t}$, and κ is the curvature of the curve. Such a derivation can be found in typical calculus textbooks, but for this assignment the student had to give the derivation in his own words. Another assignment asked the student to write a "book review" of the *De Motu*, such as might have appeared in the *Philosophical Transactions of the Royal Society*.

Lessons Learned

Students presumably learn in high school that "Newton's second law" is $F = ma$. Many college-level textbooks on differential equations show how to deduce elliptical planetary orbits from $F = ma$ and Newton's law of gravitation, using the techniques of differential equations. Some of the students in my course had perhaps seen this development. Newton's own approach was of course quite different.

The famous three "laws of motion", which are so prominent in Newton's *Principia*, do not appear in the *De Motu*, although of course they are included among the extracts from the *Principia* in Dana Densmore's book. In William Donahue's translation, "Law 2" reads "That the change of motion is proportional to the motive force impressed, and takes place following the straight line in which that force is impressed." [2, p. 13.] I pointed out to the students that Newton says "change of motion", not "rate of change". It seems to me that, in modern notation, what Newton is saying is not $F = ma$, but rather $\Delta\mathbf{p} \propto \mathbf{I}$, where $\mathbf{p}$ is the momentum (Newton calls it "quantity of motion" in his Definition 2) and $\mathbf{I}$ is what is now called "impulse", that is, $\mathbf{I} = \int_{t_0}^{t_1} \mathbf{F}(t)\,dt$. (Newton's Latin word 'vis', translated "force", had not yet become a technical term of physics, and Newton uses it with a variety of meanings.) In my course, I referred to the two "second laws" as "the physicists' Newton's second law" and "Newton's Newton's second law".

Newton's own derivation of the planetary orbits is very geometrical, and is expressed in the classical language of proportions, as his own statement of the "second law" suggests. During the semester, I think that both I and my students developed some expertise in the application of Newton's geometrical criterion for the action of a centripetal force on the motion of a planet. I think that the students enjoyed looking at the subject from Newton's point of view. I certainly did.

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Appendix A

Course Outline

Introduction and Historiography of Mathematics (1 class period)
Ancient astronomy (1 period)
Kepler's laws and the principles of mechanics (3 periods)
The background of the Principia (1 period)
The De Motu (23 periods)
Material from the Principia (7 periods; Densmore, pp. 1–54)
Exams (2 periods)

Appendix B
Exercises

Exercise Set 1

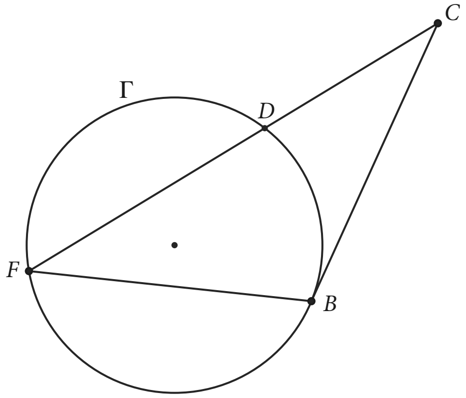
This exercise is related to Theorem 2 of the *De Motu*. This theorem occurs as Proposition 4 of Book I of the *Principia*, but Newton's proof there is different, so Densmore's guidebook doesn't give us much help here.

In the proof of Theorem 2, Newton makes use of a theorem of elementary geometry that he assumed everyone would recognize; however, perhaps we don't. Let Γ be a circle, C a point outside the circle, CB a tangent, and CDF a secant line. The theorem says that $CD \cdot CF = CB^2$. (This is Euclid, III.36; that is, Proposition 36 of Book III of Euclid's *Elements*.)

The exercise is to write down a proof of this geometrical fact.

Hint: Draw the line BD. Then $\angle F = \angle CBD$, so $\Delta CFB \sim \Delta CBD$, and we can use similar triangles.

But why is $\angle F = \angle CBD$? (This is Euclid, III.32; but can you give a proof?)



Exercise Set 2

In order to derive the centripetal force on a body that orbits in an ellipse, Newton had to make use of some of the geometric properties of the ellipse. Some of these properties should be familiar from elementary calculus. Others may be new—to us. Of course, they were familiar to all 17th-century mathematicians. They can all be found in the great work on conic sections by the 3rd-century BC Greek mathematician Apollonius of Perga. In this exercise set, we review some of the properties of the ellipse which are usually covered in elementary calculus.

1. Suppose that we have an ellipse with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where $a > b > 0$. The foci of the ellipse are the points $F_\pm = (\pm c, 0)$, where $c = \sqrt{a^2 - b^2}$.

If P is a (variable) point on the ellipse, show that the sum $\overline{PF_+} + \overline{PF_-} = \text{const.}$
[Hint: It is enough to show that $\left(\overline{PF_+} + \overline{PF_-}\right)^2$ is constant.]

2. Now show that PF_+ and PF_- make equal angles with the tangent to the ellipse at P (the "reflection property" of the ellipse).

Exercise Set 3

In Problems 2 and 3 of the De Motu, Newton makes use of some further geometric properties of the ellipse.

1. We continue to work with an ellipse having equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

where $a > b > 0$. A diameter of the ellipse is a line-segment passing through the center (i.e., the origin) and bounded by its two points of intersection with the ellipse. The major axis $-a \le x \le a$ and the minor axis $-b \le y \le b$ are two important diameters.

Suppose that we have a diameter d whose endpoints are $(\pm x_0, \pm y_0)$. Apollonius introduced the idea of the diameter d' *conjugate* to d. By definition, this is the diameter which is parallel to the tangent to the ellipse at the point (x_0, y_0).

Find the coordinates of the points where the conjugate diameter d' intersects the ellipse.

(By symmetry, we may assume, without loss of generality, that the point (x_0, y_0) lies in the first quadrant, so that $x_0 > 0$ and $y_0 > 0$.)

2. Since d' is a diameter of the ellipse, we could also find the diameter that is conjugate to d'. Show that the diameter conjugate to d' is the original diameter d. [Hint: How is the slope m' of the conjugate diameter d' related to the slope m of the original diameter d?]

Remark: Note that the diameter conjugate to the major axis is the minor axis, and conversely. Usually, however, conjugate diameters are not perpendicular to one another.

3. Apollonius, of course, did not write the equation of the ellipse as we do. In fact, the idea that a geometric curve could be represented by an algebraic equation was invented by Descartes in 1637. However, Apollonius did have an analogous relation. It was expressed in the geometric language of proportions, and was known as the *symptom* of the curve. If we were to express this relationship in the language of algebraic equations, it would look something like this. Note that our equation can be written in the form

$$y^2=\frac{b^2}{a^2}(a^2-x^2)=\frac{b^2}{a^2}(a-x)(a+x).$$

In this equation, $a - x$ and $a + x$ are the lengths of the two segments of the major axis into which that axis is divided by the point $(x, 0)$. Thus, in words, the "symptom" says that the square of the y-coordinate is *proportional* to the product of those two segments of the major axis.

Now, Apollonius did not usually express the symptom of the ellipse in terms of the rectangular coordinates we are so fond of. Suppose that d and d' are any pair of conjugate diameters. Given a point P on the ellipse, draw a line through P, parallel to d', and meeting the diameter d in the point T.

In this problem, we will derive the theorem of Apollonius (Conics, Book I, Proposition 50): that the above symptom of the ellipse is still correct when it is referred to the oblique diameters d and d'. In other words, PT^2 is proportional to the product of the two segments of the diameter d into which that diameter is divided by the point T.

Actually, it seems rather difficult to do this by straightforward analytic geometry; the algebraic expressions we encounter become very messy. The following sequence of steps provides an alternate approach.

a) Introduce the new coordinates $\tilde{x} = x/a$, $\tilde{y} = y/b$. Show that in $\tilde{x}$-$\tilde{y}$ coordinates, the ellipse becomes a circle—in fact, the unit circle $\tilde{x}^2 + \tilde{y}^2 = 1$.

b) If ℓ and ℓ' are parallel lines in the x-y plane, show that the corresponding lines $\tilde{\ell}$ and $\tilde{\ell}'$ in the $\tilde{x}$-$\tilde{y}$ plane are also parallel.

c) Show that the conjugate diameters d and d' are transformed, in the $\tilde{x}$-$\tilde{y}$ plane, into diameters $\tilde{d}$ and $\tilde{d}'$ of the circle, which are mutually perpendicular.

d) A circle, of course, is a special case of an ellipse; and the symptom of an ellipse applies also to a circle. In fact, if, from a point P on a circle, we drop a perpendicular to a diameter, meeting the diameter at T, then PT^2 is equal to the product of the two segments of the diameter cut off by T. (This is contained in Euclid, VI.13.) Prove this fact. Thus, the symptom holds for a circle, and in fact the constant of proportionality is 1.

But in the case of a circle, it is obvious that this relationship does not depend on which diameter of the circle we choose. It applies, in particular, to the diameters $\tilde{d}$ and $\tilde{d}'$.

e) To transfer this relationship back to the x-y plane, we need to know how the transformation $\tilde{x} = x/a$, $\tilde{y} = y/b$, affects distances. Let P and Q be two points in the x-y plane, and let $\tilde{P}$ and $\tilde{Q}$ be the corresponding points in the $\tilde{x}$-$\tilde{y}$ plane. Show that the distance $\tilde{P}\tilde{Q}$ is obtained from the distance PQ by multiplying by a factor which depends only on the slope of the line PQ.

f) Use these facts to deduce that the symptom holds for the original ellipse when referred to the conjugate diameters d and d'.

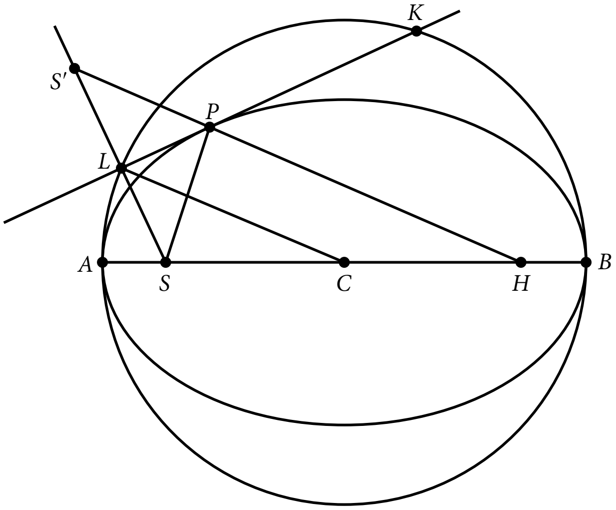
4. Given a pair of conjugate diameters d and d', draw the tangent lines to the ellipse at the end-points of the respective diameters. Then we have circumscribed a parallelogram about the ellipse. If we take for the two diameters the major axis and the minor axis, this parallelogram will be a rectangle, and it is easy to see that its area is $4ab$. Show that the area of the circumscribed parallelogram will in fact always be $4ab$, no matter which pair of conjugate diameters is chosen. (This is Lemma 2 of the *De Motu*.) [*Hint*: How do you calculate the area of a parallelogram?]

5. The *latus rectum* of an ellipse is the chord through one of the foci, perpendicular to the major axis. Show that the length of the latus rectum is $2b^2/a$.

Exercise Set 4

1. In the *Scholium* following Theorem 4 of the *De Motu*, Newton uses the following geometric theorem. Let APB be an ellipse with major axis AB, center C, and foci S and H. Let P be a point on the ellipse, and LPK be tangent to the ellipse at P. Drop a perpendicular SL from S to the tangent LPK. Then $LC = ½AB$.

   [Hint: Extend the line SL to S', where $SL = S'L$. Then the line $S'H$ passes through the point of tangency P. Why? Furthermore, LC is parallel to $S'H$. Why? From this, conclude that $LC = ½S'H$. Why?]



2. One difficulty with Newton's theory, as we have seen it so far, is that the sun and planets are not geometrical points! Newton was very proud of the following result, which is Book I, Proposition 74 of the *Principia*: the gravitational force exerted on a body by a homogeneous sphere is the same as if all the mass of the sphere were concentrated at its center. Use calculus to give a modern derivation of this result.

   [Hint: Suppose that the sphere has radius R and density δ (a constant). Consider a point-mass m. We assume that the gravitational force $\Delta\mathbf{F}$ exerted on m by each little particle of the sphere is given by Newton's gravitational formula

$$\Delta\mathbf{F} = \frac{Gmm'}{d^2}\mathbf{u},$$

   where G is the gravitational constant, m' is the mass of the particle, d is the distance between the particle and m, and $\mathbf{u}$ is a unit vector pointing from m to the particle.

   We can suppose that the center of the sphere is at the origin of coordinates, and that the mass m is on the z-axis. Write down a formula for the $\Delta\mathbf{F}$ exerted by each particle of the sphere, and express the total force $\mathbf{F}$ as a triple integral. Then just evaluate the integral. You might wish to use spherical coordinates. The integral is a little messy, so persevere! It can be done.]

The History of Statistics
A Discussion-Intensive Seminar on 20th Century Development and Beyond

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Course Overview

Many students are required to take statistics courses during their undergraduate careers, which usually present a prescribed curriculum of descriptive statistics, inferential methods, and modeling, such as two-sample *t*-tests and least-squares regression analaysis. Students learn what to do with data, but not necessarily why current statistical practices embrace particular methods and ways of thinking over possible alternatives. For example, students may not learn why statisticians and other scientists usually use a cutoff value of $p = 0.05$ to decide whether or not a result is statistically significant, nor do they generally know how long we have been doing hypothesis tests this way and who started us on this path. I designed this three credit hour, junior level seminar course to explore the history of statistics through the lens of these big ideas. In order to make this course flexible and accessible for a mixed audience of statistics minors, statistics majors, and mathematics students who might be looking for an interesting elective, the course prerequisite is one introductory-level statistics course.

To achieve its goal of exploring the overarching idea of why, this course focuses on some of the people who shaped statistical thinking and practice in the 20th century, such as Karl Pearson (1857–1936), Ronald Fisher (1890–12962), John Tukey (1915–2000), Florence Nightingale David (1909–1993), and Gertrude Cox (1900–1978). We explore them not only as mathematicians and scientists, but as individuals. The readings and activities are centered around several themes.

- How did statistics evolve as a discipline and who are some of the key players?
- What roles have communication and computation played in the growth of statistics?
- How did major events like the industrial revolution and World War II affect statistics?
- How did scientific and social issues like the eugenics movement shape the discipline?
- What role have women and minorities played in statistics and statistics education?
- What is statistical significance and why is it important (or not) to science and research?
- What is the future of statistics, as a scientific discipline and an educational endeavor?

Overall, the course does not delve too deeply into any particular statistical methods or proofs of certain critical concepts, such as the Law of Large Numbers or the Central Limit Theorem. This is in part because of the broad diversity of mathematical and statistical skills among students in the course. Instead, it focuses primarily on themes of people, communication, computation, society, and historical events that shaped "modern" 21st century statistical practices, as well as what the future of statistical thinking and practice might be.

Course Design

Typically, the course meets once per week for fifteen weeks, with each class period lasting 150 minutes (2.5 hours). It developed as an extension of other courses, including a one hour capstone seminar for statistics majors, an independent study course for statistics majors and minors, and a "history of statistics" theme used in a summer science and mathematics program for advanced high school students. Student enrollment has been a mix of majors and minors, and students tend to have taken at least two statistics courses beforehand. The statistics minor requires students to earn twelve credit hours in statistics, which most students fulfill by taking four three-credit courses. The minors are most often psychology majors, with some students from other science or social science disciplines like biology, geology, or sociology.

The largest portion of face-to-face class time is taken up by discussions of the course readings, so it depends heavily on students completing assigned reading outside of class on a weekly basis. The primary textbook is *The Lady Tasting Tea: How Statistics Revolutionized Science in the Twentieth Century* [46] by statistician David Salsburg (1931–). Most of the textbook chapters are read in order, but a few chapters have been regrouped so that related themes can be discussed together; for example, Chapters 5 and 7, which address Ronald Fisher's career, or Chapters 15 and 19, which focus on women in statistics (see Appendix A: Course Outline). Supplemental readings, also intended to be completed before class, add depth or additional perspectives. The amount of time required for the class discussion varies from class period to class period, depending on the amount and content of that week's readings, as well as students' prior knowledge and interest.

Class discussions use a Socratic seminar style [17, 31]. The focus of this method is on developing shared knowledge and understanding of the material, not debating or proving a point, though disagreement can certainly occur as part of the process. Another requirement of the Socratic style is that students must explicitly acknowledge and attempt to synthesize others' contributions before adding their own. They speak mostly to one another while the faculty member serves as a facilitator, redirecting the conversation as needed [27, 28]. This method should be introduced and explained on the first day of the semester, as it is a novel concept to most students, especially those who are accustomed to speaking in terms of "the" solution to a problem. In keeping with the overall course theme of communication as well as the Socratic framework, we explicitly consider the audience for whom the readings were originally written, readers' individual interpretations of the language, and what we know about the author's background and intentions [17]. Specific questions about how the texts connect to course themes, or reminders to students to explicitly consider those points, can be interjected as needed to steer the discussion.

As an example, consider two questions posed to students when they read the Author's Preface: "How would you define the term *science*?" and "Do you consider yourself to be a *scientist*?" The subtitle of the textbook is "How Statistics Revolutionized Science in the Twentieth Century," which suggests a particular perspective on history by the author. Students are also assigned to read a brief interview with the author in which he talks about both himself and the book [36]. At first glance these questions might seem simple—everyone in college knows what science is, right? Further, our university strongly encourages undergraduate research, which many of my students have done. The sample of online student responses below shows a diversity of thoughts, definitions, uncertainties, and potential biases:

- I would not consider myself a scientist because I never use mathematical formulas to determine events that happen. If I do I am not aware that I am using them. I am not certain.
- I can't consider myself a professional scientist by any means, but I relish the findings of science. I suppose this makes me a science enthusiast. I must disqualify myself from the status of being a complete

scientist because I don't run any experiments, only collect data haphazardly for my own enjoyment, and certainly don't lend effort toward constructing professional hypotheses (though I can make a casual conjecture that is based on what I observe).

- Yes, in a way, in that I try to use scientific ideas and methods in forming my thoughts about the world. Also, anthropology is totally a science, even if social sciences aren't "real" science. But not an experimental scientist, like physics or chemistry or such things.
- I would not consider myself a scientist because I have not published any results from my own personal experiments or studies. I have, however, done research, ran experiments, and gathered results for graduate students and professors. In a sense, I would call myself a scientist that is not recognized by the scientific world.
- I feel like everyone is a scientist in their own way. We test our everyday ideas in different ways and although these are not sound experiments, there is still experimentation done nonetheless.

These sample online responses touch on some of the course themes, such as communication (e.g., publication of results is one criterion given to be a scientist) and the potential overlap between scientific and social issues (e.g., whether social science concerns are "real" science). The students are not given each other's written responses for the discussion, though optionally they can be allowed to view them later. Instead, they must contribute their ideas orally. They discuss until they reach a consensus about what constitutes science (or what is "real" science?) and who they *as a class* consider to be scientists. Their exchange should also include an assessment about whether their definition is consistent with the author's perspective and whether notions of science and scientists have changed over time. Students tend to converge on the idea that they are scientists by virtue of their intention to systematically explore the world. However, that is not necessarily the "right" answer. Whatever they decide, they should be prompted to be aware of how that might affect their future conversations and perceptions.

While the class discussions are engaging and often quite lively, they almost never fill the entire class period. Students may reach a consensus quickly, find themselves discussing in circles, or simply grow tired. It is necessary (but also fun) to mix in hands-on activities to illustrate or explore concepts from the readings and help students connect with the experiences of the people we are learning about. While the focus of the course is not methods or proofs, a few class activities delve more deeply into the statistical methods mentioned in the text. These explorations can be adapted depending on the composition of the class; for example, a group with mostly statistics majors may have more experience and enjoy more in-depth statistical explorations. Allotting time for a handson activity, sometimes in the middle of class, can give students a break from speaking. However, if a class discussion is going well, skipping or delaying an activity so as not to break the group's flow can be the better choice. Flexibility is required, and some activities can be introduced before their corresponding chapter or chapters.

Drawing from *The Lady Tasting Tea*, there are many possibilities for hands-on activities. Near the fourth week, students might do paper and pencil simulations and find square roots by hand to replicate the process William Sealy Gosset (1876–1937) used to develop Student's *t* [25, 46]. About the sixth week, they can research the median lethal dose (LD-50) for common substances like caffeine or sugar, which connects to the research of Chester Bliss (1899–1979). Sometimes this becomes an impromptu competition to find the most unusual or the deadliest item. For more depth, the LD-50 activity can be accompanied by a discussion of logistic regression and why it is more difficult to find the extreme LD values. Watching a clip of the mathematician character Ian Malcolm in the movie *Jurassic Park* and playing *The Chaos Game* [48] usually makes students eager to discuss chaos around the seventh week. We spend part the eighth week working problems using Bayes' Theorem [52], which is named after Thomas Bayes (1701–1761). Around the tenth week we could explore how to use slide rules, which gives insight to the work of many people, including the nameless (to us) women hired as "calculators" (or "computers") in the early part of the 20th century. There is an online slide rule [15] that is easy to use if students have access to a computer lab or laptops.

Most but not all of the activities are inspired by the primary textbook. For example, one of the first activities is a variation on the well-known “draw a scientist” [9] or “draw a mathematician” [43] activity, in which students draw a statistician. We analyze their images for recurring themes and stereotypes. For example, most of the students draw men—even the women. They also tend to portray their statisticians as working at a computer. Both of these observations connect to the course themes (communication, women and minorities, computation), as well as the initial discussion about science described above. Students may reflect back on the omnipresence of computers in this activity and their own experiences when completing the course Letter Writing Project (see Assignments).

Finally, roughly one class period is devoted to working on projects (see Appendix A: Course Outline), which are otherwise largely completed outside class. There are at least a few minutes built into the beginning of every class for announcements and a brief recap of the previous week or two, often via targeted questions that the students answer. This refreshes their memories of where we are in the textbook’s timeline and helps get everyone focused on that day’s events, since a week between meetings can be a long time for some students. Students receive a verbal reminder about overall course themes at each face-to-face meeting as well. Finally, a few dedicated minutes at the end of each class allows time for students to ask questions about projects or other assignments.

Resources

Textbook

This course is structured around the text *The Lady Tasting Tea: How Statistics Revolutionized Science in the Twentieth Century* [46] by statistician David Salsburg (1931–). This easy-to-read book, which is often used with AP Statistics students [38], takes a nonmathematical approach to the history of statistics. The author focuses on people who influenced the development of statistics and includes not only “big names” like Ronald Fisher, who has been referred to as “the hero of 20th Century statistics” [16], but also interesting lesser-known figures like Chester Bliss, an American who worked in the Soviet Union during the height of Joseph Stalin’s (1878–1953) purges. Salsburg’s conversational writing style generally appeals to students, and many can read the entire book in a single day. At the same time, there is a great deal of material to spur interesting conversations and investigations. For example, the author’s choices about who to include (which he mentions in the Afterward), the representation of women (his wife urged him “don’t forget the women” [36]), the inclusion of minorities (there aren’t many), as well as his subjective characterizations of people (e.g., referring to Karl Pearson as a “dispirited old man” [46, p. 10]), are excellent fodder for discussions about potential biases in communication and diversity in the statistics community. The bibliography contains many additional sources aimed at readers without mathematical training, which the author says he chose because they are works where “other statisticians have tried to explain what excited them about statistics” [46, p. 312]. Students read some of these, such as the interviews described in the next section. Before each class meeting, students read one to three chapters in *The Lady Tasting Tea* and answer related questions online. These questions are typically due twenty-four hours before the class meeting so I can review and analyze them for particular points or observations that students made related to the chapters, how their observations and comments tie in to our overall course themes, and items for potential discussion prompts. Thus, to some extent, course meetings cannot be entirely planned before the start of the semester.

Supplemental Readings

This course relies heavily on journal articles, magazine articles, book excerpts, and websites to add depth. *Biometrika*, the seminal journal co-founded by Pearson, is discussed in the second chapter of *The Lady Tasting Tea*. Other articles about this publication show the evolution of statistics research [11, 55], including controversial ideas like Pearson's investigations in eugenics (e.g., [42]). Some readings more broadly explore the eugenics movement (e.g., [2, 20]), which influenced several early statisticians. Students read a few seminal methods papers on topics typically found in introductory classes, such as the first published articles on regression [21] and Student's t [51], with an eye toward discussing how the results were communicated. These articles often differ in style from research papers that students are used to reading or writing (e.g., the APA Style [3] introduction, methods, results, and conclusions format). Another source is *Significance* magazine [49], which offers a wealth of accessible articles on topics including the future of statistics [22, 26, 35, 58] and perspectives on statistical significance [32, 45]. The journal *Statistical Science* has published many interviews with statisticians, including women and minorities [13, 19, 33, 47]. Some feature the person's spouse [18] or discuss personal influences like religion [14, 47]. These details are often left out of more formal articles about statisticians' achievements. Such interviews let students experience these people's voices or see pictures of them as children or with their families, which helps some students better identify with historical figures. There are two nice articles from *The American Statistician* about women in statistics [8, 59], as well as a few articles about women and minorities in mathematics from *The Encyclopedia of Mathematics in Society* [23]. The *Encyclopedia* is aimed at a general audience like the primary text and also has many useful entries for topics and time periods covered by *The Lady Tasting Tea*. A more complete listing of supplemental readings can be found at my History of Statistics Seminar webpage [54].

Web Resources

There are some good websites for teaching a history of statistics course focused on people, including *Figures from the History of Probability and Statistics* [1], *Life and Work of Statisticians* [34], *MacTutor* [39], *Mathematics Genealogy Project* [37], and *Statisticians in History* [4]. The proliferation of digital archives means that many historical books and papers are available online, such as the *R.A. Fisher Digital Archive* [56] and Karl Pearson's book *The Grammar of Science* [41]. Documents like the racial policy pamphlet [7] found at the *Nazi Propaganda Archive* and artifacts in *Image Archive on the American Eugenics Movement* [10] are engaging resources to teach about the eugenics movement. Even bloggers can offer contemporary perspectives, like statistician Gerard Dallal's (1948–) discussion about the historical evolution of the $p = 0.05$ convention in significance testing [12]. *Retro Calculators* [44] and the interactive *Derek's Virtual Slide Rule Gallery* [15] are interesting sites to spur discussions about computers in statistical practice. Additional websites can be found at my History of Statistics Seminar webpage [54].

Course Website

Course content is organized and presented using my university's Moodle course management system. Moodle has many features that facilitate student interactions outside face-to-face meetings, such as message boards and a wiki module. All supplemental readings are posted on Moodle and students answer questions about the readings using Moodle's questionnaire module. Faculty who do not have Moodle could replicate most of these features with some combination of websites, message boards, wikis, or blogs available for free on the web.

Assignments

Weekly Reading Questions

Each week, students answer questions about assigned textbook chapters, which must be submitted online at least twenty-four hours before the class meets. They are used to check that students are doing the reading and facilitate in-class discussions of course themes. Many questions ask students for their opinion based on facts gleaned from the reading, which they can also support with ideas found in supplemental readings. See Appendix A: Course Outline for a weekly breakdown of chapters and sample questions.

Family Tree Wiki Project

Typically on the first day of class, students use the *Mathematics Genealogy Project* [37] website to trace back several generations of my mathematics family tree. They complete a worksheet on which they fill in names to identify the pictures provided for each ancestor and record other information about their schools and degrees. Although the Mathematics Genealogy Project focuses on PhD students and advisors, the undergraduates in this course begin the tree with themselves, then me. Later we create an interlinked Family Tree Wiki. Students pair up to write structured entries for two or three mathematicians, depending on the number of students. The pages for mathematicians include not only professional accomplishments, but also information about families, early lives, and hobbies. Students also list five contemporary historical events to place these people in a broader context and they must provide at least three web or print references for their mathematician(s). Finally, they peer review the Wiki entries using a rubric. Once the mathematicians are complete, each student writes an entry for a classmate. Their entries include information about their family, early life and K–12 education, honors and awards, future plans, and why they are majoring or minoring in statistics. In follow-up discussions, we make comparisons between the members of the class and the mathematicians in our tree, in particular any similarities they may share and facts that might have surprised the students. I am lucky to have a number of well-known mathematicians and statisticians in my mathematical family tree, some of whom connect directly to *The Lady Tasting Tea*. Students are often excited to discover names they recognize from other classes or the book, such as Joseph Lagrange (1736–1813), Leonhard Euler (1707–1783), and Jerzy Neyman (1894–1981). Connections within the Wiki can be expanded by including entries for the schools people attended and the countries in which they were born or worked.

Letter Writing Project

The history of mathematics and statistics is full of collaborations. The story of the origins of probability theory generally centers on letters exchanged between Blaise Pascal (1623–1662) and Pierre de Fermat (1601–1665) [40], some of which have been translated and published in various sources [57]. It could be argued that letter writing is a lost art in the 21st century, given the availability of many kinds of electronic communication and file sharing. To connect students to the kind of experience statisticians would have had in the early part of the 20th century, they are asked to solve a probability problem with a partner via hand-written letters. They are not allowed to type their letters, nor to discuss the problem in any other way. Letters are exchanged once per week for six to eight weeks. One week Partner A gives a letter to Partner B. The next week, Partner A receives a reply—a painfully slow process for most students! Eventually they are given a class period to work with each other and their classmates. Time permitting, they can also be allowed to collaborate solely through electronic means for a period of time between writing letters and the final face-to-face collaboration. When the project is complete, they reflect on the experience by answering several questions. Some of these questions address perceptions of the general positives and negatives of collaborative letter writing versus other modes of communication. Other questions require them to think about whether having to write letters versus being able to speak to one another changed the way they actually thought about both the assigned problem and problem-solving in general, and whether it influenced their perspective on the way the people they read about solved problems.

Online Forum Discussions

This is a seminar course, so everyone is expected to speak in class. However, there are varying degrees of comfort and ability among students, so online discussions using Moodle forums allow for a different type of back and forth exchange. The students are split into small groups of about four people and can only view and discuss the assigned topic with their group, using the Socratic style as in class. These groupings are shuffled for each discussion, which take place over about three weeks. Students are required to post at least once per week, and they must back up statements with reputable citations. Online discussions tend to require students to focus on a broader range of sources than the weekly discussions. Some past topics include: (1) Statement: Karl Pearson being in control of *Biometrika* for so long was more beneficial than harmful, given the level of development of statistics as a discipline in the early 1900s and (2) Question: Is it ethical to celebrate Ronald Fisher as a hero, given his strong associations with and contributions to eugenics? Do his lasting contributions to the field of statistics overshadow their potentially questionable origins?" In another case, students watched a video by "mathemagician" Arthur Benjamin (1961–) called "Down with Calculus! Up with Statistics!" [6]. They had to discuss whether they agreed with him or found flaws in his assertions.

