NOTES

USING THE PHILOSOPHY OF MATHEMATICS IN TEACHING UNDERGRADUATE MATHEMATICS

Bonnie Gold,
Carl E. Behrens,
Roger A. Simons
Editors

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Preface

When POMSIGMAA, the Special Interest Group of the Mathematical Association of America, was founded, its purpose as stated was "to stimulate an interest in the philosophy of mathematics in the wider mathematical community, including undergraduate and graduate students, high school, college and university faculty, and mathematicians in industry; to inform this wider community of concepts, issues, and recent developments in the philosophy of mathematics; and to encourage research, and the communication of ideas and results, in the philosophy of mathematics by people with a substantial understanding of both mathematics and philosophy. . . . Since one's philosophy of mathematics affects (sometimes consciously, sometimes unconsciously) one's teaching style and attitude toward students, this research and discussion can be expected to have a large effect on the teaching of mathematics." In alignment with that mission, soon after the founding of POMSIGMAA its officers started discussing putting together a volume, similar to those put together by the history of mathematics folks, sharing with the mathematical community how philosophy of mathematics can be used in the classroom.

This book, edited by three past chairs of POMSIGMAA, with chapters contributed by many other members of POMSIGMAA as well as by quite a range of other mathematicians, brings that discussion to fruition. We solicited authors via the POMSIGMAA listserv as well as through assorted other discussion lists in order to have a wide variety of courses, philosophical viewpoints, and types of activities represented. As a result, there are chapters relevant to almost every level and type of undergraduate mathematics course, but only one or two in most cases. However, if you find the activity or approach from a particular chapter intriguing, you can probably find a way to adapt it to at least one of the courses you teach.

We hope that this volume (as with its correlates in the history of mathematics) spurs the production of more such volumes with a wider range of ideas, and that you find the volume interesting and useful.

Acknowledgements

All three co-editors are grateful for the opportunity to work with each other on this project. The three of us have been more thorough and more broadly knowledgeable than any one of us would be. We have enjoyed working together and learning from each other. We also greatly appreciate the MAA Notes committee and especially thank Stephen F. Kennedy, MAA senior acquisition editor, for their helpful suggestions. We also have individual acknowledgements.

Carl Behrens: I would like to thank Bonnie for asking me to take part in the thrilling adventure of putting together this extraordinary collection. More, I'd like to thank each one of our contributors for demonstrating how vibrant a role the philosophy of mathematics plays in today's classroom. My own somewhat controversial philosophical position has been fleshed out and enhanced by exposure to so many descriptions of classroom experiences in dealing with philosophical questions. Varied as they are, they all show in one way or another that integrating philosophy into standard curriculum not only answers unexpected questions students may have but also helps students understand better the mathematics they are pursuing. I look forward to more and more teachers finding discussing the fundamentals of mathematical science not only rewarding but pleasurable.

Bonnie Gold: I would especially like to thank Steve Kennedy who, when the then-chair of the Notes committee went missing-in-action, took over his role and read each chapter and each committee member's comments, compiled them, and patiently negotiated with me about what needed to be fixed, and in general brought the book to completion. I would also like to thank my son, Robert Lipshitz, as well as my colleague at Monmouth University, David Marshall, for help with TeX: I am a relative TeX novice, and my two co-editors even more so. Robert helped me put the individual chapters together in a book and helped with various formatting problems (including some due to authors' individual TeX macros that were incompatible with the rest of the book). David patiently explained smaller issues when I bounced into his office with “another stupid TeX question.” More broadly, I would like to express appreciation to many POMSIGMAA members who have supported contact between the mathematical and philosophy of mathematics communities. Finally I would like to note that, without the influence of Stanley Tennenbaum when I was an undergraduate at Rochester, I might never have developed in interest in the philosophy of mathematics.

Roger Simons: I am grateful to the people who stimulated my curiosity with issues in the philosophy of mathematics and inspired me to learn something about them. This began in some of my mathematics and philosophy classes when I was a student. Sometimes, I would think about such issues underlying mathematics while frustrated trying to prove theorems in graduate school. About three decades later, my understandings and curiosities were reawakened when I read a book review by Bonnie Gold in the *American Mathematical Monthly*. I am very grateful to Bonnie for that. Others who helped were through one of three institutions: The Logic Colloquium of the UC Berkeley’s Group in Logic and the Methodology of Science, Philip Davis’ informal philosophy of mathematics seminar at Brown University, and POMSIGMAA. I wish to thank the many contributors to those formal and informal discussions. I also thank my wife, Patricia, and the rest of my family for their support and encouragement.

Introduction

1 What this Book is About

What are mathematical objects? Where do they reside? How are we able to perceive them? These are philosophical questions about mathematics whose best known answers are controversial. What place might questions like these have in a regular mathematics class? How might they help students learn mathematics?

The chapters in this book offer a variety of answers to the last two questions. They describe ways of introducing some philosophy of mathematics into a very wide variety of regular mathematics courses, spanning all college levels, to provide students with different outlooks on mathematics. This seems to help students' motivation and effectiveness in learning the regular mathematics content of the course.