Final Paper

The final paper has two parts. First, students write a thousand-word obituary for a statistician of their choice, similar to obituaries we read in class (e.g., [5, 24]). Then they discuss one of that person's statistical achievements in roughly another thousand words. This is the only course assignment in which students must go into some depth with regard to statistical methodology—to the extent of their abilities. Calculus is not a course prerequisite, so there is a limit to how much some students can discuss advanced statistical methods. However, course readings introduce a broad array of people and discoveries to choose from, and no student is incapable of finding some topic about which they can write.

Tests and Exams

Given the discussion-based, integrative seminar style of this course, there are no traditional tests or exams. Students are sometimes surprised by this, especially mathematics students who are often used to being assessed via homework, quizzes, and tests.

Lessons Learned

This course and the courses that inspired it are simultaneously one of my greatest pleasures and greatest challenges. I spend a great deal of my time teaching introductory-level statistics courses to non-majors. Even in those I tend to include historical people and facts. However, a course like History of Statistics allows me to do more than just dip my toe into the waters of history before returning to the how-to routine of data and methods. I learn new things each time I revisit *The Lady Tasting Tea*, which I first read for pleasure, and there are always new books, articles, and websites to discover. My mathematics family tree has also grown over the last few years as more historical connections are added to the database.

Digital archives put a wealth of documents at our fingertips, so in some ways a course like this has become easier to teach than it would have been even a few years ago. On the other hand, a seminar course by its very nature never plays out the same way twice. This kind of class requires significant preparation each time it is offered: selecting new supplemental readings, reading responses to online response questions, being prepared to facilitate discussions for whatever student personalities are enrolled that semester, adjusting for varying class sizes, and making sure there are sufficient activities to fill the classes where discussions are shorter—something you may not know until you are in the classroom. It also requires you to be familiar with multiple sources rather than a single textbook, and to make readings and resources accessible for students in a way that does not require reams of copies or violate educational fair use. Password-protected course management systems and websites help a great deal.

Managing lengthy discussions can be a significant challenge, one that statistics instructors do not necessarily face in other classes. The Socratic seminar style may feel unnatural to both you and the students at first. Initially they tend to want to talk to the teacher rather than to one another, and it takes a while before they habitually use the syntax of acknowledging one another. It can also be difficult for me to let go of being the one who provides a "right" answer. I have also had students who developed poor attitudes, treating the course as less challenging or important because we are mostly talking in a casual setting rather than listening to lectures or computing. While I want students to be comfortable, some misinterpret the atmosphere or the fact that I am stepping back from a more typical teacher role during discussions. Setting clear expectations in the first class meeting and reinforcing them is important. It also helps to be in a classroom where students can physically sit facing each other. Try to avoid a lecture-style arrangement, to the extent that your classroom space permits.

Class size also definitely affects the discussion dynamics. My experience leads me to say that about ten students is the ideal number for this course, which is enough for a variety of perspectives but not so many that it is difficult for everyone to contribute. I had twenty students in the spring of 2014, which was frustrating and unwieldy. One solution for a larger class is to split the students into smaller discussion groups, either talking at the same time in separate parts of the room with the facilitator rotating among groups, or taking turns speaking. I used this method with great success for fourteen students in the spring of 2016. Some Socratic seminars are designed to have rotating speakers [53]. A few students can monopolize the conversation if flow is left unchecked. Giving students "talking chips" [50] can help until they have more practice, but often it is possible to verbally guide them. Keeping track of participation with so many students is also more difficult. Tally sheets and rubrics are available (e.g., [29, 30]). However, some students may not respond positively to either chips or being "checked off" in this manner.

A once-per-week meeting schedule can be problematic at times. Some students prioritize this course differently than they would a three-credit course that meets more often. They seem to be inherently driven by the sense of immediacy that comes from seeing an instructor multiple times per week. Other students simply grow fatigued or lose focus when discussions carry on too long or there is too much material to absorb in one sitting. The temptation is strong to overload them on supplemental readings—there is so much interesting material!—but that ultimately leads to shallow reading and discussions. In the spring of 2016 this course met three times per week for fifty minutes, and the classes seemed to end far too quickly. In the future, I might be inclined to ask for meetings twice per week, with one day for discussions and the other in a computer lab for activities and projects. In addition, I will likely include questions about the supplemental reading in the online questions rather than just *The Lady Tasting Tea*, to ensure that students do the reading and to better help all of us focus and make thematic connections.

Instructors who prefer a one- or two-credit course could easily use the primary text and a subset of course themes to which they could tailor supplemental readings. For example, they might want to focus on personal histories and communication, using primarily statistician interviews and obituaries as supplemental readings, such as those in the bibliography of *The Lady Tasting Tea*. Based on feedback I have received from students in this course's various iterations, even just reading the primary textbook gives them a new perspective on why they have been doing what they have been doing in their major or minor program.

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Appendix A
Course Outline

Week 1

In most cases, students are not able to do any reading before the first day of class. Therefore, I use this time to make introductions, discuss expectations, explain the course structure and assignments, make pair assignments for the Letter Writing Project, and have students work on and discuss the first Family Tree Wiki activity (as described in Assignments above).

Week 2

Author's Preface

- How would you define the term "science"? In other words, science is …
- Do you consider yourself to be a scientist? Why or why not?
- How do scientists share their work or results with one another?

Week 3

Chapter 1: "The Lady Tasting Tea"

- How many cups of tea would you expect the lady to be able to identify correctly, if she were just guessing? What is the probability that the lady could identify all ten cups of tea correctly just by guessing? How many would she have to identify correctly to convince you that she wasn't just guessing?
- Salsburg says, "What makes scientists interested in their work is usually the excitement of working on a problem." Do you think Ronald Fisher was excited to be studying agricultural data? What was the problem that drew his interest?

Chapter 2: "The Skew Distributions"

- With reference to the article "On Jewish-Gentile Relationships," Salsburg says that Karl Pearson "concluded that the racial theories of the National Socialists … were sheer nonsense …" After reading Pearson's article, would you agree with Salsburg? Explain. (Note: a Nazi propaganda pamphlet of the period asserted: "The environment can only influence what is already present in the genes.")
- What are the pros and cons of *Biometrika* being largely under the influence of a single man (Pearson) for so long?

Week 4

Chapter 3: "That Dear Mr. Gosset"

- William Sealy Gosset is described by Salsburg as "an unassuming man" and a mediator between Karl Pearson and Ronald Fisher (called "two towering and feuding geniuses" by the author). To what extent do you think an individual's personality contributes to the credit he or she receives for professional contributions and whether or not they are later labeled as "important"?
- How old were Fisher, Gosset, and Pearson in 1908, the year Student's "The Probable Error of the Mean" appeared in *Biometrika*? How might their relative ages have influenced the relationships described in the chapter?

- Do you think Gosset had any idea how important his kitchen-table work on the mean would become? What do you think were his motivations?

Week 5

Chapter 4: "Raking Over the Muck Heap"

- Did Karl Pearson behave ethically in his handling of Ronald Fisher's work, with respect to the publication in *Biometrika*? What about with respect to his job offer? Do you think Pearson's behavior can be explained as a function of the time period?
- How did Pearson's disdain for Fisher affect the early development of statistics?

Chapter 5: "Studies in Crop Variation"

- Consider the lengthy quote from Fisher on p. 51. How does this relate to or reflect upon his relationship with Karl Pearson and his later successes at Rothamsted?
- How did the relationship between Fisher and Pearson end? Do you think Fisher's sentiment would have been different if he had achieved his successes?

Chapter 7: "Fisher Triumphant"

- On p. 71, Salsburg says, "Maximum likelihood rules the world, and Pearson's methods lie in the dust of discarded history." What was the fundamental difference between Pearson's and Fisher's views on statistics? Does the fact that we don't use Pearson's approach mean that his work was not worthwhile?
- Later in life, Fisher seems to treat Pearson's son Egon with a disdain similar to the treatment he received from Karl Pearson. Do you believe this is justified? Why or why not?
- Fisher became the chairman of the University of London's Department of Eugenics. Reflect on Britain's side during theWorld Wars. Does the existence of this department surprise you?

Week 6

Chapter 6: "The Hundred-Year Flood"

- What are the similarities between L. H. C. Tippett and William Sealy Gosset, with respect to their statistical work and motivations?
- Salsburg refers to Emil Gumbel as courageous. Do you agree?
- Do you think you could have done what Hiram Bingham did? Why or why not?

Chapter 8: "The Dose that Kills"

- I have read this book several times, and each time I get to this chapter, I want to ask Chester Bliss: "Man, what were you thinking?!" Of course, this comes in part from growing up while the Soviet Union was still a major superpower (and a little hindsight, as with most things we are reading about). What do you think he was thinking, taking the job in Russia, and then staying in the face of later events?
- Describe Chester Bliss's response to the government committee sent to investigate him as a possible American spy. Does the committees reaction surprise you? Why or why not?
- Salsburg participated in experiments to determine the LD-01 of a chemical that causes cancer in mice. What was the sample size used? Is such research allowed today? Is it ethical, then or now?

Chapter 9: "The Bell-Shaped Curve"

- In our Mathematics Family Tree, many of our ancestors were Russian or studied in Russia (e.g., Sierpinski). Others were Polish, German, and/or Jewish. How did WWII change the landscape of mathematics and, by extension, statistics in the world?
- Salsburg described how research was done by many mathematicians (in his view) during the early 1900s. Does this match with your vision of how research is done today? Explain.
- As a whole, why did the regimes in Russia and Nazi-occupied Germany have issues with mathematicians and statisticians? Is their treatment (on any level) comparable to Karl Pearson's treatment of Ronald Fisher?

Week 7

Chapter 10: "Testing the Goodness of Fit"

- How is chaos theory a return to the determinism we read about at the beginning of the book?
- It seems pretty clear that Salsburg is a Jerzy Neyman fanboy. How do you think this might affect his discussion of Neyman in chapter 10 and other people who interacted with Neyman (e.g., Pearson)?

Chapter 11: "Hypothesis Testing"

- What is the frequentist approach to statistics? Are the methods you've learned in your other courses frequentist-based?
- Salsburg briefly mentions that David Blackwell had professional difficulties because of his race (which he neglects to mention is African American), but he uses the anecdote primarily to give insight into the character of Jerzy Neyman. What role did Blackwell later play in the college department created by Neyman? What were some of his other contributions to statistics? (You'll have to do a little web searching and/or consult the supplemental readings.)
- Describe Ronald Fisher's behavior toward Jerzy Neyman. Does this surprise you? Why or why not?
- Until this chapter in the text, we have hardly seen any discussion of women in the field of statistics. Where have you seen them before and in what context? How are women involved in this chapter?

Chapter 12: "The Confidence Trick"

- Why can't we be 95% "sure" when using a standard confidence interval?

Week 8

Chapter 13: "The Bayesian Heresy"

- Why is the chapter titled "The Bayesian Heresy (my emphasis)"?
- What are the similarities and differences between the Bayesian Hierarchical and Personal Probability approaches to probability? Why is each of these considered Bayesian?

Week 9

We take a break from reading in Week 9. Students get an opportunity to discuss with one another in person the problem posed in their Letter Writing Project, and we discuss it as a class. They also peer-review each other's mathematician Wiki entries and conduct interviews to collect the information they need to create the student Wiki entries.

Week 10

Chapter 14: "The Mozart of Mathematics"

- What does Salsburg mean when he refers to Andrei Kolmogorov as a "Mozart"?
- How might statistics have advanced differently if the Soviets had not rejected the notion of "random variables" during the middle of the century and had instead embraced more of Kolmogorov's and other statisticians' work?
- After describing the Failures of Soviet Statistics, Salsburg noted that perhaps there is a lesson to be learned. What is this lesson?

Chapter 15: "The Worm's-Eye View"

- What does the title of this chapter mean?
- Does Salsberg talk about Florence Nightingale David the same way as he did Karl Pearson's other students, like William Sealy Gosset? Is it significant that we're about halfway through the book before we hear about her (or any woman besides the titular lady tasting tea)?
- Jerzy Neyman was surprised that David did not have a PhD by the time he began working with her. What do you believe were the reasons for this?
- How did WWII change the opportunities available to women, both in and out of statistics?

Chapter 19: "If You Want the Best Person ..."

- In this chapter, Salsburg explicitly acknowledges the lack of women in the narrative prior to his presentation of Florence Nightingale David. What do you think of his discussion of the women behind the men in the early days of statistics and the notion of women as "calculators" (or "computers")? Should this have been included earlier? Discuss.
- How was Gertrude Cox's educational and professional path similar to and different from the men we have already read about? What about Janet Norwood?
- On page 205, Salsburg says that both Cox and Norwood were "primarily administrators and teachers," though he notes that other women played a role in theoretical statistics in the latter half of the 20th century. How might this statement change a reader's perception of these two statisticians' contributions to the field, relative to the men?

Week 11

Chapter 16: "Doing Away with Parameters"

- Why is it not okay to just throw out outliers in data?
- What role do assumptions (e.g., independence, random sample, etc.) play in statistical analysis? Why are they important?
- Frank Wilcoxon was sure that his bizarre findings must have been discovered previously. Why was he hesitant? What does his hesitancy suggest about the nature of research in statistics in the 1940s?

Chapter 17: "When Part is Better than the Whole"

- What role(s) does money play in science and the development of a scientific discipline (including mathematics and statistics)?
- Are there any similarities between the comments by Margaret Martin on page 173 with those of F. N. David on page 157?

- On page 174, members of the U.S. Chamber of Commerce find that when they use opportunity samples, the findings by the Department of Labor are not true. Explain the findings and why their specific opportunity sample did not work.

Chapter 18: "Does Smoking Cause Cancer?"

- Does smoking cause cancer?
- Is there such a thing as cause and effect?
- Do you find any irony in the objections that Fisher raised to studies suggesting smoking causes cancer?

Week 12

Chapter 20: "Just a Plain Texas Farm Boy"

- Why was mathematical statistics considered to be "disreputable" in Wilks's time?
- Why do you think Salsburg characterizes abstract mathematics as "cold"?

Chapter 21: "A Genius in the Family"

- What is the "curse of dimensionality"?
- Persi Diaconis was advised by the eminent Harvard statistician Frederick Mosteller (who also dabbled in magic) not to do any magic tricks in his classroom because "The kids end up thinking you're a performer, and they stop believing you're a scholar." Do you think this is true, and what does it say about our expectations of mathematicians/statisticians?
- I. J. Good and Persi Diaconis are portrayed as being fortunate amidst difficult circumstances. Having said this, what demographic characteristics do both have that likely allowed them the chance for success?

Chapter 22: "The Picasso of Statistics"

- Why might John Tukey be a better candidate than Ronald Fisher for "Hero of 21st Century Statistics"? (Or is he not better? Discuss.)
- Salsburg describes John Tukey as one who always challenged assumptions. Why is this important in statistics, or in any field of research?

Week 13

Chapter 23: "Dealing with Contamination"

- What does "robust" mean in the context of statistical analysis? Why is this an important concept?
- What is a contaminated distribution?

Chapter 24: "A Genius in the Family"

- What is "statistical quality control" in manufacturing?
- What cultural factors may have led U.S. business managers in the 1970s to be reluctant to listen to W. Edwards Deming, whose methods had been developed in large part in Japan?
- Pick two major events in this chapter and discuss how they might have differed if women had the same social status as they do now.

Chapter 25: "Advice from the Lady in Black"

- Salsburg said "the attempt to formulate a problem in terms of a mathematical model forces the scientist into understanding the question that is really being posed." He is echoing the sentiment expressed in the quote immediately above by Stella Cunliffe (p. 265), which talks about the role of the statistician in applied research. What do they mean?

- Consider the excerpt from Stella Cunliffe on page 260 where she is informed of appropriate company behavior. Was this expectation also held of women in public? Do you believe it would be held at other companies?

Week 14

Chapter 26: "The March of the Martingales"

- How do the two conditions stated at the bottom of page 270 relate to a patient's status or change of status in a clinical trial? For example, what if we were rating the health status of a person with congestive heart failure on a numerical scale?

Chapter 27: "The Intent to Treat"

- What is "intent to treat" methodology?
- Is it potentially unethical to use p-value based statistical hypothesis tests in clinical trials? If not strictly unethical, is it otherwise problematic? Discuss.

Chapter 28: "The Computer Turns Upon Itself"

- What is the "fundamental theorem of mathematical statistics," in plain English?

Week 15

Chapter 29: "The Idol with Feet of Clay"

- What does the phrase "the idol with feet of clay" mean with respect to statistics?
- What is your answer to the question: Can statistical models be used to make decisions?
- What is your answer to the question: What is the meaning of probability when applied to real life?
- What do you think is the future of statistics in the 21st century?

"Afterword"

- Reading the author's explanation and rationale for the historical "path" he elected to follow, do you think he made the good choices? What might you have done differently?

History of Probability, using primary sources

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Course Overview

It began with a gambler's question: if a game between two players is interrupted before either player has won, how should the stakes be divided? The young Blaise Pascal, in Paris, wrote to Pierre de Fermat, in Toulouse, about the problem, and thus initiated a correspondence that is one of the classical sources of what we now call the mathematical theory of probability. In this chapter, I describe a course in the history of mathematics that took up the history of probability theory, and was based on the Fermat-Pascal correspondence and on other primary sources.

At the University of San Diego, the History of Mathematics course is a junior or senior level course. The prerequisite is third-semester calculus. The course satisfies an upper-division writing requirement. It also satisfies a requirement for students working toward a single-subject teaching credential. At USD, these single-subject students are mathematics majors, so that most of the students in the History of Mathematics course are upper-division mathematics majors, a sizable fraction of whom intend to be secondary school teachers.

Course Design

The course meets three times weekly, for 55 minutes. The class is mainly in lecture format. There are regular writing assignments, about one per week.

Even when we restrict from the history of mathematics as a whole to the history of probability theory in particular, the subject is too extensive to cover completely in one semester. In the course described here, I covered the period from the time of Cardano and Tartaglia, in the middle of the sixteenth century, to the time of Abraham de Moivre, whose *Doctrine of Chances* was first published in 1718.

After a few days spent discussing basic issues in the historiography of mathematics, the first part of the course was devoted to a discussion of the concept of probability and to the main explanations that have been given as to what probability actually is, namely, the frequentist view of probability, on the one hand, and the subjective, or personal, view, on the other. Since most of the students in the class had not previously taken a course in probability, this seemed to be a necessary foundation, although the historical origin of the issues involved came somewhat later than the chronological period covered in the course. (The theorem named after Thomas Bayes was published, posthumously, in 1764. It was rediscovered, independently, by Laplace, in 1774.) Even so, many mathematical ideas occur in embryonic form, so to speak, long before they become clearly formulated. Thus, at some level it makes sense to ask, as I did in one of the writing assignments, whether Jacob Bernoulli's view of probability was "frequentist" or "subjective".

Most of the semester was spent studying primary sources. Primary sources are not easy reading, and our students nowadays are not very good readers. Thus, they need a lot of coaching to get through the readings. When I present the material in class, I generally begin by telling the students about the setting from which the source comes: when it was written and where, what were the circumstances at the time, and how it fits into the development of the subject to which it is a contribution. I tell them something about the author of the reading and some of the circumstances of his life. Usually, we have to go through the readings very slowly, paragraph by paragraph. Since the students have difficulty in seeing the "big picture", I try to show them what the author is up to, and where he is going. Although of course we work with English translations of the primary sources, I sometimes have to "translate" the translations, by cutting through an author's long-winded discussion to express what he is saying in simple English. The writing assignments for the course give the students an opportunity to formulate the ideas in their own words.

The first primary source that we studied was a short note by Galileo, "Sopra le scoperte dei dadi" ("Discoveries concerning dice"). Galileo considers a game in which three dice are thrown. He explains very clearly how to find the probabilities for the various possible sums. In particular, the probability of throwing a 9 is 25/216 and the probability of throwing a 10 is 27/216. Galileo remarks that "it is known that long observation has made dice-players consider 10 and 11 to be more advantageous than 9 and 12" [3, p. 192]. I pointed out to my students that the difference in the probabilities of getting a 9 and getting a 10 is only 1/108. How many throws would be needed in order to recognize this difference? In order, for example, to have a 95% chance that the number of 10s would even be *greater* than the number of 9s, it would require on the order of 7500 throws. While many gamblers of Galileo's time might well have played that many throws, it does not seem likely that they would have kept an accurate record of the numbers of 9s and 10s that they obtained. Consequently, I am somewhat skeptical about Galileo's claim.

A similar question arises in connection with the Chevalier de Méré, who observed to Pascal that the probability of throwing a six in four throws of a single die was greater than the probability of throwing two sixes in 24 throws of two dice (see [3, pp. 88–89]). In her book *Games, Gods and Gambling*, F. N. David comments, "The Chevalier de Méré was obviously such an assiduous gambler that he could distinguish empirically between a probability of 0.4914 and 0.5" [3, p. 89, footnote].

The next source that we studied was the correspondence between Fermat and Pascal. Working through these letters was a challenge both for me and for the students, for reasons that, of course, are common in the study of primary sources. Here, Fermat and Pascal are trying to find their way in a subject that, at that time, was new, and for which there were not any generally recognized principles. They don't always give their arguments in detail, the terminology they use is not always clear or consistent, and sometimes their arguments are not correct. In a few cases, Fermat disagrees with Pascal's arguments, and gives his own corrections. One point of contention was the following. Suppose, for example, that there are two players, and that one of the players needs one more point to win the stake, while the other needs two points. Fermat remarks that the game will be decided in at most three throws, and that, consequently, each player's chance of winning can be determined by considering the various possible outcomes for those three throws. Roberval, however, objected to Pascal that the game will end as soon as either player attains the necessary number of points, which may happen in fewer than three throws, so that Fermat's method is not correct. On this question Fermat and Pascal, initially, come to different conclusions. One of the writing assignments for the course asked the students to give their own explanation of this problem. (A reviewer has pointed out to me that the Pascal-Fermat correspondence has been treated in a recent book by Keith Devlin [5].)

The next primary source we studied was the fourth and last part of Jacob Bernoulli's posthumously published book *Ars Conjectandi*. In this part of the book, Bernoulli discusses a problem that, nowadays, we would consider to be a problem in elementary statistics. Suppose that an urn contains tokens of two different colors, in an unknown proportion. If you draw a certain number of tokens from the urn (with replacement) and observe the proportion of each color in your sample, how can you estimate the actual proportion in the whole urn? In elementary statistics, today, one constructs a "confidence interval" for the unknown proportion, using the normal approximation to the binomial distribution, an approximation usually attributed to de Moivre and Laplace. Bernoulli, however, does not have the normal approximation available, and consequently he finds his own approximation, directly from the binomial probabilities.

Bernoulli's analysis is a difficult piece of work. Both Edith Dudley Sylla and Bing Sung, in their respective translations of the *Ars Conjectandi*, modernize Bernoulli's notation to some extent. (See Sylla's remarks, [2, pp. 123–126].) My own opinion is that, if you are going to modernize at all, you might as well go the whole hog. Consequently, when we worked through Bernoulli's arguments in class, I used modern notation for factorials, binomial coefficients, subscripts, and whatever, in order to try to make Bernoulli's argument more accessible to the students. But I also had available a facsimile reproduction of the original edition, so that the students could see what Bernoulli's own notation looked like.

Bernoulli has been criticized by some historians because his approximation of the binomial probabilities was rather loose, and consequently the sample size required for his confidence interval was very large (see [9, p. 77]). I regard this criticism as wrong-headed. When I taught this class, I was enormously impressed by Bernoulli's skill in manipulating very complicated expressions arising from binomial probabilities, in order to get a result which, at least in principle, could be applied. It is hardly fair to fault him for not having found the normal curve. Perhaps I am interpreting the students' response through the filter of my own enthusiasm, but I think that they too were impressed.

After Bernoulli, we turned to Abraham de Moivre's *Doctrine of Chances* [8]. We did not go through the whole book, of course, but we did look at some of de Moivre's applications of the method of generating functions to the theory of probability. We finished the semester by studying de Moivre's own derivation of the normal approximation to the binomial, from his *Miscellanea Analytica* [7, pp. 125–129].

Resources

The primary textbook for the course was Florence Nightingale David's *Games, Gods and Gambling* [3], available in an inexpensive Dover edition. This book gives a good overall picture of the early development of probability theory. I did not, however, go through the whole book in detail, but used it primarily for reference. Many of the primary sources that we studied are printed in David's book.

Galileo's "Sopra le scoperte dei dadi" is included (in English translation) in David's book [3, pp. 192–195]. The Fermat-Pascal correspondence is also included here [3, pp. 229–253].

An English translation of Bernoulli's *Ars Conjectandi* by Edith Dudley Sylla was published in 2006 [2], but this translation was not available at the time that I taught this course. I did, however, have a translation of Part Four of the book which had been made in 1966 by Bing Sung, at that time an undergraduate at Harvard [10]. I distributed copies of this translation to the students.

De Moivre's *Doctrine of Chances* [8], which had the benefit, for us, of being in English, is available in an inexpensive Chelsea edition, which I asked the students to buy. (This is a reprint of the third edition, of 1756.) As for his *Miscellanea Analytica*, since I had no English translation of this book, I mainly followed the presentation in Hald's *History of Probability and Statistics* [6, pp. 473f.].

Assignments

Since the course is supposed to satisfy an upper-division writing requirement, I gave a writing assignment almost weekly. The writing assignments were typically not very long—perhaps two or three pages. I prefer several short assignments to one or two long ones, on the principle that one should walk before one runs. I do not require the students to turn in rough drafts and revisions, but I do comment very thoroughly on each paper that is turned in. One of the assignments is for the students to write a short biography of a mathematician of their choice. This paper is a little longer—perhaps 4 or 5 pages. Also, one of the "writing" assignments is actually an oral presentation. In the present course, the oral presentation was on "something that my mathematician did". It did not have to be connected with the theory of probability.

For the first writing assignment, I give the students André Weil's article "History of mathematics: why and how" [12], (originally a talk given at the International Congress of Mathematicians in Helsinki, 1978); and "Mathematics: an historian's perspective" by Joseph Dauben [4], a reply to Weil. The assignment is to discuss the issues raised in these articles. The purpose of this exercise was to confront the students with two readings that take very different points of view with respect to the historiography of mathematics. Thus, the students are forced to think about the issues for themselves, and decide which of the two readings they themselves agree with—at least, that was my hope. For this reason, I deliberately did not give them any introduction to the readings, other than to tell them who the authors were. I can report that the students almost unanimously dislike Weil and register a strong agreement with Dauben. These are not my own views, so obviously I have not been coaching the students on how to respond.

Other writing assignments asked the students to interact with the primary sources we had read. One of the assignments asked the students to write up Galileo's derivation of the probabilities of the dice game in their own words. Another assignment asked the students to analyze F. N. David's claim that the Chevalier de Méré could distinguish empirically between a probability of 0.4914 and 0.5. Another assignment asked the students to clarify one of Fermat's arguments in one of his letters to Pascal. And one assignment asked the students to discuss Jacob Bernoulli's concept of probability: was it "frequentist" or "subjective"?

Lessons Learned

In retrospect, I think that the course went well. The students who had not previously taken a course in probability, when they saw Galileo, Fermat, Pascal, Bernoulli, and de Moivre struggling with the ideas of this new subject, were mostly struggling themselves. The course did not give a systematic presentation of probability and statistics in the manner of a typical college course, but, as Clifford Truesdell remarks in his history of thermodynamics [11, p. 5], "The historical method is not the easiest way to learn a science; neither is it the worst." Certainly, working through primary sources requires more "engagement" with the subject, inasmuch as the rough edges have not been smoothed off, as they typically are in modern textbooks.

My experience in this course was that the students seemed to enjoy their encounters with some of the great mathematicians of the past. The students were quite positive in their reviews of the course.

If I were to offer this course again, I imagine that I would use Sylla's translation for the Bernoulli, but I like the texts that we studied and I think that they produced a satisfactory treatment of the subject.

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4. Joseph W. Dauben, "Mathematics: an historian's perspective," in *The Intersection of History and Mathematics*, Sasaki Chikara, Sugiura Mitsuo, and Joseph W. Dauben, eds., Birkhäuser Verlag, Basel, 1994.
5. Keith Devlin, *The Unfinished Game: Pascal, Fermat, and the Seventeenth-Century Letter that Made the World Modern*, Basic Books, 2008.

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Course Outline

Introduction and Historiography of Mathematics (3 class periods)
Concept of probability (3 periods)
Cardano and Tartaglia (1 period; David, chapter 5)
Galileo (4 periods; David, pp. 192–195)
Fermat and Pascal (9 periods; David, pp. 228–253)
Jacob Bernoulli (8 periods; Sung)
De Moivre (6 periods; de Moivre, pp. 1–33, 243–254; Hald, pp. 472–480)
Oral presentations by students (3 periods)
Exams (2 periods)

Teaching Discrete Mathematics, Combinatorics, Geometry, Number Theory, (or Anything) from Primary Historical Sources

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Course Overview

Here I describe teaching four standard courses in the undergraduate mathematics curriculum either entirely or largely from primary historical sources: lower division discrete mathematics (introduction to proofs) and introductory upper division combinatorics, geometry, and number theory. They are simply illustrations of my larger dream that all mathematics be taught directly from primary sources [16]. While the content is always focused on the mathematics, the approach is entirely through interconnected primary historical sources, so students learn a great deal of history at the same time. Primary historical sources can have many learning advantages [5,6,13], one of which is often to raise as many questions as they answer, unlike most textbooks, which are frequently closed-ended, presenting a finished rather than open-ended picture. For some time I have aimed toward jettisoning course textbooks in favor of courses based entirely on primary historical sources.

The goal in total or partial study of primary historical sources is to study the original expositions and proofs of results in the words of the discoverers, in order to understand the most authentic possible picture of the mathematics. It is exciting, challenging, and can be extremely illuminating to read original works in which great mathematical ideas were first revealed. We aim for lively discussion involving everyone.

I began teaching with primary historical sources in specially created courses (see my two other contributions to this volume), which then stimulated designing student projects based on primary sources for several standard college courses [1,2,3,7], all collaborations with support from the National Science Foundation. These primary source projects (PSPs) embed primary historical sources in contextual, historical, and mathematical commentary as a guide and overview of the big story, and provide numerous mathematical exercises for students. Their pedagogy will be further described below.

Eventually we had built sufficient content into PSPs to teach several standard courses entirely from PSPs, without a textbook, and I have started expanding to other courses as well, not necessarily all in project format.

Today my first thought for teaching any course is not “I wonder what textbook I should use?” (i.e., which modern book will largely drive and determine my course), but rather “I wonder what primary historical sources I can choose to build an entire course?”

Course Design

While the four courses I describe are standard in almost any college curriculum, I will next clarify their level, content, audience, prerequisites, and to what extent I have taught them with primary sources.

1. *Discrete mathematics* at the lower division is often combined with introduction to proof, as a prerequisite to all upper division courses for mathematics or mathematics secondary education majors, and has a prerequisite of college algebra. Standard topics include basic logic and proof techniques, and some selections from algorithms (e.g., the Euclidean algorithm), discrete summation, elementary number theory, permutations and combinations, graph theory, and sets, functions, and relations. Two colleagues and I have each taught this course multiple times entirely from PSPs [4].
2. *Combinatorics* at the upper division, but without a lower-division combinatorics prerequisite, relies on lower-division discrete mathematics and student proof abilities, but nothing else. The course is an elective for an upper-division requirement for mathematics and secondary mathematics education majors. Typical topics might include advanced counting techniques, matching correspondences, and generating functions. I have taught this course entirely with PSPs.
3. *Geometry* at the upper division, but introductory in nature, relies on student familiarity with proofs, but not much geometry. This is a required course for secondary mathematics education majors, and an elective for an upper-division requirement for mathematics majors. The goal is to contrast Euclidean and non-Euclidean geometries. I have taught this course with a large historical component interspersed throughout based on a sequence of primary sources on the development of hyperbolic non-Euclidean geometry.
4. *Number Theory* at the upper division, but introductory in nature, relies on student proof abilities, but no prior number theory knowledge, nor even any abstract algebra. This is an elective for an upper-division requirement for mathematics majors. The course might typically include the Euclidean algorithm, unique factorization, linear Diophantine equations, linear congruences, Fermat's Little Theorem, Euler's generalization, the RSA cryptosystem, Wilson's Theorem, primitive roots, and the quadratic reciprocity law. I have taught the course, covering most of these topics, entirely from primary historical sources.

At home and in class we read, discuss, and interpret results and their proofs from primary historical sources, with students writing their thoughts and questions about these works, and we discuss how the various sources tie together. Regular written assignments based on the primary sources consist of probing definitions and proving results. A primary aim is to have students actually do mathematics themselves by creating some ideas and devising proofs on their own.

Primary source projects (PSPs) pervade much, though not all, of how I teach these courses [6,7]. A key feature of a PSP is a series of tasks for students as they work through the module, intended to provoke students to develop their own understanding based on the primary sources as stimuli. Some tasks ask students to fill in missing proof details, or to reflect explicitly on the nature of the mathematical process by answering questions about the level of rigor in the source. Our students thereby progress naturally in their ability to construct proofs meeting today's standards. We can also introduce our students to present-day notation, terminology, and definitions as a natural outgrowth of studying the primary sources.

Different instructors use different pedagogies in the classroom with primary sources. Personally, I use a non-lecture three-part assignment method that I apply in almost all my teaching, discussed in detail in [15], and in a handout for students (see Appendix B on homework guidelines). For each topic, I first expect students to read new material in advance of class, and to write questions about their mathematical reading for me to read before class, or, in a lower-division course, to respond to reading questions I pose. Second, I expect students to prepare mathematical work for class based on their reading, usually by attempting solutions of pre-assigned exercises. These two pre-class parts are graded based only on effort and preparation for class, very quick to assess with a plus, check, or minus.

In the classroom we first discuss student reading questions as a group, along with questions I inject. Class discussions are often challenging and tremendous fun, because the primary sources provide fabulous grist for deep and wide-ranging considerations. The majority of classroom time is then spent with students working in informal groups on the previously assigned problems, interspersed with impromptu whole-class discussions or presentations assigned by me as common questions and interesting approaches arise.

The third part of each topic's homework assignment consists of completing post-class homework on the topic, usually a very few challenging exercises not worked on in class. Students are always encouraged to discuss their ideas with others, and then expected to finish and write up their polished post-class homework entirely on their own, in their own words, to hand in for me to read and mark carefully, and possibly to request rewriting for improvement, intended to reach a high level of perfection. This post-class homework part ultimately receives a single letter grade for quality, one for each class day.

It is possible to become overwhelmed timewise with reading student papers using this approach, but it needn't be that way. One can carefully be very brief in marking the first two pre-class homework parts. In responding to student reading questions, I make only a few notes on student papers, and a few notes of my own to prepare me for class discussion. The second part, students' pre-class mathematical exercise preparation, has been fully covered in class, so I never mark the work; I only look, for at most a few seconds, to be sure the evidence shows they adequately prepared. For the third part, the post-class exercises that I mark closely, I am careful to assign only very few exercises, often only two. Once I adapted to this method, it takes no more time overall than a more traditional approach; and amazingly, students need less time from me in office hours, because they receive more individual assistance in the classroom.

The course grade is based primarily on the holistic quality of the total of the three parts of daily student work. Sometimes there will also be a midterm and final exam worth at most 30% of the grade. The key is that all the daily work must be a very large part of the course grade, the message being that it is the most important work for learning. In some upper-level courses I eventually feel no need for exams; I reserve the right to make this decision as the course progresses.

In all these courses, whether there is a preordained content syllabus or not, I use a flexible timetable (because this chapter describes four courses all with flexible timetables, there are no course outlines appended), often influenced by what explorations happen in the classroom based on student response and activities, according to the classroom methods described above. On the first day I introduce and discuss the nature and expectations of the course, and we dive into some mathematics, with students studying and discussing primary sources together. Succeeding days always have reading/writing in advance, preparatory mathematical work on exercises, in class group and whole class work, and final homework exercises, as detailed above.

Resources

When I began teaching with primary sources, I and my colleagues had little more than a handful of chosen primary sources on a few topics, some of which we had to translate ourselves, often with no annotation, context, or exercises. And indeed this is how anyone can start developing their own materials; I enthusiastically recommend it. Guidance on the pedagogical principles, and on design of materials, can be found in [5,6,7,13,16]. Today primary sources are much more easily available, and in translation as well, than when we started. The reader may be pleasantly surprised that finding promising and appropriate primary sources for teaching on a given topic is not as hard as may be feared. For those who wish to design their own materials, the bibliography [17] provides a window to many historical sources for teaching. The recent sizeable source book [20] would be a good place to find many good sources on a variety of topics (see the review [18]).

Here I describe the particular resources I have used. And as illustrations, Appendix A introduces small excerpts from selected primary source material for each course, along with connected sample exercises for students.

1. *Discrete mathematics:* This lower-division introductory course has been taught several times by multiple instructors entirely from our collections of projects based on primary historical sources, as described in [4]. The article [3] also describes the projects and much about teaching with them, and [1, 2, 3] lead to the projects themselves. Six of these projects are specifically for an introductory course in discrete mathematics:
    - *Deduction through the Ages: A History of Truth*, with sources by Chrysippus, Boole, Frege, Russell and Whitehead, Wittgenstein, and Post
    - *Sums of Numerical Powers in Discrete Mathematics: Archimedes Sums Squares in the Sand*
    - *Euclid's Algorithm for the Greatest Common Divisor*
    - *An Introduction to Symbolic Logic*, with sources by Russell and Whitehead
    - *An Introduction to Elementary Set Theory*, with sources by Cantor and Dedekind
    - *Pascal's Treatise on the Arithmetical Triangle*

    We have taught the course entirely with various combinations of four of the six projects. Instructors make their own selections.

2. *Combinatorics:* I have taught this upper-division introductory course entirely from our collections of projects based on primary historical sources. The article [3] describes the projects and much about teaching with them, and provides links to the projects themselves. Three projects are listed specifically for an upper-level course in combinatorics:
    - *Figurate Numbers and Sums of Numerical Powers: Fermat, Pascal, Bernoulli*
    - *Gabriel Lamé's Counting of Triangulations*
    - *Networks and Spanning Trees*, with sources by Cayley, Prüfer, and Borůvka

    Together these projects cover the content of a one-semester course.

3. *Geometry:* I have taught this upper-division introductory course partly from the sequence of annotated primary historical sources in the chapter Geometry: The Parallel Postulate in our book *Mathematical Expeditions: Chronicles by the Explorers* [10]. The website [11] provides sample sections from the chapter. The primary sources follow two millennia of the development of non-Euclidean geometry via excerpts from Euclid on his parallel postulate, Legendre's early nineteenth-century final attempts to prove the parallel postulate, Lobachevsky's almost simultaneous introduction of the brave new world of planar hyperbolic geometry, and Poincaré's disk model confirming its equal footing with Euclidean geometry. To the sources we have added extensive annotation, contextual, historical, and mathematical commentary as a guide and overview of the big story, and numerous mathematical exercises for students. We also included copious references to the literature for deeper understanding by both teachers and students. The primary sources from this chapter formed a major theme in the course. They were interwoven with other course content from the book *The Four Pillars of Geometry* by John Stillwell.

    Two particularly illuminating and successful uses of the primary sources were to challenge students to find the flaw in Legendre's proof of the parallel postulate, and to have students individually present to the class and argue for Lobachevsky's analysis and sequence of results leading to his non-Euclidean geometry. These presentations lasted over a number of class days, and resulted in tremendous student involvement in class discussions.

4. *Number Theory:* The majority of the content consists of guided student study of translated selections from the unpublished handwritten manuscripts of Sophie Germain (1776–1831), who developed the

first general result toward a proof of Fermat's Last Theorem. Recent discoveries in Germain's manuscripts and her correspondence with Gauss have shown that she had a grand plan for proving Fermat's Last Theorem in its entirety, and that she made very substantial progress on carrying it out [12].

I realized that students could learn almost all the topics in a first number theory course via guided study directly from Germain's manuscripts and correspondence, even though her manuscripts were not teaching documents, but rather the latest research, building on Gauss's introduction of the congruence view of number theory [19]. One particular benefit of this approach is that the course focuses largely on one grand goal, Fermat's Last Theorem, so the topics are all highly motivated. The course was essentially a detective story, learning number theory via my guidance in order to decipher Germain's manuscripts.

I stuck determinedly to a "just as needed" approach to learning the material. I never told students about number theoretic concepts or results needed to decipher Germain. Instead I helped them discover these ideas in her manuscripts, only as needed, to comprehend her writings. For instance, even Fermat's Little Theorem was a surprise discovered by students when Germain used it without mention. There is no better motivation than a "just as needed" approach.

The course ends with selections from a sequence of primary sources on quadratic reciprocity, in which Fermat studies primes of a certain quadratic form, Euler discovers general patterns in prime divisors of quadratic forms, Lagrange develops a theory of quadratic forms and divisors, Legendre asserts the quadratic reciprocity law, Gauss proves it, Eisenstein creates a geometric proof, and Gauss composes quadratic forms, foreshadowing the class group. This latter sequence of guided sources is from the chapter Patterns in Prime Numbers: The Quadratic Reciprocity Law in our book *Mathematical Masterpieces: Further Chronicles by the Explorers* [9]. If time is short, the focus can be primarily on Fermat, Euler, and Eisenstein. The website [11] provides sample sections.

While there was no textbook for the course, I did recommend that each student choose some number theory book as a bedside "security blanket" to read if they wanted something to supplement the struggle of deciphering Germain's manuscripts. I suggested that the book [14] provides an inquiry-based approach, while [8] provides a more traditional and more extensive presentation.

I am writing a book for the course based primarily on Germain's manuscripts. In the meantime, interested instructors may obtain these materials directly from me.

The response of students to a "just as needed" discovery pedagogy for learning number theory from Sophie Germain's manuscripts was absolutely phenomenal, making this course by far my most successful and rewarding in almost 40 years of teaching.

Assignments

Regular homework and related classroom work are the heart of each course. Assignments are largely mathematical in nature, based directly on the primary sources, since the course is first and foremost mathematics, set authentically in its history. Exercises often strengthen students' understanding of a primary source, and are sometimes open-ended. To give a sense for the nature of assignments, Appendix A provides sample exercises covering the four courses, with each exercise preceded by a little context and a small excerpt from the relevant primary source.

Lessons Learned

Here I have presented four standard courses taught largely or entirely from primary historical sources. Two were taught from a collection of sizeable primary source projects on diverse topics, one from a sequence of primary sources on a single topic intermingled with other material, and one was taught largely from a single author's manuscripts and letters. All approaches worked well, so there are many ways primary sources can successfully be put together to teach a course, or mixed with other material.

The particular courses chosen were largely serendipitous; I would happily aim to teach any course from primary sources. My experience is that this approach can revitalize the teaching of mathematics, by simultaneously making mathematics more authentically intellectually challenging, more genuinely meaningful, and more appealing to students. I would never relinquish my aim of offering my students the richness of studying primary sources.

While I now have the experience to enable teaching with primary sources to come naturally, the additional preparation involved in finding appropriate sources, and readying them for teaching, is large, especially for an inexperienced instructor. I have found, though, that I can teach quite easily with primary source materials prepared by other instructors, so I hope that it will become easier for anyone to teach this way as more and more such materials become available.

The accomplishments of students studying primary sources are always inspiring. And for many years we have asked our students in questionnaires what they perceive as the advantages and disadvantages. Their reactions are often insightful. The disadvantages mentioned are surprisingly few and rare, and usually balanced with a concomitant advantage. For instance, some students say that the sources can be hard to read, but that it is worth it. Others have said that the sources do not provide a modern view, but that contrasting both the older and modern views is highly beneficial. On the other hand, the advantages given by our students are many and frequent, and include the following:

- For me, being able to see how the thought processes were developed helps me understand how the actual application of those processes work[s]. Textbooks are like inventions without instruction manuals.
- The original sources can be debated to form new interpretations.
- As a student you get to see where the math we do today came from and engage in the kind of thinking that was necessary to create it.
- We learn directly from the source and attempt to learn concepts based off of the original proofs rather than interpretation of the original proof from someone else.
- It gives you the sense of how math was formed which prepares you for how to think up new, innovative mathematics for the future.

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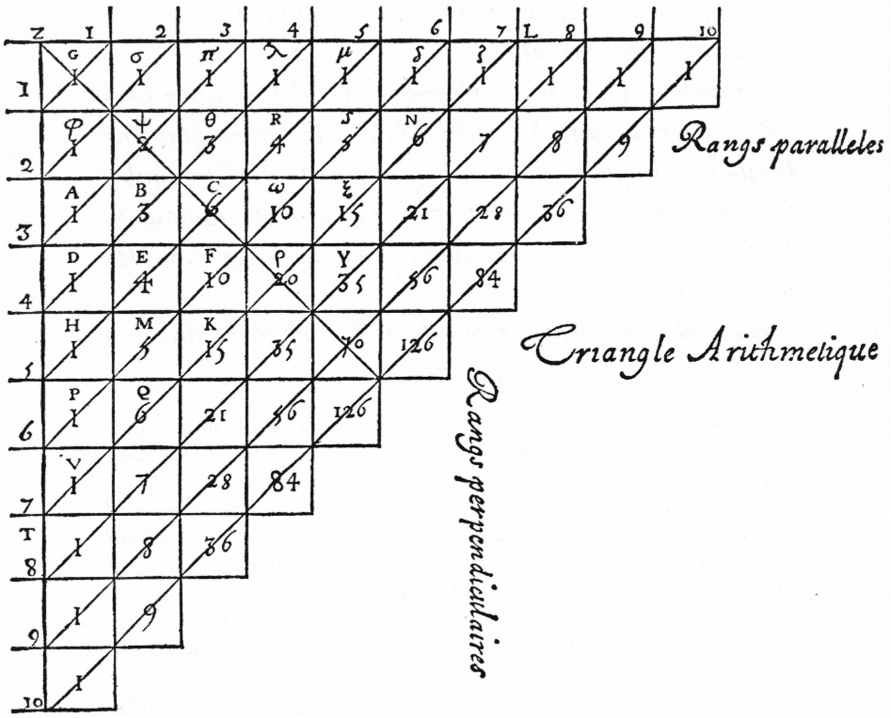
Appendix A
Sample primary source materials and exercises

To give the flavor, here I describe and provide a small primary source excerpt and related exercise from the materials for each course.

1. Discrete Mathematics

Pascal's Treatise on the Arithmetical Triangle [Pascal's Triangle] not only treats combinations and the binomial theorem. It is the first place in the literature where the principle of mathematical induction is fully explicated, justified as a proof technique, and systematically applied. Pascal's Twelfth Consequence is the most important and famous in the whole treatise. Having built up to this in previous Consequences, Pascal here presents a formula for the ratio of consecutive numbers in a base [row] of the triangle, then proven by induction in a generalizable example, after explaining why the induction method is sufficient. From this he will obtain an elegant and efficient formula for all the numbers in the triangle, essentially our modern factorial formula.



Z	1	2	3	4	5	6	7	L 8	9	10
1	G 1	σ 1	π 1	λ 1	μ 1	δ 1	ζ 1	1	1	1
2	φ 1	ψ 2	θ 3	R 4	S 5	N 6	7	8	9	
3	A 1	B 3	C 6	ω 10	ξ 15	21	28	36		
4	D 1	E 4	F 10	ρ 20	Y 35	56	84			
5	H 1	M 5	K 15	35	70	126				
6	P 1	Q 6	21	56	126					
7	V 1	7	28	84						
T 8	1	8	36							
9	1	9								
10	1									

"Twelfth Consequence

In every arithmetical triangle, of two contiguous cells in the same base the upper is to the lower as the number of cells from the upper to the top of the base is to the number of cells from the lower to the bottom of the base, inclusive.

Let any two contiguous cells of the same base, E, C, be taken. I say that

E	:	*C*	::	2	:	3
the lower		*the upper*		*because there are two cells from E to the bottom, namely E, H,*		*because there are three cells from C to the top, namely C, R, μ.*

Although this proposition has an infinity of cases, I shall demonstrate it very briefly by supposing two lemmas: The first, which is self-evident, that this proportion is found in the second base, for it is perfectly obvious that $\varphi : \sigma :: 1 : 1$;
The second, that if this proportion is found in any base, it will necessarily be found in the following base. Whence it is apparent that it is necessarily in all the bases. For it is in the second base by the first lemma; therefore by the second lemma it is in the third base, therefore in the fourth, and to infinity. It is only necessary therefore to demonstrate the second lemma as follows: If this proportion is found in any base, as, for example, in the fourth, Dλ, that is, if $D : B :: 1 : 3$, *and* $B : \theta :: 2 : 2$, *and* $\theta : \lambda :: 3 : 1$, *etc., I say the same proportion will be found in the following base, Hμ, and that, for example,* $E : C :: 2 : 3$. *For* $D : B :: 1 : 3$, *by hypothesis. Therefore*

$$\begin{array}{ccccccc} \underbrace{D+B} & : & B & :: & \underbrace{1+3} & : & 3 \\ E & : & B & :: & 4 & : & 3. \end{array}$$

Similarly $B : \theta :: 2 : 2$, *by hypothesis.*
Therefore

$$\begin{array}{ccccccc} \underbrace{B+\theta} & : & B & :: & \underbrace{2+2} & : & 2 \\ C & : & B & :: & 4 & : & 2 \end{array}$$

But $$B : E :: 3 : 4 .$$

Therefore, by compounding the ratios, $C : E :: 3 : 2$. Q.E.D.

The proof is the same for all other bases, since it requires only that the proportion be found in the preceding base, and that each cell be equal to the cell before it together with the cell above it, which is everywhere the case."

Exercise: Pascal's Twelfth Consequence: the key to our modern factorial formula

- Rewrite Pascal's Twelfth Consequence as a generalized modern formula, entirely in our $T_{i,j}$ terminology. Also verify its correctness in a couple of examples taken from his table in the initial definitions section.
- Adapt Pascal's proof by example of his Twelfth Consequence into modern generalized form to prove the formula you obtained above. Use the principle of mathematical induction to create your proof.

2. Combinatorics

Jakob Bernoulli developed the equivalent of Fermat's claims about the relationships between figurate numbers, used these to recursively develop polynomial formulas for sums of increasing powers, wrote a table for the first ten powers, and guessed the general pattern, thereby introducing the Bernoulli numbers to the world.

"*Sums of Powers*

$$\int n = \frac{1}{2}nn + \frac{1}{2}n.$$

$$\int nn = \frac{1}{3}n^3 + \frac{1}{2}nn + \frac{1}{6}n.$$

$$\int n^3 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}nn.$$

$$\int n^4 = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 * -\frac{1}{30}n.$$

$$\int n^5 = \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 * - \frac{1}{12}nn.$$

$$\int n^6 = \frac{1}{7}n^7 + \frac{1}{2}n^6 + \frac{1}{2}n^5 * - \frac{1}{6}n^3 * + \frac{1}{42}n.$$

$$\int n^7 = \frac{1}{8}n^8 + \frac{1}{2}n^7 + \frac{7}{12}n^6 * - \frac{7}{24}n^4 * + \frac{1}{12}nn.$$

$$\int n^8 = \frac{1}{9}n^9 + \frac{1}{2}n^8 + \frac{2}{3}n^7 * - \frac{7}{15}n^5 * + \frac{2}{9}n^3 * - \frac{1}{30}n.$$

$$\int n^9 = \frac{1}{10}n^{10} + \frac{1}{2}n^9 + \frac{3}{4}n^8 * - \frac{7}{10}n^6 * + \frac{1}{2}n^4 * - \frac{3}{20}nn.$$

$$\int n^{10} = \frac{1}{11}n^{11} + \frac{1}{2}n^{10} + \frac{5}{6}n^9 * - 1n^7 * + 1n^5 * - \frac{1}{2}n^3 * + \frac{5}{66}n.$$

Indeed, a pattern can be seen in the progressions herein, which can be continued by means of this rule: Suppose that c is the value of any power; then the sum of all n^c *or*

$$\begin{aligned}\int n^c = {} & \frac{1}{c+1}n^{c+1} + \frac{1}{2}n^c + \frac{c}{2}An^{c-1} + \frac{c\cdot c-1\cdot c-2}{2\cdot 3\cdot 4}Bn^{c-3} \\ & + \frac{c\cdot c-1\cdot c-2\cdot c-3\cdot c-4}{2\cdot 3\cdot 4\cdot 5\cdot 6}Cn^{c-5} \\ & + \frac{c\cdot c-1\cdot c-2\cdot c-3\cdot c-4\cdot c-5\cdot c-6}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8}Dn^{c-7} \ldots \text{ \& so on,}\end{aligned}$$

where the value of the power n continues to decrease by two until it reaches n or nn. The uppercase letters A, B, C, D, etc., in order, denote the coefficients of the final term of $\int n^2$, $\int n^4$, $\int n^6$, $\int n^8$, *etc., namely*

$$A = \frac{1}{6}, \quad B = -\frac{1}{30}, \quad C = \frac{1}{42}, \quad D = -\frac{1}{30}.$$

These coefficients are such that, when arranged with the other coefficients of the same order, they add up to unity: so, for D , which we said signified $-\frac{1}{30}$, we have

$$\frac{1}{9} + \frac{1}{2} + \frac{2}{3} - \frac{7}{15} + \frac{2}{9}(+D) - \frac{1}{30} = 1.$$

By means of these formulas, I discovered in under a quarter hour's work that the tenth (or quadrato-sursolid) powers of the first thousand numbers from unity, when collected into a sum, yield

91409924241424243424241924242500.

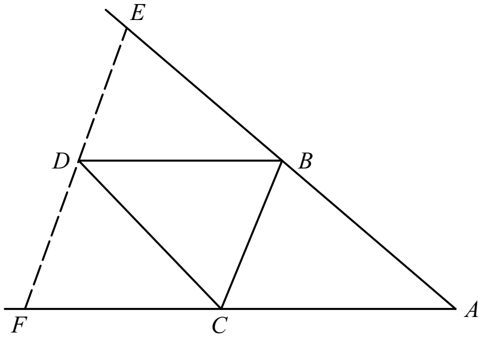
Clearly this renders obsolete the work of Ismael Bulliald, who wrote so as to thicken the volumes of his Arithmeticae Infinitorum with demonstrations involving immense labor, unexcelled by anyone else, of the sums of up to the first six powers (which is only a part of what we have superseded in a single page.)"

Exercise: Guess, as did Bernoulli, the complete pattern of coefficients for sums of powers formulas just from the examples in Bernoulli's table. Clearly the pattern is to be sought down each column of Bernoulli's table. The key is to multiply each column of numbers by a common denominator, and then compare with the arithmetical triangle (computing the sequence of successive differences in a column, and the successive differences in that sequence, etc., may also help). Can you also express the general rule for calculating the special numbers $A, B, C, D, \ldots$, which Bernoulli introduces? Hint: What happens when $n=1$?

3. Geometry

Legendre was the last serious mathematician to attempt to prove Euclid's parallel postulate, while at essentially the same time Gauss, Lobachevsky, and Bolyai were developing the non-Euclidean hyperbolic geometry that negated it. From Legendre's many published attempts, students study an entire proof that the angle sum of a triangle must be two right angles (equivalent to the parallel postulate), and are challenged to find an unsupported assumption. Legendre's text contains the passage

"*Let A be the smallest of the angles in triangle ABC, on the opposite side BC make the angle BCD=ABC, and the angle CBD=ACB; the triangles BCD, ABC will be equal, by having an equal side BC adjacent to two corresponding equal angles [pr. 7]. Through the point D draw any straight line EF which meets the two extended sides of angle A in E and F.*"



Students are first challenged, from amongst the entire proof, to ferret out that a subtlety in this passage is the key:

Exercise: Before reading our commentary after Legendre's results, find and discuss the flaw in his proof of the parallel postulate.

4. Number Theory

Germain's plan for proving Fermat's Last Theorem, and an indication of the challenges in carrying it out, are revealed at the very beginning of one of her manuscripts. But for students beginning from scratch, the text has numerous other challenges as well, such as understanding what is hidden behind the use of the word "residue", even though the word modulus, or divisor, is never mentioned.

"*The impossibility of this equation would follow without doubt if one could demonstrate the following theorem:*

For every value of p other than $p=2$, there is always an infinity of prime numbers of the form $Np+1$ for which one cannot find two p-th power residues whose difference is unity.

After having established that in fact it would result from this theorem that in the equation $x^p+y^p=z^p$, the numbers x, y, and z, could not be other than infinite, I continue on to the examination of several special propositions which, for lack of an absolute demonstration, serve to establish at least the necessity that the same numbers x, y, and z would be extremely large numbers.

I remark first that, excepting the case where N is a multiple of 3, if in the form $Np+1$ one fixes for N a constant value and if one lets the value of p vary, one will find an infinite number of prime numbers pertaining to this form, for which there will not be two p-th power residues which follow one another immediately in the order of natural numbers; and that to the contrary there can never be other than a finite number of prime numbers of the same form which enjoy the opposite property. Now since nothing prevents the successive assignment of an infinity of values to N, one can conclude from what precedes that there must exist an infinity of values of p for which the equation $x^p+y^p=z^p$ is impossible. However such a result is too vague to apply to the demonstration of the impossibility of the same equation in the case of a determined value of p. In fact if one denotes by α this *value, one could always fear that the numbers* $N\alpha + 1$, $N'\alpha + 1$, *&c. find themselves among the prime numbers of the more general forms* $Np + 1$, $N'p + 1$ *&c. for which there can exist two p-th power residues whose difference is unity; and despite the little probability that this objection is justified by examination, I have not been able to succeed at destroying it."*

Exercise:

a) Discuss very clearly the possible interpretations of Germain's claim "and that to the contrary there can never be other than a finite number of prime numbers of the same form which enjoy the opposite property" about a finiteness property, and what she concludes therefrom, namely "Now since nothing prevents the successive assignment of an infinity of values to N, one can conclude from what precedes that there must exist an infinity of values of p for which the equation $x^p + y^p = z^p$ is impossible."

b) Devise a table in terms of N and p, showing how under one obvious interpretation of her finiteness claim, the values of N and p could correspond to success or failure of Condition NC in such a way that her conclusion isn't correct, i.e., she cannot use the italicized theorem to prove Fermat's Last Theorem for infinitely many p.

c) Then modify the finiteness hypothesis by strengthening it in such a way that you can state and prove a theorem to justify Germain's conclusion. In other words, state a hypothesis making explicit what she claims in her preceding sentences, and show how it would prove Fermat's Last Theorem for infinitely many values of p, by showing how it would lead to her italicized theorem at the beginning of Manuscript A.

Appendix B

Handout for students on homework assignment guidelines

Keep this sheet

Guidelines for all regular homework assignments

Please put your name (and any nickname your prefer) on the first page, *staple* your pages together, and *do not* fold them. Use both sides of the paper if you wish, to save paper. Please *do not* write in light pencil. Please write clearly. Thank you.

Parts A, B, C of each homework are equally important.

Part A: Advance preparation. Hand this in at the beginning of class, one class period before our class discussion and work on new reading. Reading responses (a), questions (b), reflection (c), and time spent (d):

You do *not* need a new page for each part (a),(b),(c),(d).

a) Read assigned material. Reread as needed for complete understanding. Then write clear *responses* to assigned questions about the reading.

b) Write down some of your own explicit *questions* about your reading, ready to bring up in class. This may involve new or old concepts that are confusing to you, and connections to other ideas. You should also consider writing down what was well explained and interesting, what was confusing, and what you had to reread but eventually understood.

c) Reflection: Write two or three sentences *reflecting* on the process of your work; this should only take a few minutes. Write about how things went with any assignment or reading done for class, and other course work. This should reflect both your ongoing personal feelings about the course as a whole and your interaction with the material at hand.

d) Write how much *time* you worked on part A.

Part B: Warmup exercise preparation to present in class. This is due during class when we begin to discuss new material. Work individually, and then with others in your group outside class time, on a few assigned easy warmup exercises on the new material we will discuss, based on your advance reading in Part A. Write up the solutions to these individually, to hand in in class. I will ask individuals and groups to present some of these to the class, to get us started discussing new material. Be sure to hand these in before leaving class. Also always write how much time you worked on part B, and with whom.