That is the purpose of this book: to reveal a wide range of techniques, in a wide range of courses, for improving teaching effectiveness in mathematics by exposing students to the philosophy of mathematics. It's natural to anticipate that spending time and energy on philosophical issues in a mathematics class would distract students from the mathematics they need to learn. But in many cases, the perspectives learned seem to help students with their mathematical work. If you are a mathematics instructor, whatever your attitude toward, and experience with, the philosophy of mathematics may be, you are likely to find in this book useful tips and inspiration for creating ways to help students learn mathematics.

Most of the chapters describe what the author does in the course, what readings were used, what topics were discussed, what was done to assess students' resulting increased sophistication. Almost none of this can be simply transported, unchanged, to your classroom—you have students who differ in important ways from students in the authors' classes, just because American colleges and universities vary significantly. Creativity in teaching involves seeing a concept that appeals to you, and thinking about how it can be adapted to your situation. For example, one of the editors (Gold) read, in *Classroom Assessment Techniques* [1] about Content, Form, and Function Outlines used by faculty teaching graphic design to help their students learn to analyze advertisements on television—and adapted the method to teaching her students how to read a calculus textbook. (It's time-consuming, but fairly effective.) So read this book with your sense of flexible adaptation set at maximum—even if you never teach geometry, for example, perhaps you can adapt some of the ways of viewing mathematics from a chapter on geometry to your course.

There are quite a few books on using the history of mathematics in teaching undergraduate mathematics, but as far as we are aware, there is no similar compilation for the philosophy of mathematics. The officers of POMSIGMAA, the Special Interest Group of the MAA for the Philosophy of Mathematics, have discussed developing such a volume for many years. In the spring of 2013, we finally started soliciting manuscripts. We were pleased to receive drafts of papers describing a wide variety of ways to use a broad range of topics in the philosophy of mathematics to enhance student learning. They involve an extensive spectrum of courses, from developmental and introductory courses (including general education) and freshmen seminars, to courses in the core of the major (especially geometry), to senior seminars and courses for future teachers. Instructors of mathematics at any level, from high school to university graduate school, should find at least some of these papers helpful to their teaching. As the selections in this volume (all but three—those by Gold, Hersh, and Jankvist and Iversen—were written originally for this compilation) demonstrate, the teaching of the philosophy of mathematics is widespread, despite the lack of curriculum dedicated explicitly to the subject.

2 Overview of the Philosophy of Mathematics

The following brief overview of the philosophy of mathematics is intended only as a quick reference guide as you read this book. For more details on the three foundational schools, see [6]; for more details on recent developments in the philosophy of mathematics, see [8]; for individual articles on virtually any topic in the philosophy of mathematics, see [16] or [18].

2.1 Historical Overview

Classically, mathematics was viewed as the most certain form of knowledge: indeed, Plato used mathematics in many of his dialogues when discussing knowledge, and Euclid's *Elements* was regarded well into the nineteenth century as the paradigm of logical perfection. On the other hand, it was also an ancient Greek, Zeno, who introduced us to paradoxes involving mathematics. As our concept of number developed, philosophical concerns developed along with it: learning that some magnitudes are irrational multiples of other magnitudes forced the Pythagoreans to revise their implicit assumption of commensurability of all numbers. The usefulness of negative numbers and imaginary numbers forced the community to wrangle with the question of what should qualify as a number. Mathematics figured significantly in Immanuel Kant's classification of knowledge in the eighteenth century, into analytic versus synthetic, *a priori* versus *a posteriori*. He classified some mathematics as analytic *a priori* because it involves simply analysis of the definitions of the terms in the statement. However, he categorized much mathematical knowledge as synthetic *a priori* because, although the statements are not simply analytic, one does not need experience from the world to discover it: it comes just by thinking about our intuitions of space (for geometry) and time (for numbers).

Contradictions that arose as calculus was used more widely led mathematicians to seek dependable foundations on which to base our mathematical results. This work led to our current definitions of concepts such as limit, but also led to an attempt to base all mathematics on logic and set theory. When this project foundered on set-theoretical paradoxes around 1900, all attention in the philosophy of mathematics turned to foundational issues for a little over half a century. Starting in the 1960s, philosophers' attention returned to traditional philosophical questions about mathematics. They include refining issues concerning the nature of mathematical objects and the basis for mathematical knowledge. This work led to two new philosophical viewpoints, structuralism (developed primarily by philosophers) and social constructivism (developed primarily by mathematicians). Below we give a brief description of the assorted philosophies of mathematics, which are referred to by many of the authors of chapters in this volume.

2.2 The Questions

For the most part, the philosophy of mathematics focuses on four primary questions. The requirement of acceptable answers to them is that they be coherent with our more general account of the world.