Part C: Main exercises. These will be assigned after class discussion and work on new material. They will normally be due next period. Work individually and with others in your group on these. Also come to see me during office hours or at other appointment times about these. I am happy to help you. Then go home and write up your final solutions completely by yourself, without comparing with other people. The paper you hand in should be entirely your own writing, not the same as anyone else's.

Drawing it out!

The History of Mathematics through its Diagrams

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Course Overview

From the beginning, diagrams have been pivotal in mathematics. Ancient texts included diagrams as essential elements of proofs and modern mathematics still uses diagrams to support formal reasoning processes. From lettered Euclidean figures to category theory, diagrams are an ideal way to exemplify and explore themes in the history of mathematics, while shedding extra light on the mathematical cultures in which they were produced.

Our course is intended for final year (300-level) mathematics students, but also attracts intending teacher trainees and students of other numerate disciplines. The prerequisites for this course are three mathematics courses at 200 level; however due to the interdisciplinary nature of this course students with a comparable technical background in other subjects, such as computer science, physics, or even philosophy have been granted entry into this course.

We use diagrams as points of entry for discussing key episodes in the history of mathematics. Diagrams provide student-friendly access to a wealth of topics, and are a fun and interesting way to stimulate class discussions. Diagrams are simply addictive and fun to look at!

How do diagrams help us towards our main goals for the course? Our goals are for students to

1. **See mathematics as an activity carried out by people.** How did they do, think, discover, or imagine the mathematics that we see in their work? A good way to address these questions is to expose the students to primary source material. However, most historical sources are in foreign languages, use unfamiliar symbolism or technical terms, and are dense or difficult to read. On the other hand, diagrams are much more accessible, and their panoptic qualities can be exploited to highlight the ways in which mathematics in contrasting cultures was carried out.
2. **Form their own independent historical reflections.** Analyzing a text can be quite daunting for students. Diagrams on the other hand provide an opportunity for students to build their confidence when exploring historical documents as there are many levels on which a student may comment. Basic features, such as labelling, execution, visual details, are a natural point of entry. More elaborate insights follow these.
3. **Appreciate the variety of practice in the mathematical sciences throughout history.** Students can compare diagrams from different traditions, sometimes of the same mathematical proposition. They can also consider the evolution of the diagram throughout time.

4. **Think more deeply about philosophical issues in mathematics.** Diagrams raise a variety of interesting questions. Can a diagram prove a mathematical statement? Are diagrams just pedagogical aids? What is the role of intuition in mathematics? How do diagrams produce mathematical insight?

Course Design

This is a one-semester course with three classes per week: two 50-minute lectures, followed a day or two later by a 50-minute tutorial involving discussion and group work related to the week's topic. The topics chosen vary from year to year, but we proceed largely chronologically and dedicate a week or two (occasionally even three) to each distinct culture of inquiry. Typical examples include the Ancient Near East, Ancient Greece, India, the Islamic empire, Japan, Early Modern Europe, and nineteenth-century Europe. We cover fairly traditional topics that can be found in standard history of mathematics texts such as [6]. The key difference is that we look at these topics through their associated diagrams.

Classes are structured around the diagram. The mathematics from each culture of inquiry is exemplified by one or several key diagrams from that culture and salient themes that relate to the practices, traditions, or dispositions are launched via the features of the diagram. We list below a typical syllabus, indicating the relevant culture, some topics accessible via diagrams, and some useful references:

a) **Ancient Near East:** An Old Babylonian tablet depicting a square, its diagonals, and an approximation for √2 (see the Appendix B.1 for a worksheet). The Babylonian use of geometric language when solving quadratic problems. See [1, Section 1.6] and [4, Section 1.2].

b) **Ancient Greece:** Plato's *Meno*: Doubling a square. See [2, Section 2.E1] and [4, Section 2.2]; see the Appendix B.4 for a worksheet.

c) **Various early cultures:** Diagrams used to justify the Pythagorean Theorem. See [6, 3—9] and [8].

d) **Ancient India:** From the *Śulbasūtras*, geometric constructions using string. See [5, Section 4.II] and [7, Section 2.2]; see the Appendix B.3 for a worksheet.

e) **Ancient Greece:** Diagrams in Euclid's *Elements* and the instructions to construct them; see Appendix B.2 for a worksheet.

f) **Ancient Greece:** Euclid's construction of the regular pentagon. See [1, Section 2.4] and [4, Section 3.4].

g) **Ancient Greece:** Archimedes' Method and the area of a parabolic segment. See [2, Sections 4.A3 and 4.A9] and [4, Section 4.3].

h) **Medieval Islamic:** Al-Samaw'al, diagrams, and the early history of mathematical induction. This topic was based on our own research, but see [4, Section 9.3] and [5, Section 5.IV].

i) **Early Modern Europe:** Coordinate geometry and tangents: Descartes and Fermat. See [2, Sections 11.A3 and 11.C1] and [4, Section 14.2].

j) **Early Modern Europe:** Newton's discovery of the binomial series by interpolating patterns in already known expansions. See [2, Section 12.C] and [4, Section 16.1].

k) **Early Modern Europe:** Leibniz's differential triangle. See [2, Section 13.A3] and [4, Section 16.2].

l) **Edo period Japan:** Geometric problems on Japanese sangaku tablets. Again this topic was based on our own research, but see [3].

m) **Nineteenth Century Europe:** Suspicion of diagrams. Dedekind rejects geometric intuition in favour of arithmetical foundations. See [2, Section 18.C1] and [4, Section 22.2].

n) **Nineteenth Century Europe:** Infinity and Cantor's diagonal argument. See [2, Section 18.C4] and [4, Section 22.2].

A typical week proceeds as follows:

Lecture One: Overview. This gives a general introduction to the week's topic, discussing cultural context, mathematical highlights, and key personalities.

Lecture Two: The Diagrams. This focuses on one or two relevant diagrams. Typically the discussion involves deconstructing the diagrams, understanding some mathematical ideas, and analyzing the connections between the diagram and the mathematics.

Tutorial. Before each tutorial, students are given (or asked to find in the library or on the internet) a diagram related to the week's culture of inquiry. In the tutorial, the students are divided into small groups where they analyze the diagram, or solve a problem based on it, imitating the approach taken in the lectures. The small groups then report back to the whole class.

The Case Study on the Ancient Near East (see Appendix B.1) shows a typical tutorial. Students are asked to analyze a diagram and understand its mathematical content (in this case, reading sexagesimal numbers, and determining how the numbers are related). Discussing the implications of the presence, or absence, of the diagram leads nicely to the debate about the possible geometric interpretations of the famous Plimpton 322 tablet [4, Section 1.2.3]. How does the diagram reveal the sorts of mathematical objects (pure numbers? geometric configurations? quantities arising in surveying applications?) that these early mathematicians were interested in? How might the diagram help the historian determine what the numbers represented? How does the diagram testify to the sorts of mathematical practices that were carried out in this early culture of inquiry? In this way students learn to read primary historical sources and engage with their mathematical content, as well as reflecting more broadly on cultural contexts, both ancient and modern.

Resources

The course was originally based on an earlier edition of Katz's well known text [4]. As our emphasis has moved to primary sources, we also adopted Fauvel and Gray's source book [2]. A full reading list for the above Course Design is included in the References section.

Of course, today's students are well used to finding resources online. The MacTutor Website [9] is a reliable, up-to-date, resource for biographical and thematic information on historical topics in mathematics, and is particularly useful for its references. *Convergence* [10] offers a user-friendly, ever-expanding database of images and texts and, in particular, their "Mathematical Treasures" repository of images.

To accommodate the focus on diagrams we can cast our net even wider, taking advantage of the ever-growing list of original manuscripts that are now freely available online. For example, Euclid's *Elements* can be studied at a variety of levels. Thus David Joyce's website [11] offers an easy-to-use repository of all of the propositions, along with the ability to manipulate the diagrams by dragging points using a specially designed geometry applet. On the other hand, the Bodleian library site [12] houses images of a manuscript of Euclid's *Elements*. While this is mainly directed at specialists, it gives students a chance to take a direct look at the real images and how scribes incorporated them into the text. Finally Ken Saito's site [13] for diagrams in Greek and Arabic Mathematics allows students to compare versions of mathematical diagrams in different manuscript copies, learning that diagrams can evolve as a result of successive copyists making their own decisions about orientation, size or layout.

For course designers who wish to find out more about recent scholarship about the role of diagrams in mathematics and its history, we have also included in the References a selection of recent books and articles [14 to 22].

Assignments

Assessment for the course is based on fortnightly homework exercises, weekly tutorial participation, an assignment due two-thirds of the way into the course, and a take-home test at the end of the course.

We include examples of our weekly tutorial activity, which earns a small percentage (about 2% per week) for participation, and the major assignment, which is worth 30% of the grade for the course.

The tutorial discussion usually focuses on the week's particular culture of inquiry. However the very first tutorial takes a broader view; see the "Tutorial Discussion" below for the instructions we give to the students. Diagrams play many different roles in the history of mathematics. They are, of course, an integral part of the statement and proof of a Euclidean Proposition. But they may also help the reader see Archimedes' imaginary lever as he finds the area of a parabolic segment. They may help elucidate a novel idea, like Leibniz's differential triangle. They may capture a key structural element in an otherwise abstract proof, as in Cantor's diagonal argument. The idea of the first tutorial is for the students as a group to discover as many of these different roles as possible.

Tutorial discussion

Select a diagram—it can be from any field or any period. Search the internet, look in books, or take a snapshot during one of your lectures. Bring a print-out to class to share in your small group.

Be prepared to discuss it with your small group. Consider

- What is the context of the diagram?
- How does it relate to the text, or proof, or example, it accompanies?
- What are its various features?
- How was it produced?
- Could it stand alone or does it need the text to explain what is what?
- How is it helpful? What are its limitations?

Examples of other tutorial activities are given in Appendix B.

The major piece of assessment during the course, which allows students to focus on those areas that capture their interests and imagination, is an assignment in which students have to select ten diagrams and contextualize them. These can cover either the history of mathematics or a specific theme or strand within it. The assignment is given to students about a third of the way into the course, once they have gained familiarity and confidence with mathematical diagrams and is due about two-thirds of the way into the course. The "Major Assignment" brief below shows the outline that is given to students for this assignment.

Major assignment

Briefing: Your publisher has asked you to write a popular book on the History of Mathematics. As its "point of difference" this book will highlight episodes from the History of Mathematics through key images, diagrams, and pictures.

Your Task: You need to compile a proposal containing ten mathematical diagrams.

- Your images must be carefully and mindfully selected. They must span a significant time period, or encompass a theme within the History of Mathematics, for instance.
- Each diagram must have a provisional chapter heading, an explanation giving where it is from, the broader context and the mathematical significance and the impact the diagram had, where appropriate.

- You must include an introductory section of no more than 1000 words explaining the broader significance of your selections.
- Make sure you include a section on the references you used.
- Think of a clever title for your "book".

Your resulting proposal will be about 20–30 pages in length (2–3 for the introduction, 1 for each diagram, and 1–2 pages explanation for each diagram).

We hope that, in choosing their own diagrams, students will be more likely to use the critical eye for diagrams they have developed during the course, dissecting the diagrams, contextualizing them, and reflecting more deeply on their role and significance.

We tell the students that when we are marking their assignments, we will be focusing particularly on their choice of diagrams, the coherence of their overall idea, how their introduction supports the overall theme they have chosen, and the identification of key themes relating to diagrams that we have discussed in class. As part of a scholarly approach, students also need to demonstrate familiarity with primary and secondary sources, and provide a good bibliography. Students also need to address general communication issues, such as writing style, layout and presentation, visual and textual communication, and to convey their ideas to a general audience. Finally, in an era where the answer to everything is on the internet, we also look for novelty and uniqueness. The assignment is an opportunity for the students to show us everything they have learned in the course, as well as a chance to use their creativity to maximum effect!

Lessons Learned

We find this approach very rewarding but some simple lessons come to mind from our experiences.

First, although we thought that diagrams would provide the students with easier access to primary sources, we were surprised at how little students saw in the diagrams. Students do need to be told what to look for, what to notice! For example, the relationship between some diagrams and their accompanying text is not obvious to students and needs to be teased out over several sessions. Similarly, students need to experience that you often learn more from drawing a diagram than from merely contemplating an already completed diagram.

We have also learned that we ourselves need to be precise when talking to students. For us, designing the course, it was obvious what a diagram was, but because of sloppy use of language (as in the phrase "through key images, diagrams and pictures" in the assignment brief, for example) some students thought that a picture of Newton could be included as one of their ten diagrams! Some students chose interesting examples for their diagrams, such as tables and arrays of numbers. This was fine, but we needed students to justify why they considered a table of numbers a diagram.

Furthermore, in the assignments many students fail to focus fully on the diagrams, and instead give us background information and detailed mathematical explanations. They find it difficult to really engage with and explore the features of the diagram. Emphasis must be placed on analyzing diagrams regularly and, indeed, taking a more structured approach to diagrams. Appendix D gives some questions that we used (or in some cases, should have used) to get students thinking more deeply about diagrams.

We include several pages from the major assignment to show some of the selections the students came up with (see Appendix C) and some of the feedback we gave them.

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Reading list (Course design)

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2. Fauvel, J. and J. Gray, *The History of Mathematics: A Reader*, Palgrave Macmillan, Hampshire, UK, 1987.
3. Hidetoshi, F. and T. Rothman, *Sacred Mathematics: Japanese Temple Geometry*, Princeton University Press, Princeton, NJ, 2008.
4. Katz, V. J., *A History of Mathematics: An Introduction, 3rd ed.*, Boston, MA, Addison-Wesley, 2009.
5. ——— (ed.). *The Mathematics of Egypt, Mesopotamia, China, India and Islam: A Sourcebook*, Princeton University Press, Princeton, NJ, 2007.
6. Nelsen, R. B., *Proofs Without Words: Exercises in Visual Thinking*, Mathematical Association of America, Washington, DC, 1993.
7. Plofker, K., *Mathematics in India*, Princeton University Press, Princeton, NJ, 2009.
8. Pythagorean theorem and its many proofs, www.cut-the-knot.org/pythagoras/ (accessed 5th Nov. 2014).

Useful Websites:

9. MacTutor Website: www-history.mcs.st-and.ac.uk/
10. *Convergence*: www.maa.org/publications/periodicals/convergence
11. Euclid's *Elements* at David Joyce's website aleph0.clarku.edu/~djoyce/java/elements/elements.html
12. Bodleian library for Euclid: treasures.bodleian.ox.ac.uk/The-Elements-of-Euclid
13. Ken Saito's site for diagrams in Greek and Arabic Mathematics: www.greekmath.org/diagrams/diagrams_index.html

Further readings about diagrams:

14. Brown. J. R., *Philosophy of Mathematics: A Contemporary Introduction to the World of Proofs and Pictures*, Routledge, New York, 2008.
15. Catton, P. and C. Montelle, "To diagram, to demonstrate: to do, to see, and to judge in Greek geometry," *Philosophia Mathematica*, 20(1) (2012) 25–57.
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20. Mumma, J. and M. Panza. "Diagrams in mathematics: history and philosophy," *Synthese* 186 (2012) 1–5.
21. Netz, R. "Greek mathematical diagrams: their use and their meaning," *For the Learning of Mathematics*, 18 (3) (1998) 33–39.
22. Saito, K. and N. Sidoli. "Diagrams and arguments in ancient Greek mathematics: Lessons drawn from comparisons of the manuscript diagrams with those in modern critical editions," in *The History of Mathematical Proof in Ancient Traditions*, K. Chemla, ed., Cambridge University Press, Cambridge, 2012, pp. 135–162.

Appendix A
Course Outline

Week	Content	Assessment
1	Introduction Ancient Near East: square root of 2, diagrams in problem texts, cut-n-paste algebra	
2	Ancient Greece: the three great geometrical problems, Pythagoras	Homework exercise due
3	Ancient Greece: Euclid, Plato	
4	Ancient India: Sulbasutras, Trigonometry, Series	Homework exercise due
5	Medieval Islam: Al-Samaw'al, Gnomons, Astrolabes, Spherical trigonometry	
6	China and Edo-period Japan: Go-gu theorem, Sangaku	Major assignment released Homework exercise due
7	Early Modern Europe: Coordinate geometry and Tangents	
8	Early Modern Europe: Newton and the binomial series	Homework exercise due
9	Early Modern Europe: Leibniz's differential triangle	
10	Nineteenth Century Europe: Dedekind and the suspicion of diagrams	Major assignment due date
11	Nineteenth Century Europe: Infinity and Cantor's diagonal argument	
12	Philosophy of Mathematics: a revivial of diagrammatic reasoning	Take home exam released (3–5 days to complete)

Appendix B
Case studies

Case study 1: A diagram from the ancient near east

Theme: What can a diagram tell us about the mathematical interests of ancient near eastern mathematicians (ca. mid-second millennium BCE)?

YBC 7289 . Image reproduced with the kind permission of Prof. William A. Casselman.

Your task:

1. Use your sexagesimal arithmetic skills to confirm that the numbers on this tablet are 30 and 1,24,51,10 and 42, 25, 35.

Reflective questions:

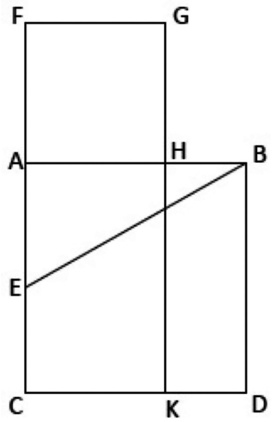
2. How are these numbers related? What are their absolute values?
3. Describe the diagram on the tablet. How carefully has it been drawn?
4. How does the diagram support the mathematical relationship? Suppose these numbers had appeared on a tablet without any diagram. How would that affect our ideas of the types of mathematics these ancient mathematicians were interested in?

Further Reading:
www-history.mcs.st-and.ac.uk/HistTopics/Babylonian_Pythagoras.html

Case study 2: Diagrams in ancient Greek geometry

Theme: Text and diagram are interdependent in Euclid's *Elements* (ca. 300 BCE).

Your task: Read Euclid's Proposition II, 11: "To cut a given straight line so that the rectangle contained by the whole and one of the segments equals the square on the remaining segment".



1. Highlight the words in Euclid's proof that refer to drawing or the diagram. How does Euclid refer to a rectangle? To a square? Are these references always the same?
2. Using a compass and a straightedge, follow Euclid's instructions for drawing the diagram. Label all the points as he instructs and leave in your traces of working.
3. Now look at the actual diagram in the text. Check each lettered point and line, and each resulting shape, and determine the role of each in the proof.

Reflective questions:

1. Compare your drawing with the diagram given in the text. What are the differences?
2. What can you conclude about the diagram that appears with the text?

Further investigation:

1. Compare the above diagram with the one using color in Oliver Byrne's edition which you can find at www.math.ubc.ca/~cass/Euclid/byrne.html. How effective is Byrne's diagram?
2. Compare the diagram with the one found in a Latin manuscript: www.sciencephoto.com/media/527257/view. What can you conclude about the diagram as it got copied by various scribes?

Case study 3: An Indian diagram from The Śulbasūtras (ca. 800 BCE)

The *Śulbasūtras* have often been seen as the first geometrical texts in India. They outline ways to construct squares, rectangles, triangles, and other elementary geometrical operations with a rope on the ground.

We will be using string (on paper) and following the instructions to see if we can draw our very own geometrical objects.

Your task: Draw the square using the instructions given in the following account:

> Having desired (to construct) a square, one is to take a cord of length equal to the (side of the) given square, make ties at both ends and mark it at its middle. The (east-west) line (equal to the cord) is drawn and a pole is fixed at its middle. The two ties (of the cord) are fixed in it (pole) and a circle is drawn with the mark (in the middle of the cord). Two poles are fixed at both ends of the diameter (east-west line). With one tie fastened to the eastern (pole), a circle is drawn with the other. A similar (circle) about the western (pole). The second diameter is obtained from the points of intersection of these two (circles); two poles are fixed at the two ends of the diameter (thus obtained). With two ties fastened to the eastern (pole) a circle is drawn with the mark. The same (is to be done) with respect to the southern, the western and the northern (pole). The end points of intersection of these (four circles) produce the (required) square. [As translated by S. N. Sen and A. K. Bag in *The Śulbasūtras*, Delhi: Indian National Science Academy, 1983, p. 77]

You will need a piece of string and some paper and a writing device of some sort.

Reflective questions:

1. What are the advantages of using string? What can you do with string that you can't do with a straightedge and compass? What can't you do?
2. How do the texts describe the various drawing procedures? What sorts of techniques do they use to explain how to draw the diagram?

Case study 4: Plato's Meno

Many Greek philosophers had a high opinion of mathematics, and used it as a paradigm for how we gain true knowledge of the world around us and make insights. Plato in particular held mathematics in high esteem. He outlined his thoughts on mathematics in one of his dialogues, the *Meno*, in which a slave boy is brought to "prove" a mathematical fact with supposedly no prior knowledge at all.

Your task: Read through the following portion from Plato's philosophical dialogue, *Meno* 81e–86c. (See [2, pp. 61—67] or online at classics.mit.edu/Plato/meno.html.)

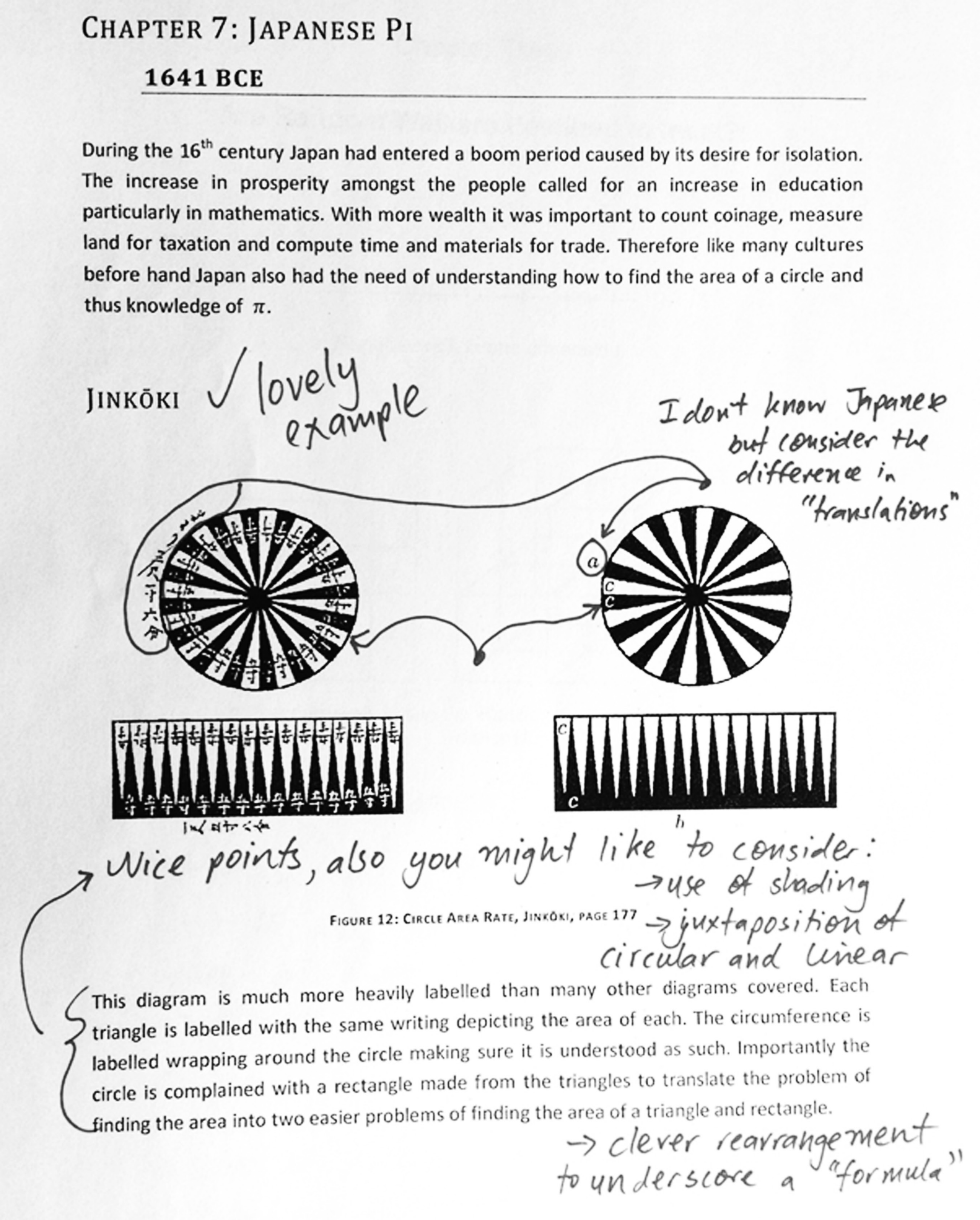
Assign someone to be Socrates, someone to be Meno, someone to be the slave boy, and someone to draw the diagram as it is referred to on the board so everyone can see it.

Reflective questions:

1. What mathematical question does Socrates pose to the slave boy?
2. Describe the answers the slave boy gives
3. Is Socrates teaching the slave boy at any time?
4. What is the significance of the diagram in this "experiment"?
5. What point(s) is Plato trying to make about mathematical knowledge?

Appendix C

Examples of student work and feedback



CHAPTER 7: JAPANESE PI

1641 BCE

During the 16th century Japan had entered a boom period caused by its desire for isolation. The increase in prosperity amongst the people called for an increase in education particularly in mathematics. With more wealth it was important to count coinage, measure land for taxation and compute time and materials for trade. Therefore like many cultures before hand Japan also had the need of understanding how to find the area of a circle and thus knowledge of π.

JINKŌKI ✓ lovely example

I don't know Japanese but consider the difference in "translations"

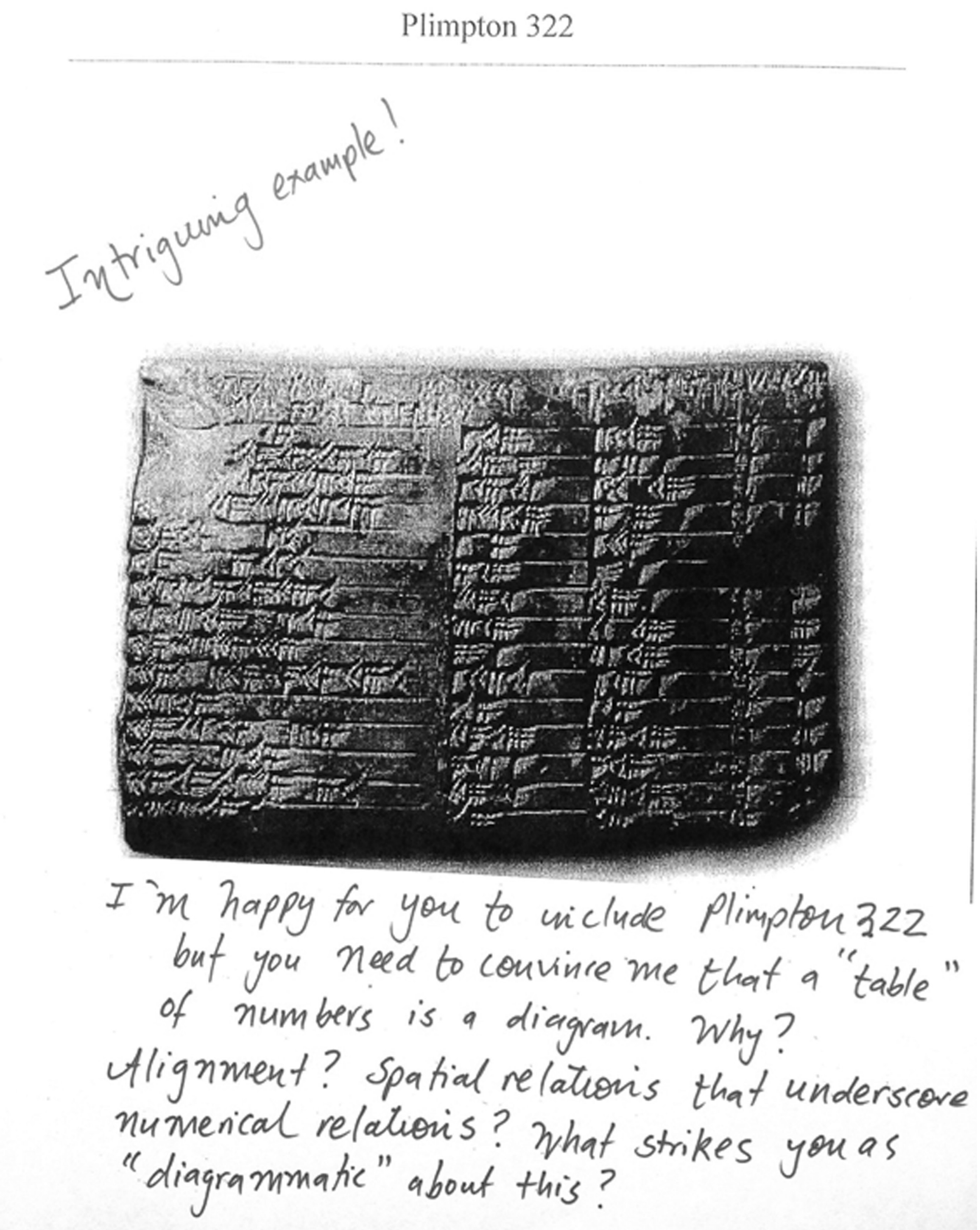
Nice points, also you might like to consider:
→ use of shading
→ juxtaposition of circular and linear

FIGURE 12: CIRCLE AREA RATE, JINKŌKI, PAGE 177

This diagram is much more heavily labelled than many other diagrams covered. Each triangle is labelled with the same writing depicting the area of each. The circumference is labelled wrapping around the circle making sure it is understood as such. Importantly the circle is complained with a rectangle made from the triangles to translate the problem of finding the area into two easier problems of finding the area of a triangle and rectangle.

→ clever rearrangement to underscore a "formula"

Figure 1. A (partially graded) sample from an assignment showing a student's choice of diagrams from Japanese sources.

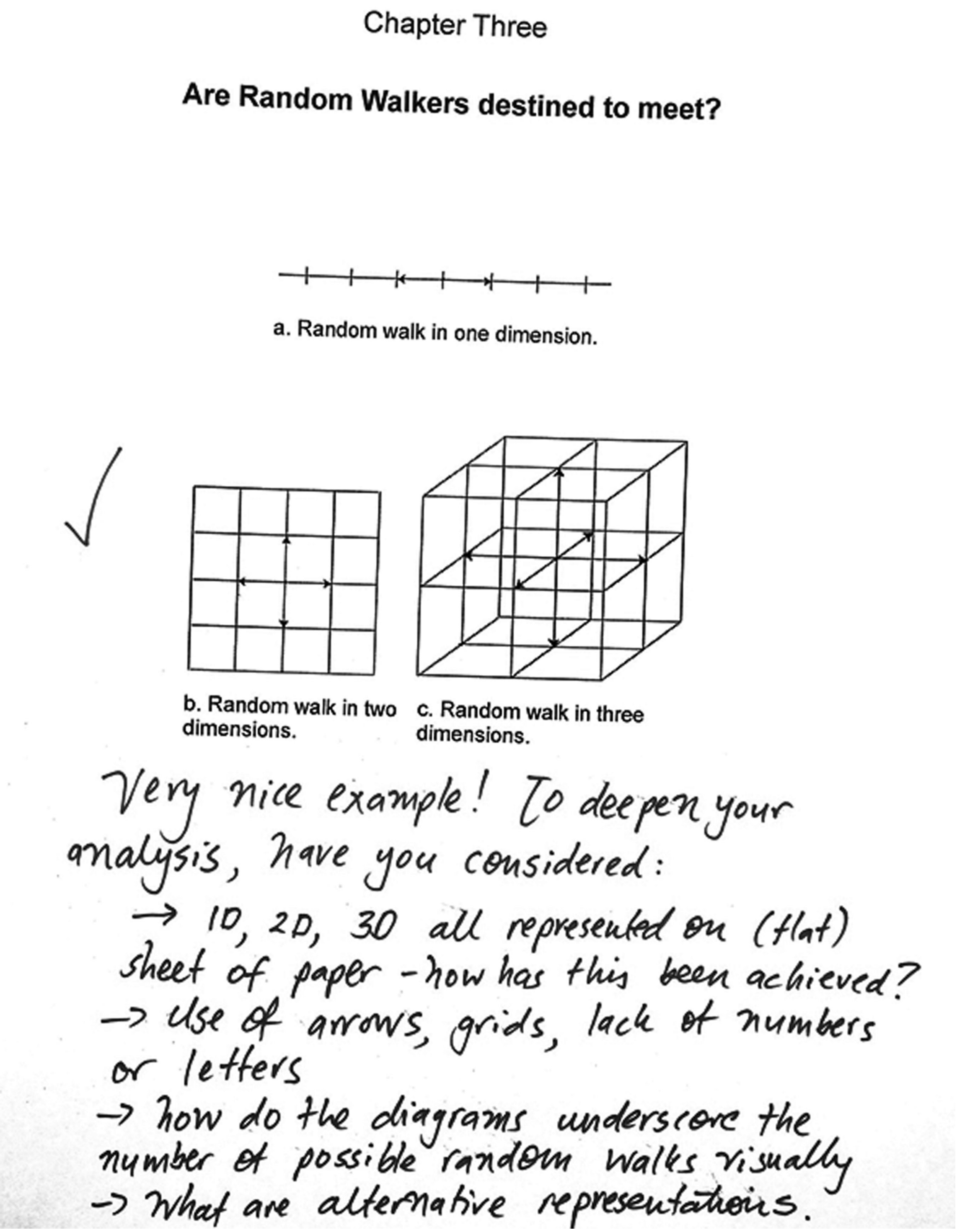


Plimpton 322

Intriguing example!

I'm happy for you to include Plimpton 322 but you need to convince me that a "table" of numbers is a diagram. Why? Alignment? Spatial relations that underscore numerical relations? What strikes you as "diagrammatic" about this?

Figure 2. A (partially graded) excerpt from a major assignment. Note the student has not justified why they believe this table of numbers to be a diagram.



Chapter Three

Are Random Walkers destined to meet?

a. Random walk in one dimension.

b. Random walk in two dimensions.

c. Random walk in three dimensions.

Very nice example! To deepen your analysis, have you considered:

→ 1D, 2D, 3D all represented on (flat) sheet of paper - how has this been achieved?

→ use of arrows, grids, lack of numbers or letters

→ how do the diagrams underscore the number of possible random walks visually

→ what are alternative representations.

Figure 3. An example from a student's major assignment. This is a clever modern example, but the diagrammatic observations were not fully exploited by the student.

Appendix D

Discussion questions

These were some of the general discussion questions that arose during the course. We have collected them and intend to use them in a more structured discussion section in subsequent offerings of this course.

Diagrams have many uses: introducing ideas, clarifying a concept, giving an example, proving a result, showing that something exists (like an intersection). Keller notes (2005; 291) a diagram can be a tool to specify a definition, a summary of a process, an object in which a procedure is carried out, or even a proof. Find examples of such uses.

What are some limitations of diagrams? For example, perhaps they are good for representing spatial qualities, but no so good for non-spatial ones. Can a diagram be general, or is it always a specific example? What role, if any, is there for figures in more than two dimensions? Can a diagram deal with infinite objects? Can you use a diagram to show a contradiction (*reductio ad absurdum*)?

Maybe diagrams are more useful for discovering or clarifying a mathematical relation, while text and symbolism are better when it comes to proof. It is sometimes said that diagrams can provide "immediate insight". What do you think this might mean, and how might they do this?

Diagrams and text represent different elements of mathematical reasoning. Are they complementary or independent? Can you think of a mathematical concept that is easy to capture in a diagram but difficult using symbols, or vice versa? Do diagrams give us a fuller picture of mathematical practice, above and beyond the written text?

Diagrams may mean different things to different audiences. How might the analysis of a diagram differ for a mathematician and a historian? Are there different roles or expectations for technical diagrams in scientific texts as opposed to "pure" mathematical ones?

Do geometric diagrams need to be metrically correct? What roles do proportion and orientation play in diagrams? Is there a difference between diagrams on the printed page and their mental representations? Does this have implications for our perception of mathematical certainty or knowledge?

What counts as a diagram? Is a table a diagram? Is a matrix a diagram?

What tools are used to draw diagrams and how has this changed throughout history? What technical terminology (if any) is invoked with mathematical drawings?

A Combined Number Theory and History of Mathematics Course for Mathematics Majors and Minors

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Course Overview

Number Theory/History of Mathematics, a sophomore-level course for mathematics majors and minors at the University of Redlands, originated over 25 years ago as an effort to ensure that our mathematics majors who plan to become secondary teachers get both the mathematics history and the number theory they need. The course has always been a popular elective for sophomore through senior mathematics majors and minors, not just those who plan to become teachers. My version of the course is an integrated number theory and history of mathematics course that uses both a standard elementary number theory text and a not-so-standard mathematics history text, *Journey Through Genius: The Great Theorems of Mathematics* (henceforward abbreviated *JTG*), by William Dunham [3]. The course also satisfies a university-wide upper-level writing requirement.

Elementary number theory and history of mathematics fit well together because number theory can be traced back to the Pythagoreans, if not to the earliest recorded numeration systems. Furthermore, the presentation of elementary number theory in most textbooks follows and even alludes to its historical development. But the two main topics are not the only pieces of the Number Theory/History of Mathematics course that blend well. Writing proofs is a theme shared by any elementary number theory course and by the particular text on which the history of mathematics portion of the course is based (*JTG*). The course contains a writing component that develops students' writing skills in the contexts of both mathematics and history.

Our Number Theory/History of Mathematics course has been rewarding for students and faculty because students are able to improve their knowledge and skills in many areas: history of mathematics, number theory, proof-writing, and more general mathematical writing and speaking.

Course Design

The prerequisite for our Number Theory/History of Mathematics course is at least one of our sophomore-level Linear Algebra or Conjecture and Proof in Discrete Mathematics courses, ensuring that students have had some practice with mathematical proof. The prerequisite also guarantees that students will have taken at least two semesters of calculus, with nearly all having taken three semesters. Finally, students must have satisfied the university's lower-level writing requirement in order to obtain upper-level writing credit for the course.

When I teach this course, I try to weight number theory and history of mathematics equally in content, assignments, and grading, and to integrate the two fields whenever possible. Integration occurs most naturally when we study the chapters in *JTG* titled "Euclid and the Infinitude of Primes" (Chapter 3) and "A Sampler of Euler's Number Theory" (Chapter 10), and when students choose results in number theory as topics for their "Great Theorem" term papers and presentations. Writing and speaking assignments in the course lead up to this final paper and presentation, to be discussed in more detail below.

Which number theory topics are taught? In my view, an undergraduate elementary number theory course taught using a traditional textbook should cover at least through quadratic reciprocity. How much should half a number theory course cover? I try to cover material through the basics of primitive roots, giving short shrift to some of the number-theoretic functions along the way. I do this in a fairly traditional way, by assigning reading and homework problems corresponding to every class meeting and using class time to introduce new concepts, sometimes through discovery activities, and to go over more challenging ideas, methods, and proofs. This study of number theory is directly supported by the two chapters of *JTG* noted above. Furthermore, students often choose as topics for their final projects and presentations results from number theory that we have not covered in class, including:

- the Chinese Remainder Theorem
- perfect numbers and the Euclid-Euler formula
- formulas for Pythagorean triples
- Fermat's Last Theorem $n = 4$ case
- writing integers as sums of squares
- Euler's Criterion
- quadratic reciprocity.

These topics are popular because they head the list of suggested topics for the final project and because students see an advantage in using their number theory text as one of the resources for their projects. Student presentations then fill in the remainder of what I would consider to be a 3- or 4-credit elementary number theory course. One could expand what I have described so far to become a course in the history of number theory. However, my colleagues' and my aim has always been a more general history of mathematics course.

The history of mathematics text for the course, *JTG*, consists of twelve beautifully crafted chapters, each featuring a "great theorem" of mathematics, together with a history of the ideas and events that led up to the theorem and an epilogue describing the influence of the theorem on later mathematics and mathematicians. Furthermore, nearly every great theorem is proved. In rare cases, only the idea of the proof is given. My goal is one chapter of this text per week, except in weeks with many student presentations and/or with in-class mid-term exams. In the relatively short 13-week semester at my institution, this means we cover at most nine chapters adequately and another lightly.

Because *JTG* is so well and entertainingly written, I can count on students to read it. Most students can understand the proofs, but of course there are those who struggle with them. Since learning to read and understand proofs (as well as devise and write them) is one of the most important goals of the course, I often ask students to work in groups to understand selected proofs from both of our texts, number theory and mathematics history, and to present them to the rest of the class in an understandable way. This requires identifying logical structure and strategies, working out details, and devising a helpful, effective way to present the proof to the class. *JTG* has no exercises, but I do write and assign problems that require students to explain, extend, or apply results from their reading or, occasionally, to support or critique a historical thesis offered in their reading of *JTG*. I also occasionally assign exercises from a relatively challenging set of exercises, "Problems for Great Theorems" [6], prepared by the author of *JTG* for high school teachers participating in summer workshops he conducted during the early 1990s and available online from MAA *Convergence* [13].

JTG features primarily mathematics of ancient Greece (Hippocrates' quadrature of lunes, Euclid's *Elements* and especially his proofs of the Pythagorean Theorem and of the infinitude of primes, Archimedes' formula for the area of a circle, and Heron's formula for the area of a triangle) and western European mathematics from the Renaissance forward (Cardano's solutions of cubic equations, Isaac Newton's estimate of π, the Bernoullis and Euler on infinite sums, Euler's number theory, and Cantor's treatment of the infinite). I assign additional reading and exercises to fill in gaps in history inevitable in an episodic text based on selected great theorems—or any text, for that matter. These readings include some, but never all, of the following already pared down list:

- An introduction to earliest evidences of counting and number systems
- Arithmetic in ancient civilizations
- "Geometric algebra" in ancient Greece
- Completing the square in the medieval Islamic world and in Renaissance Italy
- Origins of combinatorics, to include Pascal's triangle
- Mathematics in the Americas, especially Maya mathematics.

In the limited extra reading I am able to assign, I have shifted over the years from material directly relevant to the mathematics that makes up students' major and minor courses, such as origins of combinatorial formulas and the ideas of calculus, to information I feel is even more essential. Every mathematics major and minor—indeed, every college graduate!—should know and be able to share with their friends and family what is currently known about the origins of counting, number systems, and arithmetic. Of course, the history of such topics as completing the square, permutations and combinations, and limiting processes (in addition to that in *JTG*) also is appropriate and interesting, and I try to include it, too.

Finally, a screening of the PBS *Nova* television program, "The Proof," in which Andrew Wiles and colleagues recount in engaging and dramatic fashion Wiles' lifelong interest in and eventual proof of Fermat's Last Theorem, is completely appropriate for this course (see [9] or [10]). The program conveys what research mathematicians do every day, along with their frustration and delight in doing so, and students find it to be interesting and at times touching.

A more detailed course outline is given in the Appendix. Textbooks for the course are discussed in the next section and, in the section after, the writing component of the course is described in detail.

Resources

The ideal textbook for the number theory half of Number Theory/History of Mathematics is, in my view, David Burton's *Elementary Number Theory* [1] because it is written at the right level for my students and because it integrates the history of number theory throughout. Another number theory text with a strong historical component is James Tattersall's *Elementary Number Theory in Nine Chapters* [5]. I would also recommend considering Joseph Silverman's *A Friendly Introduction to Number Theory* [4], which takes a non-traditional, exploratory approach to number theory with some attention to its history. After using Burton's text for many years, I currently use none of the three texts recommended above due to their high prices, but rather a far less expensive text, Underwood Dudley's *Elementary Number Theory* [2], available from Dover Publications.

William Dunham's *Journey Through Genius: The Great Theorems of Mathematics* [3] consists of twelve beautifully crafted chapters that introduce students to mathematics and its history in an entertaining and accessible way. As noted above, each chapter features a "great theorem" of mathematics, together with a history of the ideas and events that led up to the theorem and an epilogue describing the influence of the theorem on later mathematics and mathematicians. Nearly every great theorem is proved; for a very few theorems, only the idea of the proof is given. The book is available as a Penguin paperback; Dunham's challenging set of exercises to accompany the text is freely available from the MAA website [13].

Writing Assignments

I have outlined above a rather traditional course (or half-course) in elementary number theory, with reading and homework assignments corresponding to most class sessions, and a history of mathematics course (or half-course) that not only emphasizes mathematics, but also seems to be modeled after a mathematics course with its weekly reading and homework assignments. What sets this Number Theory/History of Mathematics course apart from most mathematics courses is a writing component that develops students' writing skills in the contexts of both mathematics and history. Students write both history and mathematics throughout the course and, especially in their "Great Theorem paper" (see below), they integrate the two subjects in their writing. Completing mathematics problems, especially devising and writing mathematical proofs, requires communicating clearly in complete sentences, as does discussing historical issues in writing. For the latter, I start students off very gradually by asking such questions as, "Did the ancient Egyptians have a usable formula for the volume of the frustum of a pyramid? Give at least one argument for and one argument against (in no more than one page, please)."

Over the years, fairly early in the course, in order to gauge and begin to improve students' writing and research skills, I have used an assignment in which students evaluate websites on mathematics history and an assignment in which they write short biographies of mathematicians (the latter titled "Getting acquainted with a mathematician and with library resources"). More recently, the students' first multi-page essay has been a "Great Little Theorem paper" on Fermat's Little Theorem. This paper is to be a "mini-chapter" (at most six pages) inserted into *Journey Through Genius*. (Fermat's Little Theorem is actually introduced in Chapter 10 of *JTG*, though it is not the "great theorem" of this chapter.) It is to be written in the style, tone, and format of *JTG* and is to include:

1. A prologue that includes biographical information about Pierre Fermat and an introduction to the mathematics involved in Fermat's Little Theorem and its proof, including some really good examples.
2. Any lemmas not already stated and proved in part (1), along with further examples, and the statement and proof of Fermat's Little Theorem.
3. An epilogue that should include at least one of Wilson's Theorem, Fermat's Last Theorem, Euler's generalization of Fermat's Little Theorem, or an application of Euler's Theorem to cryptography (specifically, the RSA cryptosystem).

This mini-chapter is written in three pieces and goes through at least that many drafts. It has been a much more challenging assignment for students (and instructor) than the biography assignment it replaced. However, it provides much better preparation for the main writing assignment for the course, the "Great Theorem paper."

In their "Great Theorem paper," students research and write their own new chapter for *Journey Through Genius*. This chapter is to be written in the format and lively style and tone of *JTG*, except that it doesn't have to be quite as long (papers have averaged about 12 pages). Instructing students to use Dunham's prologue–great theorem–epilogue format and his lively, nearly over-the-top writing style may sound restrictive, but it helps turn organization, style, and audience into more tractable problems, allowing students to focus on the challenges of finding, understanding, and clearly communicating mathematics and its history. The mandate—or freedom—to be a bit silly in their writing also helps students overcome writer's block and avoid plagiarism. The “Great Theorem papers” are written in at least three pieces and go through at least two additional drafts. They are graded on the accuracy of the history and the mathematics, difficulty of the mathematics, organization and clarity, use of appropriate and illustrative examples, use of mathematical notation, and use of the English language (style and grace, as well as grammar and spelling). Students also make 10- to 15-minute oral presentations of their “chapters.”

In addition to the number theory topics listed above, suggested topics include:

- properties of Pascal’s triangle
- Stirling’s formula
- Wallis’s formula
- definite integrals and geometric series
- Fundamental Theorem of Calculus
- logarithm as an integral
- logarithms of negative numbers
- de Moivre’s Theorem
- Cauchy’s “wrong” theorems and proofs
- Rolle’s Theorem and Mean Value Theorem
- Königsberg Bridge Problem and Euler circuits
- Euler’s formula and generalizations
- Four Color Theorem (for maps or graphs)
- Lagrange’s Theorem (for finite groups)
- Fundamental Theorem of Algebra
- your favorite theorem!

These “new chapters” of *Journey Through Genius* have been a delight to read, and students have been quite proud of their work. This assignment has been used at many colleges and universities since *JTG* was first published in 1990. Although it takes much effort by students and instructors, I have heard nothing but praise for it.

Lessons Learned

The Number Theory/History of Mathematics course presented here combines not only number theory and history of mathematics, but also mathematical and historical writing. I have found that these topics complement each other well and that, with enough planning, teaching the course—and, I hope, taking the course, too—rarely feels like a juggling act. However, I do have to rein myself in, accepting that we will be able to study and understand only a certain amount of elementary number theory, explore only a selected stream of developments in the history of mathematics, and improve our writing (and speaking) skills via a limited set of assignments. The temptation to try to teach two or three courses in one is great because of the course’s dual topics of number theory and history of mathematics and its focus on writing. However, as with any course, understanding fewer ideas well is more important than skimming over too much material with little understanding.

A 3- or 4-credit history of mathematics course based on *Journey Through Genius* could incorporate more assignments from “Problems for Great Theorems” [6], in addition to writing assignments leading up to each student’s own chapter on a great theorem of mathematics. It certainly could include all 12 chapters of *JTG*, as well as extra readings to fill selected historical gaps or even an entire second text, such as George Joseph’s *The*

Crest of the Peacock: Non-European Roots of Mathematics [11] or Jacqueline Stedall's *Mathematics Emerging: A Sourcebook 1540–1900* [15].

My now retired colleague Mary Scherer originated the Number Theory/History of Mathematics course at the University of Redlands in an attempt to offer future mathematics teachers the content they needed to satisfy California's requirements for secondary mathematics teachers, yet to keep the credits required for these students' mathematics majors below our university's credit cap for majors. The course was an immediate success due to the interesting topics and to Scherer's excellent teaching. Her idea could possibly be adapted to other areas of mathematics besides number theory, with a combination of discrete mathematics and history of mathematics or of geometry and history of mathematics seeming to me to hold the most promise.

My department's Number Theory/History of Mathematics course is the primary reason I became interested in the history of mathematics, which is now my main area of scholarship. In order to teach number theory at my institution, I needed to learn enough mathematics history to teach the combined course! Fortunately, the MAA was offering an NSF-sponsored Institute for the History of Mathematics and Its Use in Teaching (IHMT), initially during 1995–97. I was fortunate enough to participate in IHMT—as the student with the most to learn—during 1996–97. The Institute was led by Victor Katz and Fred Rickey, to whom I am grateful for instruction and encouragement during the Institute and ever since.

References

Required texts ([3] and one of [1], [2], [4], or [5]):

1. Burton, David, *Elementary Number Theory* (7th ed.), McGraw-Hill, 2010.
2. Dudley, Underwood, *Elementary Number Theory*, Dover Publications, 2008; original publication: Freeman, 2nd ed., 1978.
3. Dunham, William, *Journey Through Genius: The Great Theorems of Mathematics*, Wiley, 1990; also available as a Penguin paperback, 1991.
4. Silverman, Joseph, *A Friendly Introduction to Number Theory* (4th ed.), Pearson, 2012.
5. Tattersall, James, *Elementary Number Theory in Nine Chapters* (2nd ed.), Cambridge, 2005.

Sources for additional exercises, readings, and video ([9] and [10] are the same program):

6. Dunham, William, "Problems for Journey Through Genius: The Great Theorems of Mathematics" (or "Problems for Great Theorems"), MAA *Convergence* (July 2015): www.maa.org/press/periodicals/convergence/problems-for-journey-through-genius-the-great-theorems-of-mathematics.
7. Katz, Victor, and Karen Dee Michalowicz (eds.), *Historical Modules for the Teaching and Learning of Mathematics*, MAA, 2005.
8. Ore, Oystein, *Number Theory and Its History*, Dover Publications, 1976; original publication: McGraw-Hill, 1948.
9. "The Proof," PBS *Nova* episode (1997): www.pbs.org/wgbh/nova/proof/.
10. "Fermat's Last Theorem," BBC *Horizon* episode (1995-96): www.bbc.co.uk/programmes/b0074rxx.

Resources for instructors and for student research:

11. Joseph, George, *The Crest of the Peacock: Non-European Roots of Mathematics* (3rd ed.), Princeton, 2011.
12. Katz, Victor, *A History of Mathematics: An Introduction* (3rd ed.), Addison-Wesley/Pearson, 2009.
13. MAA *Convergence*: www.maa.org/publications/periodicals/convergence.
14. *MacTutor History of Mathematics Archive*: www-history.mcs.st-and.ac.uk/.
15. Stedall, Jacqueline, *Mathematics Emerging: A Sourcebook 1540–1900*, Oxford University Press, 2008.

Appendix

Course Outline

Week 1 – Early number systems and arithmetic; *JTG*: "Hippocrates' Quadrature of the Lune"; *Number Theory*: Integers, division algorithm, Euclidean algorithm.

Week 2 – *JTG*: "Euclid's Proof of the Pythagorean Theorem"; *NT*: Unique factorization.

Week 3 – *JTG*: "Euclid and the Infinitude of Primes"; *NT*: Linear Diophantine equations.

Week 4 – Exam #1; *NT*: Congruences.

Week 5 – *JTG*: "Archimedes' Determination of Circular Area"; *NT*: Linear congruences.

Week 6 – "Great Little Theorem paper" first piece: begin in the middle with Fermat's Little Theorem, proof, and examples; *JTG*: "Heron's Formula for Triangular Area" (briefly); *NT*: Fermat's and Wilson's theorems.

Week 7 – "Great Little Theorem paper" second piece: at least one of prologue or epilogue; *JTG*: "A Gem from Isaac Newton"; *NT*: Number theoretic functions.

Week 8 – Exam #2; "Great Little Theorem paper" complete draft.

Week 9 – "Great Little Theorem paper" final draft; "Great Theorem paper" first piece: statement and examples of the great theorem; *JTG*: "The Bernoullis and the Harmonic Series"; *NT*: Euler's Theorem and RSA cryptosystem.

Week 10 – "Great Theorem paper" second piece: proof of great theorem and at least one of substantial historical prologue, preliminary definitions and lemmas, corollaries of great theorem, or substantial historical epilogue; *JTG*: "The Extraordinary Sums of Leonhard Euler."

Week 11 – "Great Theorem paper" with third and fourth pieces; *JTG*: "A Sampler of Euler's Number Theory" (briefly); *NT*: Primitive roots.

Week 12 – Exam #3; "Great Theorem" presentations; "Great Theorem paper" penultimate draft; *NT*: Fermat's Last Theorem and "The Proof".

Week 13 – "Great Theorem" presentations; "Great Theorem paper" final draft.

Week 14 – Final Exam.

Imagining the Acceptance of Complex Numbers

An Approach Through Primary Sources

Stephen Kennedy
Carleton College

Course Overview

The class[1] is offered for junior and senior math majors but has as prerequisites only three semesters of calculus, linear algebra, and our introduction-to-proof course. Having been exposed to David Pengelley and Reinhard Laubenbacher's books [7, 8] and an MAA workshop they organized several years ago on teaching history of mathematics with primary sources, I am convinced of the value of reading primary sources. So, a significant portion of the course consists of reading primary source material. I also want my students to experience something of the initial fumblings and contingencies and evolution of mathematical concepts, so I decided to focus on one theme rather than attempt a big-picture approach to the entire history of our subject.

After surveying primary sources available in English on a variety of topics I eventually settled on the acceptance of complex numbers as my theme. The topic has an attractive narrative, is not overly demanding in terms of mathematical prerequisites, and offers many easily available primary sources.

The goal of the course is to have the students essentially relive the discovery and acceptance of complex numbers through careful reading of the original works of the major players in that drama. We start with Cardano, finish with Argand and, in between, follow a strand of ideas and papers connecting these two across the centuries. I also incorporate a substantial dose of reading from Jacqueline Stedall's masterful *From Cardano's Great Art to Lagrange's Reflections: Filling a Gap in the History of Algebra* [11]. A "primary-sources-only" course would have cheated my students out of the experience of reading her meticulous scholarship and erudition.

Course Design

The course meets twice a week, Tuesday and Thursday, for one hundred minutes each time during Carleton's ten-week spring term. As it is a designated seminar (which, at Carleton, means a heavy dose of student-led discussion), I assign a pair of "experts" to give a presentation and lead a discussion on each of the readings.

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[1] Carleton College does not offer a history of mathematics course in its regular curriculum. We do, however, have a regular rotating seminar course in which a faculty member gets to teach an upper-level course on a special topic of his or her choice. In 2013 the seminar rotated to me and I decided, after assessing student interest, to try teaching the history of mathematics. I have no particular expertise in the subject.

Each student is the expert for three different readings; each reading has two experts assigned to it. The experts are responsible for presenting the most important elements, ideas, and algorithms in their reading to the class in a forty-five-minute time slot. We have two expert presentations in each class meeting. I assign partnerships for each reading and I make certain that each student is responsible for no more than one Stedall chapter; thus each student is responsible for mastering at least two primary sources. The experts know their assignment several weeks in advance and their task is to master the content of their reading and present the major ideas in it to the class.

Each class period is divided in half with one pair of experts in charge of each half. The entire class is expected to have read the relevant material for each class and to have brought intelligent questions. I strongly encourage the experts to engage the details. Nearly every presentation includes careful working of carefully chosen examples to illustrate the mathematics being written about. If the reading does not include an appropriate example, I ask the students to create one of their own—or I do that for them. During the presentation I sit among the students and am part of the discussion. I always prepare a list of the major points that I want the class to notice and I ask the presenters specific questions if they neglect an important point. I often end up spending five or ten minutes lecturing after a presentation, either trying to clear up a murky point, or trying to point out connections between the current reading and a previous one.

Because the students have not yet had time to become experts on a reading, I lecture for the entire first week. After explaining the structure of the course and my expectations of them I try to give a bird's-eye view of 5000 years of mathematical progress and evolution. I begin with Egypt and Babylon and give examples from each culture to show how differently they conceived of mathematical objects and how incomplete our understanding of their mathematics is. I talk about Greek axiomatics and rigor and the distinction between number and magnitude. I show them the first few proofs from Euclid and throw in the Pythagorean proof for fun. We discuss Arab innovations in algebra and their role in transmitting Greek mathematics to Europe. Because we are about to study it intensively, I skipped over the Italian renaissance of algebra and talked about Fermat and Descartes and the precursors of calculus. We examine the ideas of Newton and Leibniz and their different notations for, and conceptions of, the calculus. I show them examples of mathematics discovered by Euler and Gauss and describe the century and a half from Newton to non-euclidean geometry as a sort of golden age for mathematics. I talk about the century from Bolyai to Gödel as one of revolutionary reconfiguring of our fundamental notions of what mathematics is. As much as possible I try to illustrate these mathematical ideas by using examples, problems, and proofs from the time period we are discussing. I also try to emphasize by whom and by what means mathematics was being done. That is, mathematics is a human, and therefore social, institution. Egyptian mathematicians were a priestly caste of secretive semi-sorcerers and today we are an organized profession with journals and societies and strange customs. I try to, telegraphically, track this institutional evolution along with the evolution of mathematics itself.

Between the first and second class meeting I ask them to read Benjamin Wardhaugh's *How to Read Historical Mathematics* [15]. In post-course evaluations students reported that they had found it extremely useful.

The second week begins the student-led presentations from the readings and we start with algebra in the Renaissance. We read the first chapter of Stedall which discusses the contributions of Cardano, Bombelli, and Viète. We read extensively in Cardano's *Ars Magna* [2] in which Cardano, not allowing negative coefficients, solves his cubics for many different cases, e.g., "On the Cube Equal to the Square and Number" or $x^3 = ax^2 + c$. Modern readers quickly lose interest in the details of all the cases, but don't forget to look at his later chapters on quartics. Bombelli has not been translated from Italian, but we read a nice *Mathematical Intelligencer* [6] article that gives one the feeling of reading him.

In our third week we examine the beginnings of symbolic algebra. Stedall, in her second chapter, treats Viète, Harriot, Girard, and Descartes. Her explanation of Harriot's works are particularly insightful and made me wish the class was reading some of them. We read some of Viète [13] here, but the portion read is pretty conceptually thin and concerned mostly with terminology.

In the fourth week we engage more significantly with Viète and read some Descartes [4] and Wallis [14]. Watching the transition as mathematicians became more comfortable with symbols than with geometric diagrams is fascinating. It is particularly interesting to see Viète realize (p. 174) that solving a certain cubic is related to the triple-angle formula for cosine. This, of course, recurs in de Moivre [9] and is crucial to the eventual discovery of the geometric interpretation. The relationship between polynomial roots and factors is in Descartes. Make sure to tell your students that Wallis, who is struggling for a geometric insight, is making a mistake before they spend days struggling to understand him.