1. Are there (and if there are, what are) mathematical objects, and what is their nature? (These are called "ontological" questions.)
   So, just what is a number? A function? A topological space? A functor? Where do they reside: in people's heads? In the physical world? In some abstract world outside of time and space? What properties do they have, and which of them do they share with other objects people are familiar with?
2. How do we gain (or even, is there) knowledge of mathematical objects? (These are called "epistemological" questions.)
   If mathematical objects reside in our heads, why do proofs, rather than examination by psychologists, give us knowledge of them? Is there any objective knowledge of mathematics at all? If mathematical objects reside in an abstract realm, how do we, physical beings, have any contact with them? If they reside in the physical world, why isn't experimentation the most fundamental and trusted way to acquire knowledge of properties of mathematical objects, as with other physical objects?
3. What do mathematical statements mean? How should we interpret mathematical statements? (These are called "semantic" questions.)
   When a mathematician says, "Every finite integral domain is a field," how should we interpret this statement? Possible interpretations include: (1) At face value: there are objects (somewhere) that meet the requirements of

being finite integral domains, and every one of them has multiplicative inverses for each non-zero element of the object. (2) As an "if-then" statement: if there were (or are) mathematical objects that satisfy the definition of a finite integral domain, then each one would also satisfy the definition of a field. (3) As a statement about possible worlds: in all possible worlds, everything that is a finite integral domain is also a field. There may be other ways to interpret it also.

4. What is the relationship between mathematics and the physical world? (This includes questions about the applicability of mathematics.)
   If mathematical objects are just in our heads, or in an abstract, non-spatio-temporal realm, how can they be so useful for solving problems in our physical world? And why does it often happen that, when physicists see a need for a mathematical concept, it has already been developed in response to an internal need of mathematics? This is what Wigner [20] called the "unreasonable effectiveness of mathematics."

2.3 The Different Philosophies of Mathematics, and How They Answer the Questions

There are quite a few different responses to the questions. Below are very brief descriptions of the main schools of the philosophy of mathematics, and how they answer most of them. You will find more details on the foundational schools in [6] and more information on the other topics in our earlier collection [8].

Platonism
Platonism, also called "mathematical realism," is the view that mathematical objects are real, objective, and mind-independent abstract objects existing outside of space and time. Mathematics describes them and their properties. Different versions of platonism differ on where they are, given that they are outside of space and time. They may be in Plato's "realm of the forms" (this is where the name "platonism" comes from). One current response to this question is that "where are mathematical objects found?" is not a coherent question: it's a category error, similar to "Does this tree have blue eyes?" In most versions of platonism[1], mathematical objects do not interact causally with the physical world. That then raises the question of how we gain knowledge of mathematics, since we are part of the physical world. Gödel, the best-known modern platonist, asserted that we have an ability to perceive mathematical objects through our intuition in an analogous way to the way we perceive physical objects. One advantage platonists have is with semantics: mathematical statements are interpreted as meaning exactly what they appear to mean. Mathematical statements describe real facts about mathematical objects. On the other hand, platonists have some difficulty accounting for the applicability of mathematics. Mathematicians tend to believe that, due to the foundational difficulties, platonism has been largely disregarded in modern times. Quite the contrary, platonism remains an active topic of discussion among philosophers (and many mathematicians remain at least closet platonists), though often in a new form, called "structuralism" (see section on Structuralism, below).

Logicism
Logicism was the first foundational school developed in response to the "crisis" in the foundations of mathematics. Since logical knowledge appeared to be the only thing even firmer than mathematical knowledge, and is at the base of all mathematical knowledge, if everything in mathematics could be reduced to logic (which, for the logicists, included set theory)—that is, if mathematical objects are just logical objects—then logic would seem to give mathematics the certainty that the community wanted to regain. For logicists, mathematical semantics involves reducing mathematical statements to logical statements. However, mathematical concepts cannot be reduced strictly to logic alone: infinity, for example, is simply not a concept of logic. Reducing mathematics to set theory seemed more feasible, but that led Frege to the Russell paradox and other paradoxes involving the set of all sets. Russell and Whitehead's *Principia Mathematica* attempted to avoid these problems via a type theory that prevents building the set of all sets, but it had a range of other problems. Gödel's incompleteness theorems showed that it was not possible to prove the consistency of their system within the system, leading most mathematicians to lose interest in logicism, although in the last forty

[1] Although the roots of platonism can, of course, be traced back to Plato, most modern versions have diverged so far from anything he would have recognized as his views that many modern authors, including this volume's editors, use lower-case "p" when referring to it.

years there has been renewed interest among philosophers in returning to Frege's ideas and revising them to avoid the paradoxes.

Intuitionism and Constructivism

The problems with the foundations of mathematics at the turn of the twentieth century led L. E. J. Brouwer to formulate a new philosophy of mathematics called "intuitionism." For intuitionists, mathematical reality is not fully fixed and independent of us, but remains to be determined through our mathematical activity. The only kinds of sets that can be intelligibly spoken of are those that we can imagine ourselves completing, if only in principle. Infinitude is always potential, never actual, thus following Aristotle. We don't know that "P or $\neg P$" is true until we either know that P is true, or know that $\neg P$ is true. We construct mathematical truths by proving mathematical theorems. Once proven, they become true, but are neither true nor false until then. Similarly, to prove that a mathematical object exists, one must give a construction that would generate it, together with a proof that the object thus generated has the required properties. An intuitionist would not say "there is an infinite number of primes" but rather that, given any number, there is a prime number larger than that number. To a classical mathematician, this is the same thing, of course. So some deductions that a classical mathematician accepts, the intuitionist does not. For example, an intuitionist cannot deduce P from $\neg\neg P$. Thus, out go proofs by contradiction. (On the other hand, an intuitionist can deduce $\neg\neg P$ from P.) To deduce $P \vee Q$, an intuitionist needs a proof of P or a proof of Q. To deduce $(\exists x)P(x)$, an intuitionist needs a way of generating an x for which you can prove that $P(x)$ holds. Because completed infinities are rejected, the intuitionistic real numbers are rather different from the real numbers as developed by Cauchy and others, and many theorems mathematicians regularly use in calculus (such as the intermediate value theorem) are not theorems intuitionistically. This led most mathematicians to reject intuitionism. In the 1960s, Errett Bishop developed his own version, which he called "constructive mathematics," in which he derived versions of classical theorems (such as the intermediate value theorem) which are classically equivalent to the standard theorems, but which are constructively true. He called this "finding their constructive content." This effort gained some new adherents, but it is still not a popular approach to mathematics.