The power of algebra is finally revealed in the works of Euler and Newton during our fifth week. Stedall's third chapter is titled "Descartes to Newton" but contains information on a host of minor characters' contributions: Hudde, Dulaurens, Tschirnhaus, Wallis, Gregory, Rolle, and Collins. The Newton [10] and Euler [4] readings during this week are expository treatments of algebra. We read them because they are accessible readings by these authors, not because they made significant advances; though Euler is clearly grasping at the truth, at one point he considers imaginaries as potentially "a distinct species of number." This is great mathematical drama.

We spend the sixth week reading Euler. The section of Euler's *Elements of Algebra* we read contains his exposition of solving quadratic, cubic, and quartic polynomials. He engages fully with the complex numbers while still denying their existence. At one point he says: "It is true, that these values of x are imaginary, or impossible; but yet they deserve attention." The Euler article [5] is incredibly rich containing, essentially, the fundamental theorem of algebra, the fact that complex roots of real polynomials occur in conjugate pairs, that the complex numbers are closed under the operations of arithmetic and root extraction, and the polar form of complex numbers. In hindsight, we should have read the entire paper and spent several class periods on it.

The seventh week of the course presents us with the eighteenth-century uncovering of the nature of complex roots. We read two chapters of Stedall and de Moivre's [9] contribution, the latter of which would be, chronologically speaking, better read before reading the Euler above. The Wessel excerpt we read, from Smith's *Source Book* [16], contains no diagrams! During week eight we read Argand [1] in its entirety. This, the interpretation of complex numbers as a plane, comes largely as an anticlimax to the students, probably because they already understood everything in it.

In addition to becoming expert on three readings, each student is expected to study every reading and come to class prepared to discuss it. In practice, when I taught this class the students rarely came to class with questions—either because they had not done the reading, did not understand it, had not bothered to think deeply about it, or did not want to put their colleagues on the spot with a tough question. In truth, probably all those reasons applied, in varying mixtures, to every student. I knew going in that I would have to be the one asking most of the questions during the presentations and I used the opportunity to direct attention to ideas and passages that I suspected were difficult, or that were significant mathematically or historically, or that I knew would be important to understanding and appreciating upcoming readings. In order to force the students to honestly engage every reading I administered a midterm exam during the sixth week. I designed the exam to test the students' understanding of the readings and I included questions about passages that had not been discussed in class.

Resources

Stedall's [11] book is an incredible resource and guide for me and for the students. I cannot recommend it highly enough. I also leaned heavily on Stillwell [12] for interpretation, explanation, and references to primary sources when designing the course. I did not ask the students to read it, but I used it to inform my contributions to our in-class discussions. It was convenient to find the Newton [10] and Argand [1] readings on Google Books. Descartes' *Geometry* is available at archive.org. Wallis's book is available by subscription at Early English Books Online [14].

Assignments

The major student work in the course is to become the expert on their assigned readings. Each student receives written feedback and a grade from me within two days of presenting. The grade reflects my evaluation of both the student's understanding of what he or she has read and his or her clarity of exposition (usually reflected in mastery of details). These grades each count as 15% of the student's final course grade

In the midterm exam I ask the students to implement various algorithms we have studied, to identify various passages we have read, and to interpret and explain some passages. I am completely upfront with the students that the reason for the exam is to give them an opportunity to show me the level of commitment they have brought to the job of understanding the daily readings. The exam grade counts for 20% of a student's final course grade. Each student also receives a class participation grade that counts as 10% of his or her final grade; this participation grade reflects his or her contributions to discussion in which he or she is not the assigned expert.

The final graded component (25%) is a ten-page paper and twenty-five minute presentation by each student on some historical topic. I present the students with a list of possible topics from which to choose. About one-half of these are primary sources, the rest are secondary sources. Most of the topics are related to the theme of the course—the early development of algebra. In practice, students proposed topics outside that theme: one read a paper by Cantor (and portions of Joseph Dauben's book for context), another read a paper by Cauchy, a third read Lakatos's *Proofs and Refutations*. With seventeen students the presentations occupied more than eight hours; I used the last two weeks of class and our assigned final exam period for these.

Lessons Learned

In my syllabus I wrote.

> We are going to study, in some detail, the discovery, evolution, and ultimate acceptance of complex numbers by the mathematical community. We are going to do that by study of primary and secondary literature. It is difficult to read three-hundred-year-old mathematics. The writers had notations, conventions and, most importantly, conceptions different from ours. That is the point—we want to learn about those differences and how our present notations, conventions, and conceptions evolved to their current state of perfection.

I think the students missed the intended irony in that last line. But, they got the rest. They understood that these writers thought about these mathematical entities and concepts differently. And they enjoyed struggling to see things from Cardano's and Euler's points of view. Especially Euler—his style is so clear and graceful, he takes you from simple beginnings to great heights effortlessly. Next time I will have more Euler in the syllabus.

It turns out that the Stedall chapters contain much more information than can be summarized in one forty-five minute session. In a future iteration of the course, I intend to incorporate lectures from me on the topics in Stedall and leave the students just the primary source readings to present. I will still require the students to read Stedall because it's so illuminating and insightful that it would be a shame for them to miss it, but I will be the "expert" on those readings. I think I would prefer that the students struggle to understand a few primary sources while I provide context and background from secondary sources. I will also try to reach farther back in time to exhibit some of the Arab roots of algebra. An English translation of al-Khwarizmi is available online at www.wilbourhall.org. Also, I'm not certain giving a midterm examination on the readings was the right way to encourage the students to engage with all the readings. Next time I will consider using an online forum and requiring every student to post a reaction to each day's reading, containing at least one question for the experts, in advance of every class discussion. Not only will this encourage them to engage with every reading, but also, if I bring their questions to class, it could help make the in-class discussions more lively.

As you might expect, some of the students take the work of being the expert on a reading quite seriously, visit me often, and gain a deep understanding of their assignment. Others do less. One thing I will change in the next iteration is to add a required meeting with me for experts three or four days before their presentation. This will force the students to get to work earlier and also give me a chance to make sure they are not missing one or more important elements of their reading.

In the design of the course and choice of theme, I am constrained by

i) The disparate backgrounds of the students in the course—there were no upper-division mathematics prerequisites, but some students had taken several such courses, other students had zero or one.

ii) The availability of primary sources in English or English translation—every Carleton student is proficient in some foreign language, but it's not all the same language.

iii) My desire to find a topic that exhibited significant increase in depth and evolution of understanding over a period of time.

iv) My desire to have a focus about which I already possessed some expertise—this was the most restrictive constraint!

I considered several possible themes: calculus, the infinite, Euler's formula $v-e+f=2$, and the discovery of non-Euclidean geometry. The acceptance of complex numbers combines a compelling narrative, availability of primary sources in translation, and a clear evolution in understanding that combines symbolic and geometric reasoning. But, it might have been a mistake to make understanding this acceptance as the primary goal. My goal should have been to have the students struggle to see some mathematical ideas evolve and emerge. I should have trimmed some of the more difficult readings above (e.g., Newton, Viète) and spent much more time with the details of others (e.g., Euler, Cardano).

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3. Rene Descartes, Geometry, available for download at archive.org/details/geometryofrene00desc
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5. ———, Investigations on the Imaginary Roots of Equations, E170, translated by Christine Stevens, available at the Euler Archive, eulerarchive.maa.org.
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15. Benjamin Wardhaugh, *How to Read Historical Mathematics*, Princeton University Press, Princeton, NJ, 2010.
16. Caspar Wessel, On the Analytical Representation of Direction: An Attempt, excerpted in *Source Book in Mathematics*, D.E. Smith, McGraw-Hill, New York, 1929, pp. 54–67.

Appendix A

Course Outline

Week One: Background and context-setting lectures by instructor.

Week Two: Renaissance algebra in Italy and France: Chapter 1 of Stedall, Chapters 1–6, 11–14, 17, 37 and pp. 237–243 of Cardano [2] and Mazur and La Nave on Bombelli [6].

Week Three: The beginnings of symbolic algebra: Stedall Chapter 2 and pages 11–32 of Viète [13].

Week Four: Algebra is being constructed: Viète [13] pp. 159–183, pp. 322–333, 344–348 and Descartes [3] Book 3 pp. 152–220 and Wallis [14] pp. 171–175 and pp. 264–269.

Week Five: Elementary algebra comes of age: Stedall Chapter 3, Newton [10] pp. 145–184, pp. 347–361, and Euler [4] pp. 38–44, 186–189, 216–229.

Week Six: Algebra in the hands of a master: Euler [4] pp. 244–288 and Euler E170 [5] sections 1–20 and 60–78. Also midterm exam.

Week Seven: Complex numbers are just numbers: Stedall Chapters 4 and 5, de Moivre [9] and Wessel [16].

Week Eight: Geometry reveals all: Argand [1].

Weeks Nine and Ten: Student presentations of final projects.

Mathematical Impossibilities

Alex M. McAllister
Centre College

Course Overview

This themed history of mathematics course explores mathematical questions that are provably impossible to answer from among the independence of the parallel postulate and non-Euclidean geometries, impossible unmarked straightedge and compass constructions, the unsolvability of the quintic by radicals, the independence of the continuum hypothesis, Gödel's incompleteness theorems, and the algorithmic unsolvability of the halting problem. I teach two versions of Mathematical Impossibilities during our January term. One version is a sophomore-level mathematics course with a prerequisite of Calculus II, while the other is a first-year seminar with no prerequisites. I also share ideas for offering Mathematical Impossibilities during a full term with a transition to proofs course as a prerequisite.

Course Design

Mathematical Questions: Answered, Open, and Impossible

A defining element of most mathematics courses is a focus on "successful" mathematics. Namely, we study questions with known solutions and the various ideas and techniques that enable us to find these answers. However, in the midst of our mathematical successes, some mathematical questions do not have answers. Some of these questions are "open" and lie at the frontier of mathematical research. In contrast, other unanswered questions possess the distinctly different character of being provably impossible to answer. From this perspective, mathematical questions resemble Neapolitan ice cream's blend of three different flavors.

- *Answered questions*, which we teach in most of our courses. Examples include what is the slope of a tangent line? and what is the probability of being dealt a full house?
- *Open questions*, which we optimistically grapple with in our search for further truths. Examples include Goldbach's conjecture, the Riemann hypothesis, and the P vs NP problem, among a host of other more or less challenging open questions.
- *Impossible questions*, which we cannot and never will be able to answer within some desired context. Examples include the impossibility of proving or disproving Euclid's fifth parallel postulate from neutral geometry and the unsolvability of the quintic by radicals, as well as questions that arise in the study of geometric constructions, set theory, modern logic, computing, and the social sciences.

As suggested by the examples given above, I think of two types of questions as falling under the unifying umbrella of "impossible." The parallel postulate is an example of an undecidable statement that is impossible to either prove or refute from a given set of axioms. In contrast, the unsolvability of the quintic by radicals is an example of a non-existence result showing the impossibility of obtaining a solution using a particular set of tools. In both cases, we encounter firm boundaries to what mathematicians can accomplish and inherent limitations to our corporate, human endeavor.

This course studies impossibilities with interesting stories, intriguing mathematics, and difference-making revelations that informed and changed our understanding of mathematics. For each rendition of Mathematical Impossibilities, I teach three or four impossibilities chosen from among

- the independence of the parallel postulate and non-Euclidean geometries
- impossible unmarked straightedge and compass constructions
- the unsolvability of the quintic by radicals
- the independence of the continuum hypothesis
- Gödel's incompleteness theorems
- the algorithmic unsolvability of the halting problem.

These six mathematical impossibilities are widely recognized as fundamental and many students find them quite interesting. The origins of some of these questions are ancient and efforts to address them span millennia of work by mathematicians. However, others were only articulated in the last century, and all the proofs of these impossibilities were developed during the 19th and 20th centuries. In this sense, Mathematical Impossibilities explores a long span of the history of mathematics, including a decidedly modern slice of mathematics.

Impossibility Choices

My development of a particular offering of this course begins with the selection of four mathematical impossibilities to study from among the six highlighted above. For each of these impossibilities, I have identified lists of topics for study, which are provided later in this section and in the Appendices (in greater detail). I make selections from among these possible topics based on the interconnections among the impossibilities, the particular audience of students, and the time constraints of teaching in a January term or full term. While I feel free to include any of these topics, I can only incorporate some of them into each rendition of the course. Personally, I find it challenging to make choices among these many competing goods because these ideas so engaging, intriguing, and fun!

During my January-term courses, I teach three or four of these mathematical impossibilities, depending on the particular audience of students and on the interconnections among the topics we study. For the sophomore-level mathematics course with a prerequisite of Calculus II, we study the unsolvability of the quintic by radicals, the independence of the parallel postulate and non-Euclidean geometries, Gödel's incompleteness theorems, and the independence of the continuum hypothesis. Appendix A provides a course outline for this version of Mathematical Impossibilities. For the first-year seminar with no prerequisites, the emphasis is on critical thinking and both oral and written communication, and we study the independence of the parallel postulate and non-Euclidean geometries, the independence of the continuum hypothesis, and the algorithmic unsolvability of the halting problem.

During a full term, I would teach at least four impossibilities, although a fifth might be possible if the selections are sufficiently intertwined. For example, our study of non-Euclidean geometries includes a discussion of consistent sets of axioms (which are free of contradictions) and the notion of consistency is central to understanding Gödel's incompleteness theorems. Similarly, our study of the independence of the continuum hypothesis includes learning Cantor's diagonalization method to prove the uncountability of the set of real numbers, and this method is a key element of proving the algorithmic unsolvability of the halting problem. For a full-term course, I would prefer a transition to proofs course as a prerequisite. In fact, the first time I taught the sophomore-level version of Mathematical Impossibilities, all but one of my students had taken our transition to proofs course, which meant that almost all of them were already familiar with basic logic and proof writing. With their stronger background, we were able to explore each impossibility in much greater depth.

One natural approach to designing a full-term Mathematical Impossibilities course is to select four of the six impossibilities based on personal preference; I designed my January-term courses in just this way. Another approach is to cast this course as an exploration of our "trust" in mathematics in light of these impossibilities. In particular, as these results were discovered in geometry, algebra, set theory, logic, and basic arithmetic, mathematicians began asking what ideas and intuitions could serve as a foundation for mathematics, and we have never really resolved this issue. For this "trust" approach, I suggest teaching the independence of the parallel postulate and of the continuum hypothesis, Gödel's incompleteness theorems, and the halting problem.

Pedagogical Choices

Every time I teach Mathematical Impossibilites my two main goals are

1) to explore the stories of the selected mathematical impossibilities, seeking to understand the mathematics itself, the evolution of these ideas over time, and the particular people and historical contexts associated with each, and
2) to create an interactive learning environment in which my students (a) engage ideas through readings, class discussions, and homework exercises, (b) conduct some level of independent research and study, and (c) share their understanding through both oral and written means of communication.

I aspire to study these mathematical impossibilities as a whole and their individual component topics from the perspective of the evolution of humans' understanding over time. In addition to learning the actual mathematics, our study includes the stories of the particular people and historical contexts surrounding their development as well as the philosophy associated with these ideas.

I seek to have my students actively learn the ideas of Mathematical Impossibilities throughout the course. In preparation for every class, my students provide written responses to a collection of reading questions based on the textbooks, handouts, or web resources. Similarly, after each class meeting, my students complete a collection of homework exercises. Some of these exercises are mathematical in nature, while others consist of short answer or longer essay questions with a more historical, philosophical, or personally reflective flavor.

Our actual class meetings vary quite a bit in their structure. Sometimes I take the lead, implementing a highly interactive lecture. Such classes usually revolve around handouts, many of which include questions for small group work leading into the students presenting their solutions at the board. Example handouts are included in the Appendices. For other class meetings, my students take the lead and give presentations of varying complexity and length, such as teaching an assigned collection of ideas from our textbooks, presenting biographical sketches, and describing and analyzing historical events. Finally, some classes are discussion-based with the flavor of a Socratic dialogue: we circle up the desks and we explore various open-ended questions.

For each impossibility, I give an overall historical sketch as well as sharing historical vignettes throughout our study. As mentioned above, student presentations of history and mathematical content are the focus of some class meetings. At least one of these student presentations is a component of an independent research project. For these projects, my students create an annotated bibliography; develop a fact list, outline, or timeline (as appropriate); present their findings to the class; and submit a paper discussing what they learned.

The remainder of this section provides a brief description of each mathematical impossibility and highlights topics that can be studied for each. Some topics are repeated among impossibilities because of natural interconnections. The Appendices include much more detailed lists of topics accompanied by references for studying and teaching these ideas. The Appendices also share selected examples of class handouts, reading questions, and suggestions for student presentations.

Non-Euclidean Geometries: Parallel Lines do Not Exist?!

Euclid's *Elements* provides a survey of the state of the art in geometry and number theory as of 300 BCE. His fifth postulate makes implicit claims about infinity that concerned the ancient Greeks and, since as early as the time of Archimedes, they began seeking a proof of this postulate from neutral geometry. In the early 1800s, Gauss, János Bolyai, and Lobachevsky independently demonstrated that such a proof does not exist, developing new "fantastical" geometries. Even more, their work shows that the parallel postulate can be neither proven nor refuted by neutral geometry. These results highlight the context-dependent nature of mathematics, the distinction between reality and mathematical models, and undermine the notion of geometry as certain knowledge.

I build our study of the independence of the parallel postulate around the following topics. First, we explore the distinction between inductive and deductive reasoning and then I introduce Euclid's *Elements* with its characteristic axiomatic approach to mathematics. We develop proofs of multiple theorems from Euclid's *Elements*, working through a handout that is discussed in Appendix B, after which we study alternative axioms equivalent to Euclid's fifth postulate, including Playfair's axiom. Non-Euclidean geometries are introduced based on negating the uniqueness and then the existence claims of Playfair's axiom. The relative consistencies of non-Euclidean geometries are argued using the sphere model of elliptic geometry and both the Beltrami disk and Poincaré disk models of hyperbolic geometry. We study various results from non-Euclidean geometry and their applications, and discuss the "Euclidean Myth," which claims that Euclid's geometry provides an exact, undoubtable model of our physical universe.

Appendix B provides a more thorough outline of topics of study with detailed references, a discussion of and sampling from my class handouts, and a list of mathematicians for student presentations.

Impossible Geometric Constructions: Boundaries of Geometry

The proofs in Euclid's *Elements* can be interpreted as algorithms for constructing geometric figures using only an unmarked straightedge and compass. For example, we can bisect angles and construct squares using just these tools. However, several sought-for constructions were not found by the ancient Greeks, despite their best efforts, including the following.

- *Double the cube*: given a cube, find a cube with double the volume of the given cube.
- *Trisect the angle*: given an angle, trisect the angle into three equal sub-angles.
- *Square the circle*: given a circle, construct a square with area equal to that of the circle.

In the early 1800s, Gauss and Wantzel independently proved the impossibility of these three geometric constructions, provided we restrict ourselves to using only an unmarked straightedge and compass.

I build our study of these impossible geometric constructions around the following topics. We compare and contrast existence with constructability in the familiar context of finding zeros of polynomials. I introduce Euclid's *Elements* and various constructions, particularly angle bisection and constructing a square on a given line segment. We then consider the three impossible construction questions of the ancient Greeks, with an emphasis on the restriction to using only an unmarked straightedge and compass. We discuss Fermat primes, leading into the Gauss-Wantzel theorem (which identifies the constructability of regular n-sided polygons with particular factorizations of integers), and constructible numbers, followed by the nonconstructibility theorem (which states that if a cubic polynomial with rational coefficients does not have a rational zero, then none of its zeros is constructible). We apply the nonconstructibility theorem to the cubic polynomials relevant to the impossible geometric constructions and we wrap up this topic by studying alternative construction tools such as the quadratrix of Hippias and the paperfolding of Row.

Appendix C provides a more thorough outline of topics of study with detailed references, a sampling from my class handouts, and a list of mathematicians for student presentations.

Unsolvability of the Quintic by Radicals: Chasing Zeros

Since at least the time of the ancient Babylonians and Egyptians, humans have known algorithmic versions of the quadratic formula. During the Italian Renaissance, the combined work of del Ferro, Tartaglia, Cardano, and Ferrari identified cubic and quartic formulas. In light of these successes, mathematicians optimistically sought a quintic formula. In the early 1800s, these hopes were dashed when Abel proved that there does not exist a solution by radicals for quintic polynomials.

I build our study of the impossibility of solving the quintic by radicals around the following topics. We begin with the fundamental theorem of algebra ensuring the existence of complex zeros for polynomials with complex coefficients, and then consider the distinction between knowing zeros exist and actually finding zeros. We trace the known development of solution techniques through history, including work by Babylonian, Egyptian, Chinese, Indian, and Islamic mathematicians. We learn the depressed cubic, cubic, and quartic formulas providing solutions by radicals for the corresponding polynomials as developed by Italian Renaissance mathematicians. Finally, we consider permutations and Lagrange's resolvent, which enabled the work of Ruffini, Abel, and Galois, and culminated in Abel's theorem.

Appendix D provides a more thorough outline of topics of study with detailed references, a sampling from my class handouts, and a list of mathematicians for student presentations.

The Continuum Hypothesis: To Infinity and Beyond

In the latter half of the 1800s, Cantor developed a precise analysis of actual infinite numbers and proved the existence of different infinite cardinals. For example, the set $\mathbb{N}$ of natural numbers is countably infinite, while the set $\mathbb{R}$ of real numbers is uncountable. In seeking to understand the relative sizes of these distinct cardinalities, Cantor conjectured his continuum hypothesis, which asserts that the cardinalities of $\mathbb{N}$ and $\mathbb{R}$ are "adjacent" in the same way that the integers one and two are adjacent because there does not exist an integer between them. Work by Gödel in the 1940s and by Cohen in the 1960s showed that the continuum hypothesis can be neither proven nor refuted by ZFC set theory. This independence of the continuum hypothesis allows us to develop multiple, consistent set theories in much the same fashion as our multiple, consistent geometries.

I build our study of the independence of the continuum hypothesis around the following topics. We begin with the proof of the infinitude of the primes, following the argument in Euclid's *Elements*, as both an introduction to infinity and to proof by contradiction. We then discuss the insights and questions of the ancient Greeks: Zeno's paradoxes of motion, Aristotle's distinction between potential and actual infinity, and Archimedes' method by exhaustion. Working toward the definition of countably infinite, we consider definitions and examples of one-to-one, onto functions. We then study the definitions and the basic arithmetic of finite and infinite countable cardinalities, and Cantor's proof that the set of reals is uncountable using his diagonalization method. Finally, we discuss the continuum hypothesis and its independence over ZFC set theory.

Appendix E provides a more thorough outline of topics of study with detailed references, a discussion of and sampling from my class handouts, and a list of mathematicians and other topics for student presentations.

Gödel's Incompleteness Theorems: Could Arithmetic be Inconsistent?

In Hilbert's famous centennial address to the 1900 International Congress of Mathematicians, his second problem asked for a proof that basic arithmetic is consistent, or free of contradictions. In 1932, Gödel found true mathematical sentences that are not provable. An informal rendition of Gödel's sentence morphs the liar paradox "I am lying" into "This statement is not provable." Gödel's incompleteness theorems show that mathematical truth and mathematical proof are not the same and that a mathematical system sophisticated enough to include basic arithmetic cannot prove its own consistency.

I build our study of Gödel's incompleteness theorems around the following topics. We begin with issues that arise with trusting our intuitions about "space" from geometry, "collections of objects" from set theory, "reasoning" from logic, and "number" from arithmetic as possible foundations of mathematics. We then discuss the basics of propositional and predicate logic, Frege's logicism, Russell's paradox, and Hilbert's formalism, leading up to Hilbert's second problem asking for a proof that arithmetic is free of contradictions. Working toward Gödel's resolution, we study paradoxes arising from self-reference (particularly the liar paradox), the arithmetization of syntax (or Gödel-numbering), algorithmically computable functions, and the Gödel sentence (interpretable as "I am not provable"). I then state Gödel's incompleteness theorems, discussing their meanings and non-meanings, and sketch proofs of these results at a level and with detail appropriate for the students in the class.

Appendix F provides a more thorough outline of topics of study with detailed references, a reading guide and discussion questions for Nagel and Newman's *Gödel's Proof* [30], a class handout for a proof sketch of Gödel's incompleteness theorems, and a list of mathematicians for student presentations.

The Halting Problem: We Really Do Need Ctrl-Alt-Delete

For millennia, mathematicians identified the solutions of mathematical questions with the algorithms for constructing these solutions. In response to mathematicians developing existence proofs that were not constructive, Hilbert asked if there exists a mathematical problem that cannot be solved algorithmically. In the 1930s, Turing produced such a problem using simple models of algorithmic processes, which became known as Turing machines. These models enabled Turing to demonstrate the algorithmic unsolvability of the halting problem, which asks for a general process to determine whether an arbitrary Turing machine stops computing on any given input.

I build our study of the impossibility of algorithmically solving the halting problem around the following topics. We begin with a general discussion of algorithms and algorithmic processes, and then state Hilbert's Entscheidungsproblem, which asked for an algorithm that could identify a given mathematical statement as true or false. We study the definition and multiple examples of Turing machines, leading into the statement of the Church-Turing thesis, which asserts that every algorithmically computable function is computable by a Turing machine. After developing the appropriate tools and ideas, I state and prove the halting set theorem (which states that there does not exist an algorithm for determining whether a Turing machine with Gödel-number n halts on input n) by means of Cantor diagonalization applied to an enumeration of all Turing computable functions. In addition, because Turing was actively involved in England's code-breaking efforts during World War II, I usually study some basic cryptography, World War II, the Germans' Enigma code, and the code-breaking work of Polish and English mathematicians. The early development of computing machines can also be discussed in this context.

Appendix G provides a more thorough outline of topics of study with detailed references, a sampling from my class handouts for studying both Turing machines and some basic cryptography, and a list of possible student presentations.

Resources

The following books and websites are good, general purpose resources for an initial exploration of these mathematical impossibilities. Both faculty and students are encouraged to begin their study of a particular topic with these sources. They also include references that enable further, more thorough investigations.

- Boyer and Merzbach's *A History of Mathematics* [3]
- Dunham's *Journey Through Genius: Great Theorems of Mathematics* [11]
- Heath's *A History of Greek Mathematics, Volume I* [19] *and Volume II* [20]
- Katz's *A History of Mathematics: An Introduction* [26]
- *Complete Dictionary of Scientific Biography* at www.gale.cengage.com/ndsb
- *Encyclopedia Britannica* at www.britannica.com
- *The MacTutor History of Mathematics Archives* at www-history.mcs.st-and.ac.uk
- *Stanford Encyclopedia of Philosophy* at plato.stanford.edu

I am also writing a book-length manuscript that describes and explores the history of these six mathematical impossibilities. More focused references about each impossibility are provided below.

Non-Euclidean Geometries: Gray [15] or Gray [16] can serve as a textbook for teaching these ideas, depending on the audience. I also found Greenberg [17] very helpful. Other useful texts for instructors are Davis and Hersh [9], Nagel and Newman [30], Rosenfeld [33], and Row [34].

Impossible Geometric Constructions: I have not found a book that seems appropriate for a course with limited prerequisites, so I create my own handouts. For me, Jones, et al. [25] and Kazarinoff [27] have been particularly helpful.

Unsolvability of the Quintic by Radicals: Pesic [32] is an excellent exposition and can serve as a textbook for teaching this impossibility. Alternatively, Dunham [11] can be used as a textbook because it also addresses the continuum hypothesis. Other useful texts for instructors are Boyer and Merzbach [3], Grattan-Guinness [14], and Katz [26].

The Continuum Hypothesis: For textbooks, I use Maor [28] in my first-year seminar and Dunham [11] for my sophomore-level mathematics course. Other useful texts for instructors are Heath [19], Rucker [35], Sainsbury [36], Smullyan [38], and the definitive biography of Cantor by Dauben [7].

Gödel's Incompleteness Theorems: Nagel and Newman [30] is essential reading and was a textbook for my sophomore-level mathematics course, and Crossley, et al. [5] guides my proof sketches. Other useful texts for instructors are Rucker [35], Sainsbury [36], Smullyan [39], Yandell [40], and the definitive biography of Gödel by Dawson [10].

The Halting Problem: I have not found a book that seems appropriate for a course with limited prerequisites, so I create my own handouts. Useful texts for instructors are Cutland [6], Davis [8], Hamilton [18], Hinsley and Stripp [21], and the definitive biography of Turing by Hodges [22]. In addition, see Hodges' website "Alan Turing: The Enigma" at [23] and NOVA Online [31].

Assignments

Reading and Reflection Questions: Prior to each class, my students read about the topics we will discuss and provide a written response to a collection of reading questions. Some of these questions are more mathematical in nature, while others are oriented toward the associated history and philosophy. In concert with these reading questions, I often ask my students to reflect on the meaning of the ideas we are discussing: namely, how these ideas impact their understanding of mathematics, the history of math, the nature of knowledge, the natural universe, or themselves as a person and/or as a mathematician.

Homework Exercises: My students are assigned homework exercises based on the ideas we are studying, using both contemporary and modern techniques. Most of the references suggested above explore the ideas and history of each impossibility, but do not include exercises. So, for the most part, I write my own exercises based on my ideas at the time and on materials I have developed for other courses. Many of these exercises are variations or extensions of those given on the class handouts appearing in the Appendices.

Presentations: Student presentations are an important component of these courses. Before any student gives a presentation, I invest class time in discussing and teaching elements of good presentations. Our discussion revolves around handouts created by both myself and our Center for Teaching and Learning (CTL), as well as the rubric used to assess their presentations. With class sizes in the mid-teens, each student gives at least two presentations and I provide them individual feedback about how to improve their oral communication. Sometimes I also collaborate with CTL to record student presentations. After watching the video, each student evaluates their own presentation, providing both a descriptive analysis and what grade they would assign themselves based on the rubric.

These presentations vary in their complexity and length, and consist of teaching an assigned collection of ideas from one of our books, presenting biographical sketches, describing and analyzing historical events, or teaching some mathematics. I usually assign my students their presentation topics. Although, for my sophomore-level mathematics course, my students help determine their topic(s): they rank their top three choices from a list of possible topics and I maximize the overall number of preferences.

Research Projects: My students usually complete an independent research project. They create an annotated bibliography; develop a fact list, outline, or timeline (as appropriate); present their findings to the class; and submit a paper discussing what they learned.