Formalism

Formalism, the third foundational school, was started by David Hilbert in an attempt to reconcile the impasse between classical and intuitionistic mathematics. Classical mathematicians were not willing to give up the principle of the excluded middle, nor infinitary reasoning. He felt that, if we could find a proof that the use of these methods will not lead to any contradictions, then mathematicians can continue safely using them. The viewpoint is called formalism, because now one does one's proofs entirely in the formal system, without any regard for interpretations in the model from which the original axioms came. One starts with some axioms that, when interpreted in a particular way, describe a mathematical system such as the natural numbers or the real numbers. His program was to prove, through an analysis of what kinds of proofs could be developed, that one could not get a contradiction from such a set of finite axioms needed to derive mathematics and the deductions from them. This area of research is called proof theory. Formalism is agnostic on the existence of mathematical objects, mathematical truth, and the meaning of mathematical statements. However, its aim was to ensure that mathematicians could use our standard methods without fear of running into contradictions. Gödel's incompleteness theorems put an end to this as a method of justifying our infinitary mathematics. His first incompleteness theorem shows that we cannot choose an axiom system that will give us the standard theorems of number theory in a coherent way and in which we can prove all true theorems of number theory. His second incompleteness theorem shows that we cannot prove the consistency of systems as complex as Peano arithmetic from within the system. Nonetheless, proof theory, while no longer expected to provide a firm foundation of mathematics, has continued to be studied quite broadly, as metamathematics, looking into implications and interrelationships of particular axiom systems.

Nominalism

For nominalists, there are no mathematical objects—no abstract, mind-and-time-independent objects. Or at least, they believe that assuming that such objects exist can be based only on faith (like a religious belief), not reason. Since they do not want to throw out mathematics entirely, nominalists must interpret mathematical statements non-literally.

If there are no mathematical objects, there appears to be nothing to gain knowledge of. Mathematical statements appear to be simply false. This is, however, a major problem, because mathematical knowledge appears to be our most dependable kind of knowledge—if we know anything, we know certain mathematical facts. This is, in fact, the primary argument against nominalism. Nominalists deal with this problem in a range of ways. One is to say that, even though there are no mathematical objects, there are still true statements about them. One version of this is "conventionalism," which says that mathematical statements are true because they're simply linguistic conventions: "every finite integral domain is a field" is true by analyzing the definitions of "finite," "integral domain," and "field." Another variation, "paraphrase nominalism," says that mathematical statements become true when suitably paraphrased. One version of this, "if-thenism," says that we should paraphrase "every finite integral domain is a field" as "if there were any finite integral domains, each of them would be a field." This feels peculiar, that we have to rephrase all our statements, but except for the rephrasing, it allows mathematicians to continue doing mathematics as we always have. Another variation, "fictionalism," says that, while mathematical statements are false, they are true in our "story of mathematics," just as "Sherlock Holmes lived in London" is true in the stories of Conan Doyle.

Structuralism

This is a relatively recent development by philosophers of mathematics, starting in the 1970s (primarily by Michael Resnik [13] and Stewart Shapiro [15]). The primary versions of structuralism are modern forms of platonism, with mathematical objects being real, independent abstract objects (structures). Mathematical knowledge comes from our knowledge of structures, a knowledge we have outside of mathematics as well. Mathematical statements are then interpreted as statements about the structures. This has the advantage, from a philosophical viewpoint, of making questions about mathematics continuous with philosophical discussions concerning a broader class of objects. Philosophers can thus use philosophical discussions that have already occurred in reference to structures in their discussions of mathematical objects.

Structuralism was developed partly in response to Bourbaki's "mother structures" and several mathematicians' characterization of mathematics as the study (or science) of patterns (or structures). It is also partly a response to Paul Benacerraf's challenge in "What Numbers Could Not Be" [3]: "To be 3 is *no more or less* than to be greater than 2 and less than 4. Any object can play the role of 3" (emphasis in the original). The point of structuralism is that indeed, this is exactly what mathematical objects are: they are structures, or places in structures. For many mathematicians studying more modern mathematical concepts (groups, topological spaces, categories), it is not individual mathematical objects that are important, but how they interrelate: which groups are isomorphic to which others, for which ones there is a homomorphism from one to another, etc. This is less true for classical analysts and number theorists. They are only interested in one or two particular structures (the integers, and maybe the p-adic numbers for number theorists, the real numbers (or $\mathbb{R}^n$), or the complex numbers (or $\mathbb{C}^n$) for analysts). In a sense, in those structures, each object has its own characteristics. But in another sense, it doesn't: when doing number theory, we really do not care, in any deep sense, what the integers are. If someone wants to believe in them as sets of some kind, that is fine. All that is important about them is their relationships: that $1 < 2 < 3 \ldots$, that each integer has a unique successor, that $2 \times 3 = 6$, etc.