Lessons Learned

When I began designing Mathematical Impossibilities, I thought of the entire process as an experiment. I ask my students to explore some deep and subtle mathematical topics and, before offering the course, I was hopeful, but unclear, about how well they would manage. Overall, my students consistently rise to the challenge and they learn both more and better than I had originally expected. At the same time, I also adapt to and accept a spectrum of results in the depth of my students' understanding.

For student presentations, I recognized the need for some pre-presentation vetting prior to each student's initial presentation. Some students are simply outstanding, while others struggle with both the content and the process of giving their presentation. As might be expected, first-year students generally require more mentoring than upper-level students, although I found that mathematics majors and minors can exhibit a pretty broad spectrum of oral communication skills. I use various approaches to improving first presentations, such as having my students read and discuss what makes a good presentation, or submit fact lists or outlines prior to their presentation. Some students respond best to a personal conversation about how to improve.

The next time I teach Mathematical Impossibilities I plan to incorporate a Sigma-Delta component into the feedback process for presentations, where "sigma" is a play on the sigma-notation Σ for the addition operation and our use of "+" for positive numbers, and "delta" for change is drawn directly from calculus. After each student's presentation, the rest of the class will complete an online Sigma-Delta Survey that can address academic content, clarity of the ideas, mechanics of presenting, or the speaker's presence and engagement with the class, with the goal of providing detailed peer feedback to help each student reflect on and improve their oral communication skills. After reviewing this peer feedback, I will share the results of these surveys with the presenter.

Advanced planning is essential for my success in teaching Mathematical Impossibilities. During a full term, I would distribute my preparation throughout the term, but the compact nature of a January-term course requires significant pre-planning, including some work during the previous summer (at least for me). My preparation includes personal reading about the various topics accompanied by reflection about how I could teach specific topics to the audience of students in a particular class. I also create various reading guides, class handouts, and exercise sets prior to teaching the class.

Despite my pre-planning for the sophomore-level mathematics course, I ran out of time at the end of the term while teaching the unsolvability of the quintic by radicals. I made the choice to gloss over the evolving presentation of the quadratic formula in western Europe, the work of Chinese, Indian, and Islamic mathematicians, and Lagrange's work foreshadowing Abel's theorem. I hope to explore these topics more fully the next time I teach this impossibility.

Another mathematical topic that could be included in this course is Arrow's impossibility theorem. This result addresses the challenge of combining the individual preferences of the members of a group into a choice for the entire group. Voting is the most common approach to creating a single corporate choice, and May proved that majority rule is the best voting system for exactly two candidates. However, groups often need to make choices among three or more candidates. In 1950, Arrow identified a collection of desirable criteria for voting systems, each of which seems reasonable on its own, but found that a dictatorship is the only "voting" system for three or more candidates that simultaneously satisfies all of these criteria. I believe this impossibility from the social sciences would be interesting to many students.

Finally, these Mathematical Impossibilities courses are still very much works in progress. I first learned many of these ideas from the perspective of a logician trying to understand their mathematical content. Over time, I developed an interest in the rich history surrounding them and I am still very much a student of that history. I aspire to present this history as best I can each time I teach this course and, in between course offerings, I continue my studies in hopes of offering a better course the next time around.

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Appendix A

Course Outline

The following outline of Mathematical Impossibilities corresponds to a sophomore-level mathematics course with a prerequisite of either Calculus II or a transition to proofs course. For this group, I chose to study the unsolvability of the quintic by radicals, the independence of the parallel postulate and non-Euclidean geometries, Gödel's incompleteness theorems, and the independence of the continuum hypothesis. I would give three midterm exams and a final exam, one after each of these impossibilities.

Week 1: Introduction to the course; finding zeros of linear, quadratic, depressed cubic, cubic, and quartic polynomials; and proofs for the quadratic and depressed cubic algorithms.

Week 2: Unsolvability of the quintic by radicals; recollections and overview of the historical story; and student biographical presentations for the unsolvability of quintic by radicals.

Week 3–4: Basic Euclidean geometry: definitions, common notions, postulates, and statements (some with proofs, all with geometric illustrations) of theorems 1, 2, 3, 5, 9, 10, 11, 13, 15, 16, 23, 27, 28, 29, 30, 31, 32, 34, 37, 41, 46, and 47 (the Pythagorean theorem).

Week 5: Variations on the parallel postulate; proof that Wallis's postulate is equivalent to the parallel postulate; negations of Playfair's axiom; mathematical models as a means for proving (relative) consistency; the sphere model of elliptic geometry and both the Beltrami disk and Poincaré disk models of hyperbolic geometry; and some results and proofs from hyperbolic geometry.

Week 6: Recollections and overview of the historical story, and student biographical presentations for the independence of the parallel postulate.

Week 7: Discussion of Nagel and Newman's *Gödel's Proof* [30] (see Appendix F for details); paradoxes and self-reference; basics of sentential logic, including translations and truth tables; and Knights and Knaves puzzles from Smullyan [39].

Week 8: Arithmetization of syntax and meta-mathematics; sketch of proof of Gödel's incompleteness theorems based on Crossley, et al. [5]: computable functions, correspondence lemma, ω-consistency, the Gödel sentence G, a proof that Peano Arithmetic does not deduce either G or ~G, and proof sketches of Gödel's incompleteness theorems; and the Gödel-Rosser sentence.

Week 9: Recollections and overview of the historical story, and student biographical presentations for Gödel's incompleteness theorems.

Week 10: Perspectives on infinity; paradoxes and resolutions; one-to-one, onto functions; series; and the cardinality and arithmetic of infinite sets.

Week 11: Uncountability of the interval (0,1) and the set of all real numbers via Cantor diagonalization; the power set operation and Cantor's theorem; and the continuum hypothesis and its independence over ZFC set theory because of results due to Gödel and Cohen.

Week 12: Recollections and overview of the historical story, and student biographical presentations for the independence of the continuum hypothesis.

Appendix B

Details for Non-Euclidean Geometries

Topics of Study for non-Euclidean geometries: I choose topics from among the following and we study them in roughly the given order.

- The distinction between inductive and deductive reasoning: for inductive reasoning, we consider the initial evidence that all odds greater than one are prime (so, 3, 5, and 7) with the eventual "discovery" that they are not (thanks to 9), while for deductive reasoning, we state specific examples providing evidence that the sum of two evens is even, followed by the deductive proof that this claim holds for all even integers.
- An introduction to Euclid's *Elements*, the axiomatic approach to mathematics, and proofs of multiple theorems from Euclid's *Elements*: we work through a handout discussed below that is based on Greenberg [17], Heath [13], and Dunham [11].
- The ancient Greeks' concerns with Euclid's fifth postulate: we examine the fact that indefinite extension does not ensure the intersection of curves that approach each other, as in the case of $f(x) = 1/x$ and the x-axis that is discussed in Gray [15].
- Alternative axioms equivalent to Euclid's fifth postulate, including Proclus's intersecting lines axiom, the equidistant postulate, Playfair's axiom, the triangle postulate, Wallis's similar triangle postulate, and Farkas Bolyai's circle postulate: these axioms can be found in Gray [16], Greenberg [17], and Rosenfeld [33].
- A proof that at least one of these axioms is equivalent to Euclid's fifth postulate: I share the proof for Wallis' similar angle postulate in Gray [16] and Greenberg [17].
- Attempts at a proof by contradiction by Saccheri, Lambert, and Legendre: see Gray [15] and [16], and Greenberg [17].
- The negation of uniqueness and then existence in Playfair's axiom to produce non-Euclidean geometries: see Chapter 5 of Davis and Hersh [9].
- The consistency of non-Euclidean geometries by means of the sphere model of elliptic geometry and both the Beltrami disk and Poincaré disk models of hyperbolic geometry: these models are discussed in Gray [16], Greenberg [17], and Chapter 2 of Nagel and Newman [30]; for elliptic geometry, I bring beach balls and markers to class, and the students sketch antipodal points, great circles, and 270° triangles on these spheres.
- Proofs of results in at least one of these non-Euclidean geometry: my class works with hyperbolic geometry as outlined in the class handout below, which was based on Greenberg [17].
- Comparing and contrasting the results of Euclidean, elliptic, and hyperbolic geometry about intersections of lines, the upper angles of Saccheri quadrilaterals, and more, as well as the connection to Spinoza's and other philosophers' claims of certain knowledge: see Davis and Hersh [9] and Gray [15].
- The "Euclidean Myth" described in Chapter 7 of Davis and Hersh [9], which claims that Euclid's geometry provides an exact, undoubtable model of our physical universe and the process of verifying mathematical models of reality.
- The application of elliptic geometry to airplane travel and whale communication, which is discussed in Antonick [2].
- The application of hyperbolic geometry in Einstein's theory of special relativity.

- The connection between hyperbolic geometry and Escher's artwork: see Emmer and Schattschneider [12].
- The definition of special curves, including Peano's space-filling curve and Koch's snowflake curve: see Chapter 11 of Maor [28].

Presentations for non-Euclidean Geometries: My students present the life and work of mathematicians chosen from among Thales of Miletus, Pythagoras, Plato, Euclid, Proclus, Girolamo Saccheri, John Playfair, Johann Lambert, Adrien-Marie Legendre, Carl Friedrich Gauss, Farkas Bolyai, János Bolyai, Nikolai Lobachevsky, Bernhard Riemann, Eugenio Beltrami, Felix Klein, Henri Poincaré, Albert Einstein, or M.C. Escher.

Class Handouts for non-Euclidean Geometries: We work through a pretty extensive handout about the ideas and proofs found in Book I of Euclid's *Elements*, which I have created by blending Euclid's exposition from Heath [13] and Hilbert's refinements from Greenberg [17]. We begin with definitions, common notions, and postulates, and then work through the proofs of propositions 1, 2, 3, 5, 9, 10, 11, 12, 13, 27, 28, 31, 46, and 47 (the Pythagorean theorem). For the proofs of propositions, I give them step-by-step arguments (as illustrated for Proposition 1 below) and, during class and for homework, the students justify each step using definitions, common notions, postulates, and known propositions. We also carry out the corresponding unmarked straightedge and compass construction. Most students really enjoy revisiting their experiences in high school geometry. For homework, the students also prove proposition 15 and parts of propositions 16 and 47 from scratch. In addition, we discuss the statements of propositions 23, 29, 30, 32, 34, 37, 41, and 48 because of their role in the proof of the Pythagorean theorem. Commentary about the following examples drawn from class handouts is provided in italicized text.

Proposition 1: *Given a finite line, construct an equilateral triangle with this line as one side.*

Proof. We work through the following proof, justifying each step as using an assumption, definition, common notion, or postulate. We then sketch the corresponding construction.

1) Let AB be the given finite line.

2) Construct a circle with center A and radius AB.

3) Construct a circle with center B and radius AB.

4) Let C be the point of intersection of the circles.

5) Draw AC and BC.

6) $AC = AB$.

7) $BC = AB$.

8) $AC = BC$.

For our deductive work in hyperbolic geometry, I distribute a class handout with the following statements of results, each of which is accompanied by a step-by-step argument. During class, we justify each step with particular attention to when we use the hyperbolic postulate. For the following, recall that neutral geometry consists of Euclid's common notions, definitions, and first four postulates (i.e., an extended collection of postulates, not including the parallel postulate).

We assume *Postulate H*, which negates the uniqueness part of Playfair's axiom: Given line AB and point P not on AB (or AB extended), there exist lines WPX and YPZ, called the *asymptotic parallels* to AB, such that the following hold:

- YPZ is not a line,
- WPX and WPZ are parallel to AB, and

- no line through angle *YPZ* is parallel to *AB*.
  They sketch diagrams for this and the other statements on their handouts, and they present these sketches at the board.

Proposition (using H): *Assuming neutral geometry and Postulate H, every straight line that enters angle ZPX is parallel to AB.*

Proof. We work through the following proof, justifying each step as using an assumption, definition, common notion, or postulate. We then sketch the corresponding construction.

1) Let *PC* enter angle *ZPX*.
2) Suppose *PC* is not parallel to *AB*.
3) *PC* (or *PC* extended) intersects *AB* (or *AB* extended) at *D*.
4) Draw *PQ* perpendicular to *AB* (or *AB* extended).
5) *PX* (or *PX* extended) intersects *QD*.
6) But *PX* extended does not intersect *QD*.
7) Contradiction—*PC* is parallel to *AB*.

Proposition 14 (from Euclid's *Elements* but using only neutral geometry): *If two lines, not lying on the same side, make the adjacent angles equal to two right angles at some point on the line, then the two lines will be in a line with each other.*

Proposition (using H): *Assuming neutral geometry and Postulate H, the asymptotic parallels through a point to a line make equal, acute angles with the perpendicular from the point to the given line.*

I provide a step-by-step argument, and we work through the proof justifying each step.

Appendix C

Details for Impossible Constructions

Topics of Study for Impossible Geometric Constructions: I choose topics from among the following and we study them in roughly the given order.

- Comparing and contrasting existence with constructability: finding zeros of polynomials is a familiar context for making this distinction, as in Chapter 4 of Davis and Hersh [9].
- Introduction to Euclid's *Elements* and various constructions from Euclidean geometry: when teaching this impossibility after studying non-Euclidean geometries, I remind them about angle bisection and constructing a square on a given line segment; otherwise, we carry out these and other constructions to develop a familiarity with Euclid's methods, as described in Heath [13] and Greenberg [17].
- The three impossible construction questions of the ancient Greeks: I highlight the "rules" of using only an unmarked straightedge and compass, sketch suggestive diagrams, and physically illustrate cube-doubling by smashing together a couple of cubes made out of Play-Doh or modeling clay.
- Fermat numbers and Fermat primes: see Greenberg [17] and Jones, et al. [25] for this slice of number theory that appears in the statement of the Gauss-Wantzel theorem, which also allows me to highlight the surprising interrelationships that sometimes appear among seemingly unrelated mathematical ideas.
- The Gauss-Wantzel theorem, which identifies the constructability of regular n-sided polygons with particular types of factorizations of integers n: we consider specific examples and discuss constructions for $n = 3$ and 4, as in Cajori [4] and Greenberg [17].
- The definition and examples of constructible real numbers: see Kazarinoff [27].
- The nonconstructibility theorem, which states that if a cubic polynomial with rational coefficients does not have a rational zero, then none of its zeros is constructible: we apply this result to $x^3 - 2$ to demonstrate that cube doubling is impossible with an unmarked straightedge and compass, as in both Jones, et al. [25] and Kazarinoff [27].
- The definitions and examples of algebraic and transcendental numbers, and their relationship with constructible numbers: see Johnston and McAllister [24].
- The definition and application of alternative construction tools including the quadratrix of Hippias, the conchoid of Nicomedes, the cissoid of Diocles, and the spiral of Archimedes from Heath [19], as well as the paperfolding of Row [34], which is equivalent to marked straightedge and compass constructions.

Student Presentations for Impossible Geometric Constructions: My students present the life and work of mathematicians chosen from among Thales of Miletus, Hippias, Nicomedes, Diocles, Plato, Euclid, Archimedes of Syracuse, Pierre de Fermat, Marin Mersenne, René Descartes, Leonhard Euler, Carl Friedrich Gauss, Pierre Wantzel, or David Hilbert.

Class Handouts for Impossible Geometric Constructions: The following prompts and exercises appear on some of my class handouts when we study this impossibility. These prompts focus on the question of cube doubling.

Sketch an image corresponding to the following.

1) double the cube 2) trisect the angle 3) square the circle

Provide a compass and unmarked straightedge construction for the following.

1) bisect an angle 2) bisect a line 3) double a square

Algebraically show that doubling a square is equivalent to constructing the square root of two.

Similarly, show that doubling the cube is equivalent to constructing the cube root of two.

The nonconstructibility theorem: *if a cubic polynomial with rational coefficients does not have a rational zero, then none of its zeros is constructible.*

Define and give examples of the following.

1) polynomial 2) cubic polynomial 3) rational number

4) cubic with rational coefficients 5) zero of a polynomial 6) constructible number

Apply the nonconstructibilitity theorem to $x^3 - 2$. What does this tell us about cube doubling?

Appendix D

Details for the Unsolvability of the Quintic

Topics of Study for the Unsolvability of the Quintic by Radicals: I choose topics from among the following and we study them in roughly the given order.

- The fundamental theorem of algebra, which ensures the existence of complex zeros for polynomials with complex coefficients.
- The distinction between knowing zeros exist and finding zeros, including the distinction between existence and constructive results: see the discussion in Davis and Hersh [9].
- Babylonian and Egyptian statements and solutions of what we now know as linear and quadratic equations: see Katz [26] for the mathematics of Babylonian, Egyptian, Chinese, Indian, and Islamic mathematicians.
- The gradual introduction of our modern symbolic notation for algebra, highlighting Islamic and Renaissance mathematicians, Viète's pivotal work, and the algebraic geometry of Descartes and Fermat: see Boyer and Merzbach [3] and Katz[26].
- Comparing and contrasting solutions to specific versus general equations, and the quadratic formula as a solution by radicals.
- The depressed cubic, cubic, and quartic formulas providing solutions by radicals for the corresponding polynomials: both Boyer and Merzback [3] and Dunham [11] detail these formulas and their proofs, and provide good examples.
- Permutations and Lagrange's resolvent: see Pesic [32], which I supplement with the modern study of permutations as in Johnston and McAllister [24].
- Abel's theorem, its meaning, and the work of Ruffini, Abel, and Galois: see Pesic [32].
- Alternative methods for finding and approximating zeros.

Student Presentations for the Unsolvability of the Quintic by Radicals: My students present the life and work of mathematicians chosen from among Brahmagupta, Bhāskara, Muhammed ibn Mūsā al-Khwārizmī, Omar Khayyam, Sharaf al-Dīn al-Tūsī, Leonardo de Pisa (Fibonacci), Sciopione del Ferro, Niccolò Fontana Tartaglia, Girolamo Cardano, Lodovico Ferrari, François Viète, René Descartes, Pierre de Fermat, Carl Friedrich Gauss, Jean d'Alembert, Jean Argand, Joseph-Louis Lagrange, Paolo Ruffini, Niels Abel, or Évariste Galois.

Class Handouts for Unsolvability of the Quintic by the Radicals: The following prompts and exercises appear on some of my class handouts when we study this impossibility. Commentary about these examples from class handouts is provided in italicized text.

Definitions and Examples

1) polynomials 2) degree 3) zero

We collaborate in stating definitions and giving examples.

Fundamental theorem of algebra

I state this result and discuss its history.

Finding zeros of linear polynomials

I ask them for the linear formula they already know from basic algebra and then discuss the development of algebraic notion.

1) From the Egyptian Moscow Papyrus (ca. 2000 BCE): Find the number such that if it is taken one and one half times and then four is added, the sum is ten.
2) From the Egyptian Rhind Papyrus (ca. 2000 BCE): Find a quantity such that the sum of itself, its two-thirds, its one-half, and its one-seventh becomes thirty five.
3) From Babylonian Tablet YBC 4652 (ca. 2000 BCE): I found a stone, but did not weigh it; after I added one-seventh and then one-eleventh of this total, it weighted sixty gin. What was the original weight of the stone?
4) Find the zeros of each linear polynomial.

   a) $3x - 9$ b) $5x + 3i$ c) $(4 + i)x - (7 - I)$

   The students present their solutions at the board.

Finding zeros of quadratic polynomials

Together we recall the modern quadratic formula, and then I share the diverse, evolving expressions of this formula by Babylonian, Egyptian, Greek, Islamic, Renaissance, and French mathematicians.

1) From the Babylonians (ca. 2000 BCE): Find the side of a square if its area less its side is 14,30. Note: 14,30 is a sexagesimal number.

   We discuss various bases for numeric systems, worked through the Babylonians' solution and then the modern solution.

2) Find the zeros of each quadratic polynomial.

   a) $x^2 - 1$ b) $x^2 + x + 1$

   The students present their solutions at the board.

Finding zeros of depressed cubic polynomials of the form $x^3 + mx + n$.

I state the del Ferro-Tartaglia formula and discuss its history.

Find the zeros of each depressed cubic polynomial.

1) $x^3 - 3x + 2$ 2) $3x^3 - 36x + 48$

The students present their solutions at the board.

Finding zeros of cubic polynomials

I state the Cardano transformation.

Find the zeros of each cubic polynomial.

1) $2x^3 - 30x^2 + 162x - 350$ 2) $x^3 - 6x^2 + 11x - 6$

The students present their solutions at the board.

Working toward a proof of the quadratic formula, complete the square .

1) $x^2 + 2x = 0$ 2) $4x^2 + 8x = 0$ 3) $x^2 + 6x + 3 = 0$ 4) $2x^2 + 8x + 14 = 0$

The students present their solutions at the board.

Prove the quadratic formula. *We study the "complete the square" proof.*

Prove the depressed cubic formula. *See Cardano's proof in Chapter 6 of Dunham [11].*

Define a solution by radicals. *I state this definition and we consider examples.*

State Abel's theorem. *I state this theorem, discuss its meaning, and describe its history.*

Appendix E

Details for the Continuum Hypothesis

Topics of Study for the Continuum Hypothesis: I choose topics from among the following and we study them in roughly the given order.

- The infinitude of the primes based on the proof given in Euclid's *Elements*: this topic serves as an introduction to proof by contradiction (we also consider simple proofs about the arithmetic of even and odd numbers), and also provides links to the two geometric impossibilities (if also studied); see Heath [19] for Euclid's argument.
- Zeno's paradoxes of motion and a resolution using geometric series: my students act out Zeno's paradox of Achilles and the tortoise, and we discuss the geometric series resolution of this paradox by recalling these ideas from Calculus II or by sketching diagrams, depending on the course prerequisites; see Heath [19] for just Zeno's paradoxes or Sainsbury [36] for multiple paradoxes, including Zeno's and Russell's paradoxes.
- Aristotle's distinction between potential and actual infinity: see Mendell [29].
- Archimedes' method by exhaustion in comparison with the Riemann integral.
- Galileo's paradox of squares and the part-whole principle for finite sets (if A is a proper subset of B, then the cardinality of A is strictly less than the cardinality of B) which appears in his *Two New Sciences*: see Maor [28] and Rucker [35].
- Fourier series leading to Cantor's study of the infinite: Cantor's initial research in this area of analysis is discussed in Dauben [7].
- The definitions and examples of one-to-one, onto functions, including some basic proofs: these ideas appear in many math textbooks, as in Johnston and McAllister [24].
- The definition and examples of infinitely countable sets, including the resolution of Galileo's paradox of squares: we work with either one-to-one correspondences or with the beginning of infinite lists, depending on the course prerequisites.
- The basic arithmetic of finite and infinite countable cardinalities: Dunham [11] discusses transfinite arithmetic and the Hilbert hotels of Smullyan [38] provide an effective approach to teaching these ideas.
- Cantor diagonalization and applying this method to specific lists of decimal expansions of rational numbers.
- Cantor's proof that the set of reals is uncountable using his diagonalization method: this proof appears in many transitions to proofs textbooks, such as Johnston and McAllister [24], and in many real analysis textbooks, such as Abbott [1].
- The power set operation, the resulting strict increase in cardinality, and an infinite ascending chain of uncountable cardinals based on the power set operation.
- An overview of the cardinality of familiar number systems: integers, primes, rationals, reals, irrationals, algebraic, transcendental, and complex numbers.
- The idea of independence: as an example, we recall the independence of Euclid's fifth postulate over neutral geometry, because it can be neither proven nor disproven from just this part of Euclidean geometry: see Gray [15] and Nagel and Newman [30].
- The continuum hypothesis and its independence over ZFC set theory via results due to Gödel and Cohen: see Dauben [7] and Davis and Hersh [9].

Student Presentations for the Continuum Hypothesis: My students present the life and work of mathematicians chosen from among Zeno of Elea, Aristotle, Euclid, Archimedes of Syracuse, Gregory St. Vincent, Galileo Galilei, Sir Isaac Newton, Gottfried Leibniz, George Berkeley, Joseph Fourier, Peter Lejeune Dirichlet, Richard Dedekind, Georg Cantor, David Hilbert, Leopold Kronecker, Kurt Gödel, Paul Cohen, or Abraham Robinson. In addition, for the first-year seminar version of this course, I used Maor [28] as one of my textbooks and I had each student teach one or two chapters from this book to the class.

Class Handouts for the Continuum Hypothesis: The following headings and exercise prompts appear on some of my class handouts when we study this impossibility. Commentary about these examples from class handouts is provided in italicized text.

Define the following and sketch an example and a non-example of each.

1) function 2) one-to-one function
3) onto function 4) one-to-one correspondence

We collaborate in stating definitions and sketching examples and non-examples.

If they exist, identify all one-to-one functions, all onto functions, and all one-to-one correspondences with the given domain and target; if not, explain why such a function does not exist. Consider the sets: $A = \{1, 2\}$, $B = \{a\}$, $C = \{\beta, \gamma, \varepsilon\}$, $D = \{c, d\}$, $E = \{2\}$.

1) $f : A \to B$ 2) $g : B \to A$ 3) $h : A \to C$
4) $j : C \to A$ 5) $k : A \to D$ 6) $m : D \to A$
7) $n : C \to B$ 8) $p : B \to E$ 9) $q : D \to E$

I introduce these and the students present their solutions at the board.

Prove the following functions are one-to-one and onto, or identify counterexamples showing that one (or both) of these properties fail.

1) $f\colon \mathbb{N} \to \mathbb{N}$ where $f(x) = 2$ 2) $f\colon \mathbb{N} \to \mathbb{N}$ where $f(x) = 4x + 3$
3) $f\colon \mathbb{R} \to \mathbb{R}$ where $f(x) = x - 3$ 4) $f\colon \mathbb{R} \to \mathbb{R}$ where $f(x) = x^2 - 3$
5) $f\colon \mathbb{R} \to \mathbb{R}$ where $f(x) = x^3$ 6) $f\colon \mathbb{Z} \to \mathbb{Z}$ where $f(x) = x^3$

I introduce these and the students present their solutions at the board.

State the following definitions.

1) cardinality 2) infinite 3) finite

I state these definitions and we collaborate in finding examples.

Some Examples of Countable Sets

1) the set $\mathbb{N}$ of natural numbers, 2) the set of evens
3) the set of primes 4) the set $\mathbb{Z}$ of integers
5) the set $\mathbb{Q}$ of rationals 6) $2 + \omega$
7) $5 + \omega$ 8) $\omega + \omega$

We collaborate in proving these sets are countable. In 6, 7, and 8 above, I use ω to denote the countable cardinal aleph-naught.

Define "uncountable" set. *I state this definition along with some examples.*

Theorem: *The interval* $(0, 1)$ *of real numbers is uncountable.*

We prove this result, first spending time with some example Cantor diagonalizations.

Practice Cantor diagonalization. *The students present their solutions at the board.*

1) $a_1 = 0.22224455\ldots$
   $a_2 = 0.32132132\ldots$
   $a_3 = 0.00900000\ldots$
   $a_4 = 0.99999999\ldots$
   $a_5 = 0.14159267\ldots$
   $a_6 = 0.77777777\ldots$

2) $a_1 = 0.12233344\ldots$
   $a_2 = 0.44444444\ldots$
   $a_3 = 0.01210121\ldots$
   $a_4 = 0.11991199\ldots$
   $a_5 = 0.12344444\ldots$
   $a_6 = 0.75757575$

More uncountable sets. *We collaborate in proving these sets are uncountable.*

1) the interval of reals $(1,2)$
2) the interval of reals $(0,2)$
3) the set $\mathbb{R}$ of all reals
4) the set of irrationals
5) the set of transcendental numbers

Define the power set operation. *I state this definition and give a couple of examples.*

Find the power set of each set. *The students present their solutions at the board.*

1) $\{ a, b \}$ 2) $\{ 1, 2, 3 \}$ 3) the empty set   4) $\mathbb{N}$ of natural numbers

Cantor's theorem: *The cardinality of a set is strictly less than the cardinality of its power set.*

I state and prove this result, and then apply this result to the set $\mathbb{N}$ of natural numbers repeatedly to obtain an infinite ascending chain of infinite cardinals.

Appendix F
Details for the Incompleteness Theorems

Topics of Study for Gödel's Incompleteness Theorems: I choose topics from among the following and we study them in roughly the given order.

- Issues that arise with trusting our intuitions about "space" from geometry, "collections of objects" from set theory, "reasoning" from logic, and "number" from arithmetic as possible sources of certain knowledge or as a foundation of mathematics: see Davis and Hersh [9] for an overview.
- Aristotle's syllogisms: Smith [37] discusses Aristotle's logic, including his syllogisms
- The basics of propositional and predicate logic; I blend Chapter 1 of Johnston and McAllister [24] with Hamilton [18].
- Frege's logicism, Russell's paradox, and Hilbert's formalism: see Davis and Hersh [9].
- Hilbert's second problem asking for a proof that arithmetic is free of contradictions: Yandell [40] discusses work on all of Hilbert's problems.
- Defining and illustrating the properties of soundness, completeness, and consistency: see Nagel and Newman [30] for an overview and Hamilton [18] for a details.
- Paradoxes arising from self-reference, particularly the liar paradox: see Sainsbury [36] for a detailed discussion and Smullyan [39] for multiple related puzzles, of which the students particularly enjoy the "Island of Knights and Knaves."
- The arithmetization of syntax (or Gödel-numbering) obtained by assigning numbers to arithmetical statements: see Hamilton [18] and Nagel and Newman [30].
- Algorithmically computable functions providing Gödel's general recursive functions: see Cutland [6], Hamilton [18], and Nagel and Newman [30].
- Stating and discussing the Gödel sentence, which is interpretable as "I am not provable".
- The statement, meaning, and non-meanings of Gödel's incompleteness theorems: see Crossley, et al. [5], Davis and Hersh [9], Nagel and Newman [30], and Rucker [35].
- Proof sketches of Gödel's incompleteness theorems: I follow the outline given in Crossley et al. [5] with some adaptations based on Nagel and Newman [30].

Student Presentations for Gödel's Incompleteness Theorems: I give an overall historical sketch for this impossibility and share its history throughout our mathematical study. Also, my students present the life and work of mathematicians chosen from among Plato, Aristotle, Peter Abelard, Gottfried Leibniz, Augustus De Morgan, George Boole, Gottlob Frege, Gerhard Gentzen, David Hilbert, Bertrand Russell, Alfred North Whitehead, Giuseppe Peano, Ludwig Wittgenstein, Moritz Schlick, Kurt Gödel, John von Neumann, L. E. J. Brouwer, Hermann Weyl, Karl Popper, or J. Barkley Rosser.