For structuralists, mathematical objects are simply positions in a particular structure: the natural number structure, the structure $\mathbb{Z}_8$, etc. It doesn't matter whether there are things in the real world that exemplify the structures: they exist independent of their instances. For philosophers, the advantage of discussion of structures over discussion of traditional mathematical objects is that structures fit with broader philosophical discussion, because structures are not used only in mathematics. There is the structure of a baseball infield (where the places are those for a first baseman, second baseman, shortstop, etc.), for example, or of a corporation (as described in its organizational chart). Philosophy has developed many competing philosophical accounts of universals that can be readily adapted to structuralism. There are also some structuralists (Charles Chihara among them) who are nominalists, and have proposed a "structuralism without structures."

Social Constructivism

The second philosophy of mathematics developed in the last forty years, social constructivism, was developed originally by mathematicians (primarily Philip Davis and Reuben Hersh [4]) and mathematics educators (primarily Paul

Ernest [5]).[2] The intention is to develop a philosophy of mathematics that describes, better than platonism, nominalism, or any of the foundational schools, how people actually do mathematics. The main tenets of social constructivism are that mathematical objects are socially constructed abstract objects, constructed by the community of mathematicians. Mathematical statements become mathematical knowledge when approved by this community. Mathematical statements, then, just as for platonists, are taken at face value, as describing mathematical objects. I will concentrate here on Hersh's version of social constructivism [9], as it is closer to most mathematicians' ways of describing what we do.

Mathematical objects are created by people in response to the needs of mathematics itself and of science and daily life. Once created, they have properties that may not have been apparent to their creator, and may be quite difficult to discover or verify. For a social constructivist, a proof is what the mathematical community, at any given moment, says is a proof. Since our notions of rigor, and what constitutes a proof, change over time, this means that something that counts as a proof for one period's mathematicians may not count in later eras. See Lakatos [11] for an archetypical example of this, involving Euler's formula relating the number of vertices, edges, and faces of polyhedra. Mathematical practices are responsible for the basic beliefs and axioms being true: the Peano axioms have been accepted as an optimal characterization of the natural numbers. These mathematical practices—for example, being taught from generation to generation—then influence human beings to acquire these beliefs. However, since mathematical objects, once constructed, are independent of their creator (much as the Supreme Court is), they have objectively verifiable properties, even if we are not always successful in verifying them. Of all the philosophies of mathematics discussed here, social constructivism (and perhaps empiricism) has the easiest time explaining why mathematics is so fruitful in explaining the physical world: we develop mathematics for that purpose. (It is still no closer to solving Wigner's riddle of the unreasonable effectiveness of mathematics, when mathematics that has been developed for another purpose turns out to be exactly what is needed in a completely different context.) Philosophers initially dismissed social constructivism as inherently incoherent, a form of psychologism (a philosophical viewpoint that there is not space here to go into; however, it appears to have seriously lost favor among philosophers). However, its popularity among a significant group of mathematicians has recently resulted in some philosophers trying to make coherent philosophical sense of the view.

Phenomenology

"Phenomenology is the study of structures of consciousness as experienced from the first-person point of view" [17]. The field was initiated by Edmund Husserl, who wrote a doctoral dissertation in mathematics (calculus of variations) from Vienna under Leo Königsberger, and began working as an assistant to Weierstrass. But his interests soon turned to philosophy, and in 1900 he published his first work in phenomenology, *Logische Untersuchungen* (*Logical Investigations*). Unfortunately, as Rota wrote in 1974 in "Husserl" [10, p. 175-6], "Acclaimed by many as the greatest philosopher of the century . . . he wrapped his thoughts in a heavy-handed, redundant, solipsistic German academic style which makes his writing all but impossible to translate . . . ." As a result, a wealth of Husserl-interpreters have arisen to interpret his work. In this description we will restrict ourselves largely to two interpreters who applied Husserl's views to the philosophy of mathematics, Richard Tieszen and Gian-Carlo Rota.

Phenomenology starts with conscious experience and attempts to analyze it while avoiding psychologism, "the view that mathematics and logic are concerned with mental entities and processes, and that these sciences are in some sense branches of empirical psychology" [19, p. 2]. We do use our mind to investigate mathematical objects such as natural numbers, but "the objects known about—natural numbers—are not themselves mental entities. Rather, they are ideal objects" (ibid, p. 8). Acts of perception are paradigmatic of intentional acts (ibid, p. 102). A central concept, for Husserl, is *intentionality*, "the characteristic of 'aboutness' or 'directedness' possessed by various kinds of cognitive acts, for instance, acts of believing, knowing, remembering, willing, desiring . . . an act of cognition is directed toward, or refers to, an object (or state of affairs) by way of the 'content' of the act, where the object (or state of affairs) the act is about may or may not exist" (ibid, p. 52). For example, consider the *cognitive act* "M knows that there is no largest

[2] The constructivist movement in mathematics is distinct from a similarly named school of philosophy that became prominent in the 1990s concerning the empirical sciences and technology. The latter arose among sociologists studying the structure of the scientific community, and some of its leading exponents questioned the validity of the method of scientific experiment as a means of obtaining knowledge. Mathematical constructivists show no evidence of having been influenced by this controversy.

prime number.” The *content* is “there is no largest prime number”; the *object* could be either the set of prime numbers, or the theorem that there is no largest prime number.