Reading and Discussion Questions for Nagel and Newman's *Gödel's Proof* [30]: When studying this impossibility, I assign readings from *Gödel's Proof* for homework and my students write their responses to the following questions for homework. For our class meetings, students share and discuss their responses, and present solutions to exercises at the board. I fill in any gaps that show up along the way, although I try to keep the focus on peer-instruction.

Foreward

1) Describe Russell and Whitehead's "noble vision."

2) State the liar paradox and Gödel's rendition of this statement in the context of *Principia Mathematica*. Why is Gödel's statement not a paradox?
3) Discuss how models for the human mind have changed since Gödel's work.

I: Introduction

1) Describe the "axiomatic method" of mathematics.
2) What impact did Gödel's work have on our understanding of the axiomatic method?

II: The Problem of Consistency

1) State three questions of the ancient Greeks that were resolved during the 19th century. How were they resolved?
2) According to *Gödel's Proof*, why was Euclid's parallel postulate not "self-evident"?
3) State three important intellectual outcomes of the development of non-Euclidean geometry. Based on these outcomes, how does *Gödel's Proof* define mathematics?
4) Define "consistent" and "model."
5) State the consistency result for elliptic geometry that follows from the sphere model.
6) How did Hilbert try to prove the consistency of Euclidean geometry?
7) State Russell's antimony. How is this relevant to the question of consistency?

III: Absolute Proofs of Consistency

1) Describe Hilbert's "complete formalization" of a deductive system. How is this helpful?
2) Define "metamathematics." Give an example contrasting mathematical and metamathematical statements.
3) How does metamathematics help us work toward an absolute proof of consistency?

IV: The Systematic Codification of Formal Logic

1) What kind of "gaps" does *Gödel's Proof* identify in Euclid's proof that there exist infinitely many primes?
2) Describe the goal of Boole's study of logic.
3) Describe the goal of Russell and Whitehead's *Principia Mathematica*.

V: An Example of a Successful Absolute Proof of Consistency

1) For a formal system, define vocabulary, formation rules, transformation rules, axioms, theorem, and proof.
2) State the vocabulary, formation rules, transformation rules, and axioms for the sentential calculus.
3) What observation follows from $p \supset (\sim p \supset q)$ being a theorem of sentential calculus?
4) Describe the "hereditary" approach to an absolute proof of consistency.
5) What hereditary property provides an absolute proof of consistency for sentential calculus? Define and give an example of this property.
6) Define "complete" formal system.

VI: The Idea of Mapping and Its Use in Mathematics

1) State the two main conclusions that follow from Gödel's work.
2) State the Richardian paradox. How is this paradox resolved?
3) Define and give an example of a "map."
4) State the second sentence in the last paragraph on page 66. This is important.

VII: Gödel's Proofs, Part A: Gödel-numbering

1) State the Gödel-numbering given on page 70.
2) How can we identify interpretations for the "meaningless" symbols of a formal system?
3) State the content of Gödel's "correspondence lemma."
4) State the three kinds of variables and their Gödel-numbering.
5) How do we Gödel-number a formula?
6) Give an example of an integer that is not a Gödel-number. Explain your answer.
7) Gödel-number the following.
    - a) $x\ (0 < Sx)$
    - b) $x\ (0 < Sx) \to (0 < S0)$
    - c) $0 < S0$
    - d) $x = x$
    - e) $x(x = x)$
    - f) $x(x = x) \to (p = p)$
    - g) $p = p$
    - h) $x(0 + x = x)$
    - i) $xy(x + y = 0)$
    - j) $x(Sx = SS0)$

VII: Gödel's Proofs, Part B: The Arithmetization of MetaMathematics

1) What is the basic idea behind Gödel's "ingenious" mapping?
2) What is meant by the "arithmetization of metamathematics"?
3) Define dem(x,z) and sub(x,17,x).

VII: Gödel's Proofs, Part C: The Arithmetization of MetaMathematics

1) Describe the five steps in Gödel's argument.

VIII: Concluding Reflections

1) State a couple of the concluding reflections that you find interesting.

Class Handout for a Proof Sketch of Gödel's Incompleteness Theorems based on Crossley, et al. [5]: After discussing Nagel and Newman [30], I present a proof sketch of Gödel's two incompleteness theorems that follows the outline given below.

- State the meaning of the predicate denoted by $Pf^{+}(x, y, z)$.
- State formal and intuitive versions of the Gödel sentence $G(g)$ for Peano arithmetic (PA). Define all symbols you use in your predicate.
- State and prove Gödel's second incompleteness theorem.
- Prove that if PA is ω-consistent, then PA does not deduce $G(g)$.
- Prove that if PA is ω -consistent, then PA does not deduce the negation of $G(g)$.

Appendix G

Details for the Halting Problem

Topics of Study for the Halting Problem: I choose topics from among the following and we study them in roughly the given order.

- Algorithms and algorithmic processes, including calculus as a collection of algorithms: Davis [8] describes our evolving understanding of the algorithmically computable.
- Hilbert's Entscheidungsproblem, asking for an algorithm that could identify a given mathematical statement as true or false: see Davis [8] and Hodges [22].
- The definition of Turing machines: see Cutland [6] and Chapter 7 of Hamilton [18] for a definition, examples, and more details about Turing computability.
- The process of the computations of given Turing machines and then defining Turing machines to compute basic functions, such as $x + 3$ and $2x$: see Hamilton [18].
- The Church-Turing thesis stating that every algorithmically computable function is computable by a Turing machine: see Cutland [6] and Davis [8].
- An enumeration of all computable functions obtained by assigning Gödel-numbers to Turing machines.
- Partial functions whose domains are a proper subset of the natural numbers.
- Cantor diagonalization applied to specific lists of computable functions.
- The statement and proof of the halting set theorem (i.e., there does not exist an algorithm for determining whether a Turing machine with Gödel-number n halts on input n) by means of Cantor diagonalization applied to an enumeration of all Turing computable functions.
- The definition and examples of equivalence relations, oracle Turing machines, the Turing equivalence relation on sets of natural numbers, and Post's problem about the relative computability of sets based on Turing equivalence: see Cutland [6].

In addition, because Turing was actively involved in England's code-breaking efforts during World War II, I usually include topics chosen from among the following.

- The basics of cryptography in the context of the Caesar cipher.
- Prime numbers, modular arithmetic, and RSA cryptography: these ideas can be found in multiple textbooks, such as Johnston and McAllister [24].
- The Germans' Enigma code from World War II and the code-breaking work of Polish and English mathematicians: see Hinsley and Stripp [21], and NOVA Online [31].

Student Presentations for the Halting Problem: My students present the life and work of mathematicians chosen from among Euclid, Muhammed ibn Mūsā al-Khwārizmī, Gottfried Leibniz, Sir Isaac Newton, David Hilbert, Kurt Gödel, Alonzo Church, Marian Rejewski, Alan Turing, Emil Post, or Andrey Markov.

Sometimes my students give presentations about World War II, including such topics as the Axis powers of Germany and Italy, the Allied powers of the United Kingdom, the United States, and USSR, the Blitzkreig, the Battle of Britain, the Holocaust, U-boats and convoys, the Enigma code, Polish cryptographers, or English cryptographers at Bletchley Park. In fact, during the first-year seminar, my students gave five-minute presentations about these topics the second day of class to help me assess their baseline oral communication skills.

In addition, students could give presentations about the development of computing machines, such as biographical presentations about Blaise Pascal, Charles Babbage, Ada Lovelace, Konrad Zuse, J. Presper Eckert, John Mauchly, or John von Neumann. They could also present the development of various "first" computers. The Turing Test for artificial intelligence might also be of interest.

Class Handouts for the Halting Problem and the study of Turing Machines: The following prompts and exercises appear on some of my class handouts when we study Turing machines and the algorithmic unsolvability of the halting problem. These exercises are based on the presentation of Turing machines in Hamilton [18].

1) Fill in Turing machine tapes to reflect each value.
   a) input-output of zero b) input-output of three
   c) input-output of four d) input-output of six
2) Trace the computation of the Turing machine (s, 1, b, s_1), (s_1, b, R, s), (s, b, 1, h) on an input of three. What function does this machine compute?
3) Trace the computation of the Turing machine (s, 1, L, s_1), (s_1, b, 1, s_1), (s_1, 1, L, s_2), (s_2, b, 1, h) on an input of three. What function does this machine compute?
4) Trace the computation of the following Turing machine on an input of two. What function does this machine compute?
   (s, b, b, h), (s, 1, Y, s_1), (s_1, Y, R, s_1), (s_1, 1, R, s_1), (s_1, b, R, s_2), (s_2, 1, R, s_2), (s_2, b, 1, s_3), (s_3, 1, L, s_3), (s_3, b, L, s_4), (s_4, 1, L, s_4), (s_4, Y, R, s_5), (s_5, 1, Y, s_1), (s_5, b, L, s_6), (s_6, Y, 1, s_6), (s_6, 1, L, s_6), (s_6, b, R, h)
5) Design a Turing machine that computes the constant function $f(n) = 2$. State the idea for the algorithm you are implementing, the formal definition of the machine using quadruples, and a test run of your machine on an input of three.
6) Design a Turing machine that computes the parity function which outputs 1 if n is even and 0 if n is odd. State the idea for the algorithm you are implementing, the formal definition of the machine using quadruples, and a test run of your machine on an input of three.
7) Practice Cantor diagonalization on example enumerations φ_n of Turing computable functions. The diagonalization function is $\Phi(n) = \varphi_n(n) + 1$ when $\varphi_n(n)$ is defined and $\Phi(n) = 1$ otherwise.

	1	2	3	4	5	6
φ_1	3	1	4	1	5	9
φ_2	2	2	2	2	2	2
φ_3	9	8		8	9	
φ_4	3		3		3	
φ_5	2	4	6	8	2	4
φ_6	2	3	5	7	11	13

	1	2	3	4	5	6
φ_1	1	3	2	5	4	7
φ_2	1		2		3	
φ_3	11	11	11	11	11	11
φ_4	1	2	1	2	1	2
φ_5		4		8		4
φ_6	2	3	2	3	3	2

Class Handouts for the Halting Problem and the study of cryptography: The following prompts and exercises appear on some of my class handouts when we study cryptography in connection with Turing's code-breaking work during World War II.

1) Complete the following table for shift right two spaces encryption.

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z

a) Encrypt the plaintext: MATH IS AWESOME.
b) Encrypt a message to send a classmate. State the plaintext and ciphertext.
c) Decrypt a message from a classmate. State the ciphertext and the plaintext.

2) State the elements in the sets $\mathbb{Z}_5$ and $\mathbb{Z}_{10}$.

3) More Clock Arithmetic: Working with $n = 12$ and $\mathbb{Z}_{12}$, compute the following:
a) 4 mod 12 b) 22 mod 12
c) 96 mod 12 d) (-13) mod 12

4) Working in Z_{10}, compute the following:
a) $2 + 3$ mod 10 b) $9 + 5$ mod 10
c) $8 + 11$ mod 10 d) $7 + 6$ mod 10
e) 2×3 mod 10 f) 9×5 mod 10
g) 8×11 mod 10 h) 7×6 mod 10

5) Find the numeric string identified with each alphabetic string and the alphabetic string identified with each numeric string.
a) ALGEBRA b) TRUE LOVE
c) 05 | 21 | 12 | 05 | 18 d) 07 | 01 | 21 | 19 | 19

6) Encrypt the plaintext PEACE using the RSA code with $p = 3$, $q = 11$, and $e = 7$.

7) Decrypt the ciphertext 30 | 45 | 25 using the RSA code with $p = 5$, $q = 13$, $e = 29$, and $e^{-1} = 5$.

Some Homework Questions for the Halting Problem and the study of cryptography: After our in-class discussion of World War II and Turing's code-breaking work during World War II, my students respond to the following.

1) Describe the operation of the Enigma machine and the obstacles to deciphering Enigma-encrypted messages.
2) Outline the events leading up to the Allies breaking the Enigma code.
3) Discuss the impact of cryptography on the course of World War II.

In fact, when I taught this course as a first-year seminar during our January term, we visited both the Holocaust Museum and the NSA's Cryptologic Museum. My students submitted written responses to the above questions based on these visits and the associated class meetings.

History of Mathematics in America

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Course Overview

How many Americans associated with mathematics—especially all levels of teachers from secondary to college to research institutions, as well as practitioners such as engineers and actuaries — can identify John Winthrop, or Nathaniel Bowditch, or Charles Davies, or Benjamin Peirce? Even if a person might have some vague awareness of one or more of these American mathematicians, what was mathematics like and what, if anything, have Americans contributed to mathematics and mathematics education during our nation's history?

The History of Mathematics in America is a one-semester course whose purpose is to have students study, first-hand, the main features of the development of mathematics and mathematics education in America[1] from the Colonial era up to the early part of the twentieth century. By design, this is a *second* course in the history of mathematics, taught at the junior/senior level and which is also appropriate for graduate students in secondary mathematics education. The prerequisites are completion of the calculus sequence, a sophomore-level linear algebra bridge course, and a one-semester junior/senior-level general course in the history of mathematics as taught from a well-known textbook such as Eves, Boyer and Merzbach, Burton, Cooke, Suzuki, Berlinghoff and Gouvêa, etc. It is not uncommon that one or more of these students either has had or is currently taking a junior/senior-level course in geometry and/or a comparable course in number theory.

I have four objectives for this course. First, I want to break the mold of the mathematics course based on a single and confining textbook, organized into bite-sized sections and chapters, and containing examples and their corresponding exercises. We all know that this is not how real mathematics is done, but this fact is a shock to too many students and we need to help these students mature. The second goal is to introduce our students to serious reading, studying, struggling, and eventually coming to terms with primary sources (all in English, of course) in mathematics. Here, we emphasize *reading,* which unfortunately is not a student habit in many mathematics courses. Then, *coming to terms* is our way of having students explain mathematical concepts, orally and especially in good written English. Third, every student should become acquainted with many American mathematical pioneers, what they accomplished, and how their mathematical works compared with those in the leading European nations. Finally and most broadly, the students' horizons should be enlarged to embrace the environments—personal, historical, and more — through which mathematics has developed in America and in which our discipline now resides.

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[1] We take a rather narrow view of "America," specifically we confine our attention to the European colonies of North America that became the United States.

Course Design

The course is conducted in the style of a seminar. For each class meeting, I choose our main focus of study and the specific readings from the Historical American Mathematical Works (Appendix D). I also assign appropriate sections and chapters to be read from our textbook (see the Resources section). For the most part, we proceed chronologically.

A 15-week semester is divided as follows:

- **Part A** covers the history of mathematics in the American colonies and approximately the first century of the United States. Much of what is studied is mathematics education, at all levels during this time. There are also some applications of mathematics that were appropriate to the development of the new nation.
- **Part B** covers the history of mathematics in the United States from the founding of the *American Journal of Mathematics* at Johns Hopkins University in 1878 into the early parts of the twentieth century.

Class activities are devoted to reading, studying, discussing, and reporting on selected portions of American mathematical works, from the early eighteenth century to the early twentieth century. Details are in the Course Outline, Appendix A.

Resources

Our textbook provides a broad narrative of American mathematical history up to the start of the twentieth century.

David Eugene Smith and Jekuthiel Ginsburg, *A History of Mathematics in America Before 1900*, Arno Press, New York, 1980 (reprint of the 1934 Carus Mathematical Monograph, MAA).

We expect the students to make regular use of the library and online resources in Appendix B. Appropriate references must always be included with such usage (I recommend a variation on one of the formats in *The Chicago Manual of Style*), and any use of Wikipedia must be accompanied with at least one additional reference. For the Instructor, in Appendix C there is a list of the major resources I have used in teaching this course. For any student who expresses an interest or a special need, these works could also be appropriate.

The heart of the course is the selections made from the Historical American Mathematical Works in Appendix D. The list presented here is ambitious. But it is *not* to be taken as gospel. Alternative primary works are certainly possible. Remember, the class studies selected *excerpts* from each book or paper.

A number of the works listed here are first editions, or first American editions. Over the years, I have had the good fortune to acquire later editions of these books. For example, my copy of Bowditch is the sixth edition of 1826. Unlike many of these works whose revisions from edition to edition were either insubstantial or often virtually nil, Bowditch did undergo a good many changes up to 1867 when its publication was taken over by the United States Government. For the depths to which we study Bowditch and these other books, I believe the students can attain the goals of this course using early, if not exactly the earliest, editions. And every day, more and more historical materials are digitized and become accessible on our databases.

Assignments

For most class meetings, I assign a portion of the textbook to be read that will coordinate with the mathematical passages I select from the Historical American Mathematical Works.

At each class meeting, we focus on excerpts from one or more of the Historical American Mathematical Works. I usually prepare one or more additional historical works that complement those assigned to the students. We emphasize, these are *preparations* to be made *prior* to the class meeting: the students (and the instructor!) are expected to be able to engage in informed discussion of these readings at this class.

Each week, the students prepare a written *report* on one or two of the selections from the Historical American Mathematical Works recently studied in class. These are expository papers, approximately 2-4 pages each, whose main purpose is to discuss and analyze the mathematics in certain of the historical works read. Additional information on the context and motivation for the mathematics and some biographical information about the authors may also be included. At the start of the semester, I give the students one or two sample reports and a style sheet.

Every student has had at least one or two opportunities to present one or two of these reports orally. In keeping with the seminar nature of this course, at this time, I consider the student to be leading the class. In particular, the use of PowerPoint, handouts, visual aids, etc. is encouraged, but up to the student.

These reports could be prepared by pairs or even larger groups of students working together, but I have never done so because each of my classes was quite small and I did not believe that the dynamics of these classes would support group work. In either form, the reports are graded, and in fact constitute the great majority of the course grade.

On several occasions, I have wished that I could have helped a student convert one of these reports into a poster to take to the MAA Southeastern Section Meeting, where each year there is a large student poster session. But the timing of the course and the meeting dates have never made this possible.

I have never wanted to take the class time to give an in-class test or a mid-term. In place of a final exam, I have the students compose a comprehensive summary (approximately 8 to 10 pages, open book and notes) on the history of mathematics in America as they now understand its main features. This paper was due at the scheduled time for the final exam, it was graded and returned to the students so that they could include it with their reports in their "book"—their class notes, handouts, etc., the collection of their work for the course.

Lessons Learned

Even though the levels of mathematics in this course are not especially advanced, the technical work has not been filtered into contemporary textbook forms. This situation can be challenging for the instructor as well as the students.

Choosing and then gathering materials such as those in our list of the Historical American Mathematical Works can pose many difficulties. Our growing database resources help, and the assistance of a colleague in the Interlibrary Loan department of your college or university library is highly recommended.

Students need varying degrees of help as they approach and work through the mathematics with which they must deal in some of these primary sources. Among my students, there is a wide range of abilities to produce acceptable expository writing and to make an intelligible oral presentation, from those who are more than competent to those who have no conception of paraphrasing from what they read. Many students have grown up using text editors, which may explain why they often do not appear to value rewriting second, third, or additional drafts nor the rethinking which goes into such efforts. Reportedly, Ernest Hemingway rewrote the conclusion to *A Farewell to Arms* 39 times!

When the *USS Montgomery* (Littoral Combat Ship 8) was commissioned on September 10, 2016, I had the opportunity to ask the ship's navigator if he knew what "Bowditch" was. Standing on the *Montgomery's* bridge surrounded by the very latest array of satellite and other electronic equipment, he immediately responded, "Bowditch is the navigator's Bible, sir!" Should the need ever occur, he continued, he was fully confident in his ability to use his copy of the latest edition of Bowditch and celestial navigation to guide the *Montgomery* in any ocean of the world.

Those of us who teach in American colleges and universities have been, and many of us still are, well-acquainted with some of the most up-to-date world-wide frontiers in mathematics. This is the research we did in graduate school. In the American colonies and the United States, achieving these heights has been quite a journey. A heritage worthy of study.

Appendix A
Course Outline

Part A of the course takes approximately weeks 1–10 of the semester. Each of the following topics takes very roughly one week of class time; the interests and abilities of each class are the main reasons for variations.

- Banneker's *Almanack* is a good place to begin because the Smith and Ginsburg textbook starts with a discussion of seventeenth and eighteenth century almanacs. Additional information about Banneker is easily available, he has other notable mathematical achievements, and he was African-American. John Winthrop was Hollis Professor of Mathematics and Natural Philosophy at Harvard for more than 40 years, he was elected a Fellow of the Royal Society of London. Both of these selections call attention to the close relationship between mathematics and astronomy in the eighteenth and into the nineteenth centuries.
- A good deal can be learned by comparing Greenwood's *Arithmetick* and Pike's *New and Complete System*.
- The first few chapters of Bowditch blend pedagogical material with practical items of special interest to navigation. Bowditch can be studied in conjunction with one or two of Mansfield's *Essays* to illustrate the nature of higher mathematics in the United States, from the time of the Revolution to the early decades of the nineteenth century in contrast with that of Lagrange, Gauss, and others in Europe.
- The study of the Adams, Colburn, and Ray textbooks highlights a revolution in pedagogy. Since many of Ray's books were published in conjunction with McGuffey's *Readers*, his work relates to the general history of education in America.
- The geometry textbooks listed here range from the conservative Simson to the reforms in "Davies' Legendre" and further to the revolutionary book by Bonnycastle. Davies' *Descriptive Geometry* illustrates some applications which were important at that time.
- In algebra, the range is from translated European texts to works written by American authors. We also note the growth of the commercial production of mathematics textbooks in America.
- Abel's *Trigonometry* was written for surveyors and other practioners, whereas Hassler's *Trigonometry* was a mathematics textbook many of whose features are notably modern.
- Hutton's *Course* was imported from England and Ryan's *Calculus* was closely derived from the influential work of Sylvestre François Lacroix. They are additional examples of higher mathematics in the United States in the first part of the nineteenth century.
- Adrian's *Analyst* and *The Analyst* of Joel Hendricks show us the start and the progress of American mathematics journals. Also, Hendricks' *Analyst* was the precursor of the *Annals of Mathematics,* which continues today as one of our most prominent research journals.

Part B of the course, budgeted for approximately weeks 11–15 of the semester, is intended to show students how, in the last quarter of the nineteenth century and in the early twentieth century, mathematics in the United States bifurcated; teaching mathematics continued to be of concern, and the growth of mathematical research gained its own prominence. There are two criteria for the works selected for this part of the course. First, the primary source reading must be accessible to the students who meet the minimum course prerequisites (e.g., no modern algebra or analysis), and second, the mathematical topics should be of interest to the students.

- Research mathematics in the style of that in Europe was introduced into American higher education with *The American Journal of Mathematics*.
- The concrete nature of Gibbs' *Vector Analysis* is in contrast with the abstraction of Peirce's *Linear Associative Algebra*.

- Even though Klein was not an American mathematician, the occasion of his Evanston lecture was a notable event in research mathematics in this country. Klein was a mentor to some of the leading American mathematicians of the turn of the twentieth century and he had a significant influence on American mathematics. We study as much of Birkhoff's paper as the students can tolerate at the end of the semester to try to get a sense of the diversity of higher mathematics in the United States in the first third of the twentieth century.
- The papers of Bennett and Jones and of Coxford provide material to discuss the formation of professional societies in the United States.
- The semester ends with the "Nazi Purge" from the *Manchester Guardian Weekly.* We point out the banished German mathematicians, like Richard Courant and Emmy Noether, who came to the United States and enriched our profession here. This activity demonstrates how American mathematics was becoming on a par with that of the European mathematical community. Also, these last readings provide our students with an opportunity to think retrospectively across the two centuries of American mathematics we have studied.

Appendix B

Student Resources

1. Charles C. Gillespie (Editor), *Dictionary of Scientific Biography*, 16 vols., Charles Scribners' Sons, New York, 1970-1980.
2. Allen Johnson and Dumas Malone (Editors), *Dictionary of American Biography*, 20 vols. and Supplements, Charles Scribners' Sons, New York, 1928-1937.
3. Clark A. Elliott, *Biographical Dictionary of American Science*, Greenwood Press, Westport, CT, 1979.
4. ScienceDirect.
5. Early American Imprints, Series I: Evans, 1639–1800 (commonly known as "Evans").
6. John O'Connor and Edmund Robertson, *The MacTutor History of Mathematics Archive*. www-history.mcs.st-andrews.ac.uk/index.html.

Appendix C

Instructor Resources

1. Joe Albree, David C. Arney, and V. Frederick Rickey, *A Station Favorable to the Pursuits of Science: Primary Materials in the History of Mathematics at the United States Military Academy*, American Mathematical Society, Providence, RI, 2000.
2. Florian Cajori, *The Teaching and History of Mathematics in the United States*, Government Printing Office, Washington, 1890.
3. Florian Cajori, *The Early Mathematical Sciences in North and South America*, Richard G. Badger, Publisher, The Gorham Press, 1928.
4. Louis C. Karpinski, *Bibliography of Mathematical Works Printed in America Through 1850. And Second Supplement*, Arno Press, New York, 1980. (Reprint of the University of Michigan Press, 1940).
5. Kenneth May, *Bibliography and Research Manual of the History of Mathematics*, University of Toronto Press, Toronto, 1973.

Appendix D
Historical American Mathematical Works

1. Benjamin Banneker, *Banneker's Almanack and Ephemeris for the Year of our Lord 1793,* Joseph Crukshank, Philadelphia, 1793. (Evans 24071)
2. Isaac Greenwood, *Arithmetick Vulgar and Decimal,* Printed by S. Kneeland and T. Green, Boston, N.E., 1729. (Evans 3120)
3. John Winthrop, "Relation of a voyage from Boston to Newfoundland, for the observation of the Transit of Venus, June 6, 1761," Printed by Edes and Gill, Boston, N.E., 1761. (Evans 41678)
4. Nicolas Pike, *A New and Complete System of Arithmetic,* Printed by John Mycall, Newburyport, MA, 1788. (Evans 21394)
5. [Jared Mansfield], *Essays, Mathematical and Physical,* Printed by William W. Morse, New Haven, 1800.
6. Nathaniel Bowditch, *The New American Practical Navigator,* Printed by Edmund M. Blunt, Newburyport, MA, 1802.
7. a. Daniel Adams, *The Scholars Arithmetic.,* Printed by Adams and Wilder, Leominister, MA, 1801.
   b. Warren Colburn, *Intellectual Arithmetic, upon the Inductive Method of Instruction,* A New edition of Warren Colburn's First Lessons, Reprinted by Fred K. Brown & Co. Boston, 1863 reprint of the 1822 edition.
   c. Joseph Ray, *Ray's Arithmetic … Inductive and Analytic Methods of Instruction. Prepared for the Eclectic Series,* Winthrop B. Smith, Cincinnati, 1843.
8. a. Robert Simson, *The Elements of Euclid, viz. the First six Books,* Printed by Thomas and George Palmer, Philadelphia, 1803.
   b. Charles Davies, *Elements of Descriptive Geometry,* H. C. Carey and I. Lee, printed by J. and J. Harper, New York, New York, 1826.
   c. A. M. Legendre and David Brewster, *Elements of Geometry and Trigonometry Revised and altered for the use of the Military Academy at West Point*, James Ryan, New York, 1828. (Became commonly known as "Davies' Legendre.")
   d. Charles Bonnycastle, *Inductive Geometry,* P. M'Kennie, Charlottesville, VA, 1834.
9. a. Jeremiah Day, *An Introduction to Algebra,* Howe and DeForest, printed by Oliver Steele, New Haven, 1814.
   b. Leonhard Euler, *An Introduction to the Elements of Algebra Selected [by John Farrar] from the Algebra of Euler,* Hilliard and Metcalf, Boston, 1818.
   c. L. P. M. Bourdon, *Elements of Algebra translated from the French of M. Bourdon for the use of the cadets of the U. S. Military Academy by Lieut. Edward C. Ross,* E. B. Clayton, printed by Clayton and Van Norden, New York, 1831. (Continued from 1835 as "Davies' Bourdon.")
10. a. Thomas Abel, *Substensial [sic.] Plain Trigonometry, wrought with a sliding-rule, with Gunter's lines,* Printed for the author by Andrew Stewart, Philadelphia, 1761. [Evans 8777]
    b. Ferdinand R. Hassler, *Elements of Analytic Trigonometry, Plane and Spherical*, Printed by James Bloomfield, New York, 1826.
11. a. Charles Hutton, *A Course of Mathematics Revised and corrected by Robert Adrain,* 2 vols, Samuel Campbell, New York, 1812.
    b. James Ryan, *The Differential and Integral Calculus,* White, Gallaher and White, New York, 1828.
12. a. Robert Adrain (editor), *The Analyst; or Mathematical Museum*, Philadelphia, 1808–1809, 1811, 1814. (JSTOR)
    b. Joel E. Hendricks (editor), *The Analyst: A Journal of Pure and Applied Mathematics*, 1874–1883. (JSTOR)
13. J. J. Sylvester (editor), *The American Journal of Mathematics, Pure and Applied,* Baltimore, 1878 to the present.

14. a. Benjamin Peirce, *Linear Associative Algebra,* Lithograph, Washington City, 1870.
    b. J. Willard Gibbs, *Vector Analysis*, adapted by Edwin Bidwell Wilson, Yale University Press, New Haven, 1901.
15. Felix Klein, "The transcendency of the numbers *e* and π", Lecture VII, pp. 51–57 in *Lectures on Mathematics,* American Mathematical Society, Providence, RI, 2000, originally published as *The Evanston Colloquium,* Macmillan, New York, 1894.
16. George D. Birkhoff, "Fifty years of American mathematics," pp. 270–280 in *Semicentennial Addresses of the American Mathematical Society*, vol. II, Arno Press, New York, 1980, originally published as *Semicentennial Publications*, vol. 2, American Mathematical Society, New York, 1938.
17. a. Albert A. Bennett, "Brief history of the Mathematical Association of America before World War II," *American Mathematical Monthly, Fiftieth Anniversary Issue*, part II, 74 (1938) 1–11. [JSTOR]
    b. Phillip S. Jones and Arthur F. Coxford, Jr., "Mathematics in the evolving schools. Part 4, First steps toward revision, 1894–1920," pp. 36–45 in *A History of Mathematical Education in the United States and Canada*, Thirty Second Yearbook, National Council of Teachers of Mathematics, Washington, DC, 1970.
18. Anonymous, "Nazi 'Purge' of the universities," *The Manchester Guardian Weekly*, p. 399, Friday, May 19, 1933. From Donald Fleming and Bernard Bailyn, *The Intellectual Migration, Europe and America, 1930–1960,* Belknap of Harvard University Press, Cambridge, 1969, p. 234.