Concerning our four questions, for the phenomenologist there are mathematical objects, which are neither physical nor mental, but rather abstract objects, since they are invariant over time and between people, unlike physical and mental objects. According to Tieszen, both mathematical and physical objects “are to be understood in terms of the ‘invariants’ or ‘identities’ in our experience” [19, p. 55]. Mathematical statements are taken at face value, to be referring to these objects. Rota argues against reductionist (formalist) approaches to theories of truth. According to Rota, “Mathematical truth is philosophically no different from the truth of physics or chemistry. Mathematical truth results from the formulation of facts that are out there in the world, facts that are independent of our whim or of the vagaries of axiomatic systems” [14, p. 113]. We gain knowledge of mathematical truths through a range of mental processes. Tieszen observes that the phenomenology of perception shows that we do not create or construct perceptual objects, but that we do in a certain sense create or construct our knowledge of these objects [19, p. 105]. One way we do this is by looking at ways that mathematical objects remain invariant under various transformations. For example, our knowledge of arithmetic is founded on our experiences with small sets of objects, and the fact that operations on these sets give consistent results. One can have skepticism about abstract objects, but not about the invariance of mathematical experience (ibid, p. 58). Mathematical knowledge is reliable to the extent that intentions toward mathematical objects are fulfillable (ibid, p. 63). Rota notes that often we look for proofs in mathematics only after we become fairly certain (through our explorations of examples and known relations among concepts) that a statement is true. And, in fact, when the first proof of a theorem is long or ugly, we work it over and over until finally we get a proof that is almost a triviality. He gives, as an extended example, the proofs of the prime number theorem ([14], pp. 113–116).

One place phenomenologists have an advantage over platonists is with the applicability of mathematics. According to Tieszen, Husserl analyzes the origins of mathematical knowledge on immediate perceptual acts, and then we abstract from that experience to develop our understanding of mathematical objects. So “Husserl could argue that parts of mathematics have applications because mathematics has its origins in our everyday experience in the first place” [19, p. 55]. For example, graph theory had its origins in Euler’s analysis of the seven bridges of Königsberg.

Phenomenology has a relatively small, but renowned, set of mathematical followers. In addition to Rota, Kurt Gödel looked to phenomenology as he attempted to extend our understanding (and axioms) of set theory in ways that would enable us to settle questions such as the continuum hypothesis and the axiom of choice. For an extensive discussion of Gödel’s interest in phenomenology, see Tieszen ([19]).

Empiricism

The British empiricists have been generally dismissed by philosophers over the past century and a half, for an oversimplified, crude approach to the question of how we acquire knowledge. John Locke’s *tabula rasa* gave too little credit to the ingenuity of the human mind to be popular. David Hume has been read mostly for his role as a foil for Kant’s catalog of the forms of knowledge, analytic and synthetic, *a priori* and *a posteriori*. As for John Stuart Mill, whose *System of Logic* went through eight editions between 1843 and 1872, he isn’t read at all.

Philosophical dismissal of empiricism is surprising, because the sciences that rest on it, the empirical sciences, have flourished during the same period. The basic tenet of modern science is that observed fact trumps reasoned assertion. The goal of science is the discovery of universal laws that govern the behavior of objects in an ordered universe, but any proposed laws are endlessly subject to testing and experiment, and, if necessary, revision. Generalizations are accepted as true only as long as contrary evidence is not observed. As Mill put it: “Whence do we derive our knowledge . . . ? From observation. But we can only observe individual cases. . . . From instances we have observed, we feel warranted in concluding that what we found true in those instances holds in all similar ones, past, present, and future, however numerous they may be” [12, Book II, Chapter iii, Section 3].

Such tentative confidence in their conclusions may be acceptable to empirical scientists, but Mill went further than either Locke or Hume had. For Mill, mathematical knowledge also is valid only in single instances. Any generalization is subject to the limitation that the community must “feel warranted to conclude” that it is true. Indeed, he applies this limitation even to the very process of reasoning itself. Mill’s simple model of how humans know the truth of generalizations had a fatal flaw. Like virtually all of humanity in his time, and indeed throughout history until recently, Mill saw the universe in a dualistic Cartesian mode of immaterial mind and physical phenomena. In such a world, mathematics existed in an anomalous state. According to empiricism, our knowledge of mathematical truths, like all truths, depended on observation; but if mathematics takes place in the mind, what is there to observe? Mill solved this problem by attempting to locate mathematics in the physical world. He declared, in a nominalist mode, that abstract numbers did not exist: any number, he said, must be the number of something. This patently false position led to a savage attack on Mill's mathematical philosophy by Frege and Russell, who justifiably termed it "pebble arithmetic." Rejection of his mathematics led also to rejection of his complete philosophical position, a rejection that continues in philosophy today. But modern empiricists do not need to follow Mill into the barren wilderness of nominalist denial. In recent decades it has become clear that Cartesian dualism of mind and matter is not supportable; that the activities of the mind, including abstract thought and logical ratiocination as well as imaginative speculation and artistic creation, are physical phenomena that exist in the physical world; that they are physical events that have a physical existence in place and time, in the mechanical, electrical, and chemical states of the neurons in brains of human beings.

Thus for modern empiricists (see [2]), mathematics is not a social construct assembled by humans. It is a physical phenomenon that follows universal laws, which it is the goal of mathematics to discover. Mathematical objects are physical phenomena whose behavior mathematicians study by observation and experiment, observation including the physical acts of introspection and communication. Accepting the view that mathematics is another of the empirical sciences requires giving up the cherished illusion that mathematical conclusions have a certainty of a different order than the rest of human knowledge. But it has the comfort of making mathematicians aware of what they are doing and why they are doing it.

3 How to Use This Book

We've chosen to organize this book based on the course each author is writing about, since we expect most readers will first take a look at courses they frequently teach to see if there are chapters relevant to them. However, often a method used in one course can be adapted for use in a quite different course. So after a brief description of the chapters in the order they appear, we include lists of them classified in different ways: by the philosophical topics considered; by the overall philosophical approach of the authors, if there is one; by methods of instruction in the course; and by types of assignments used for the philosophical topics in the course. (Not all chapters are specific about each issue; so not all appear in each list.)

3.1 Summaries of the Chapters, in the Order Presented in the Book

Part I: The big picture

Chapter 1: Uffe Thomas Jankvist and Steffen Møllegaard Iversen's "Philosophy in Mathematics Education: A Categorization of the 'Whys' and 'Hows' " sets the stage for the book by offering two "whys" for using the philosophy of mathematics in mathematics education: as a goal in itself, and as a tool to improve mathematics learning. They also suggest three "hows," ways to incorporate the philosophy of mathematics in mathematics courses: illumination approaches, modules approaches, and philosophy-based approaches.

Chapter 2: Bonnie Gold's "How Your Philosophy of Mathematics Impacts Your Teaching," reprinted from the *College Mathematics Journal* (2011), observes that, without necessarily being aware of it, faculty do take stances (platonist, formalist, social constructivist) on philosophical issues when they teach mathematics: on what kinds of mathematical objects there are; on which descriptions are viewed as being of the same object; on whether mathematical statements are true and how that is determined; on how we gain knowledge of mathematics; on a range of semantic issues. So we will teach more coherently if we examine what our views are and whether how we teach is consistent with what we believe.

Part II: First-year seminar and general education courses

The audience in our freshman seminar courses and general education (quantitative reasoning) courses is less sophisticated than most of our mathematics majors, but many such courses allow faculty great freedom to focus on topics that they consider important. They thus offer us an opportunity broaden students' perspective on mathematics by examining various philosophical aspects of mathematics: the nature of mathematics and the mathematical enterprise and the role of mathematics in society, broadly interpreted. Further, helping these young citizens understand what mathematics is about and how it contributes to society can make an enormous difference in society's attitude toward mathematics, science in general, and the value of understanding an issue before taking a stance on it.

Chapter 3: Susan Jane Colley, in "What Is Mathematics and Why Won't It Go Away? Philosophy of Mathematics in a First-Year Seminar," uses both fiction and non-fiction readings about mathematics, mathematicians, and culture. The course involves a mixture of some mathematics and humanistic issues such as whether mathematics is discovered or invented, what constitutes a proof, the culture of mathematics and how social forces influence its development, and mathematics and aesthetics.

Chapter 4: Mike Pinter, in "Helping Students See Philosophical Elements in a Mathematics Course" describes a course he has developed, Analytics: Math Models, which fulfills the quantitative reasoning requirements for honors program sophomores and juniors. The metaphor for the course is "seeing the unseen." He takes a humanistic and social constructivist bent. Philosophical elements are interspersed throughout the course, intermixed with doing mathematics (problem solving, logic, discrete mathematics topics).

Chapter 5: Chuck Rocca's "An Exercise in the Philosophy of Mathematics" involves three general education courses he teaches (Great Ideas in Mathematics, History of Mathematics, and Mathematics in Literature). The "exercise" is a four-week project at the beginning of each course, aimed at broadening students' understanding of the nature of mathematics.

Chapter 6: Kayla Dwelle's "Evangelizing for Mathematics" describes her work in Mathematics for the Liberal Arts, a general education alternative to College Algebra, on changing students' attitudes toward mathematics. She asks, "How do we evangelize mathematics to mathematically agnostic students?" She wants her students to understand that mathematics is intimately connected with the world, is accessible to everyone, and has aesthetic value. To help her students develop these understandings, she engages them in a wide range of activities, some outside the classroom.

There is also some discussion of the philosophy of mathematics in a liberal arts course in Linda Becerra and Ron Barnes' chapter (in Part X).

Part III: Developmental mathematics and college algebra

Developmental and college algebra courses tend to concentrate on the nuts and bolts of symbolic computation, and seem to have little time for philosophical discussion. However, at times attention to philosophical issues can enliven the course and improve student learning. We have two contributions concerning them.

Chapter 7: James Henderson finds that his developmental mathematics students have imbibed a simplistic platonism with the mathematics they have learned. In "Less is More: Formalism in Developmental Algebra" he suggests that sometimes explicitly introducing a formalist approach may be help students avoid the morass of ontological difficulties concerning the nature of the complex numbers. He shows students that they have actually done a similar maneuver earlier, in their studies of negative numbers and fractions.

Chapter 8: Lesa Kean's chapter, "Using Beliefs about the Nature of Mathematical Knowledge in Teaching College Algebra," explores how focusing her mind on introducing students to various beliefs about the nature of mathematical knowledge affected and improved her teaching in college algebra. Toward the end of the activity she shared these views on mathematical knowledge and discussed them with the class.

Part IV: Calculus and probability and statistics

There are many philosophical issues related to calculus, and so it may be surprising that we only have one chapter on calculus in the volume. Perhaps it is because our calculus courses are so full of new concepts, examples, theorems, computations, and applications that we barely have time to breathe.

Chapter 9: Margo Kondratieva's chapter, "Capturing Infinity: Formal Techniques, Personal Convictions, and Rigorous Justifications," suggests getting calculus students to start thinking about the subtleties of infinity and the need for rigor. Why, and when, can we apply the same rules to infinitely long expressions that we do to standard expressions? She introduces historical philosophical issues concerning completed infinities and limits to help students understand that there are indeed difficult questions hidden here.

Chapter 10: Dan Sloughter, in "Making Philosophical Choices in Statistics," describes the choices all faculty who teach statistics must make, not only between frequentist and Bayesian (subjective) approaches to inductive inference, but between differing frequentist interpretations (that of Fisher versus that of Neyman and Pearson) as well. The approaches have differing interpretations of the meaning of probabilistic statements and disagree about the nature and aim of inductive inference. The role of the instructor is, in part, to clarify the distinctions: what each is measuring, and what kinds of conclusions can be drawn from the results of their analyses.

There are also some comments relevant to calculus in Bonnie Gold's chapter (in Part I). Also see the chapter by Alejandro Cuneo and Ruggero Ferro (in Part XI) in which they apply their discussion of the development of mathematical concepts to calculus.

Part V: Logic, foundations, and transition courses for mathematics majors

There are, of course, many philosophical issues undergirding our transition to proof courses, as well as logic and foundations courses. We offer two chapters focusing solely on them.

Chapter 11: Jeff Buechner, in "How to Use Ideas in the Philosophy of Mathematics to Teach Proof Skills," discusses using concepts from the philosophy of mathematics in a course in logic taken by a range of majors. He wants students to learn that rules of inference depend on what kind of world they are being applied to and what kinds of objects there are in it. So he introduces intuitionistic logic to bring out philosophical questions about the nature of mathematical objects and mathematical reasoning.

Chapter 12: Gizem Karaali introduces "An 'Unreasonable' Component to a Reasonable Course: Readings for a Transitional Class." The component considers the original observation by Eugene Wigner that mathematics seems to be unreasonably effective in physics, and a range of other articles that have taken up the theme. Why mathematics is useful, and why it is useful even in unexpected places, leads to a discussion of many implicit assumptions about mathematics: whether it is invented or discovered, whether it is describing the universe or is a game played by rules we choose.

In addition, see Sally Cockburn's chapter (in Part VII), since the mathematical content of the course is logic and set theory. Paul Dawkins' chapter in the next Part looks at the role of axioms in the context of a geometry course. Also, see Linda Becerra and Ron Barnes' chapter (in Part X) for a discussion of a set theory course, and Bonnie Gold's chapter (in Part I) for some discussion of transition courses.

Part VI: Geometry

Geometry has a rich history in the philosophy of mathematics, from informal geometry to Euclid's axiomatic approach. Philosophical issues include questions about the role of proof and of diagrams, Kant's assumption that Euclidean geometry not only is the geometry of our physical space but that we inherently intuit it, questions about the parallel postulate, the development of alternative geometries, and the discovery that Euclid's postulates were incomplete for describing his geometry. Thus there is a wealth of opportunity to consider philosophical issues in a geometry course. We have three chapters that concentrate on geometry and several others that include it; yet each chapter is quite different from the others.

Chapter 13: Brian Katz, using an IBL (Inquiry-Based Learning) approach to teaching Modern Geometry, works on "Developing Student Epistemologies." His aim is to bring students closer to adopting an "expert" epistemological viewpoint in mathematics (on the nature of mathematical truth, the role of proof, examples, and so on). In describing his course he also gives a good introduction to the IBL approach. To get students to rethink their assumptions about the mathematical paradigm, they also have some readings about the foundational period, logical positivism, and Gödel's theorems.

Chapter 14: Paul Dawkins, in "Helping Students Develop Conscious Understanding of Axiomatizing," takes a formalist approach, using a geometry course to teach students how to produce mathematically acceptable proofs and to understand the mathematical process of proving. He focuses on the use of axiomatics as an insightful example of modern proof-oriented mathematical practice, which is particularly valuable for preservice teachers who will themselves be teaching proof. The course starts with a few axioms and a wide range of examples that satisfy them. He gradually introduces new axioms or guides students to formulate possible axioms. Philosophical concepts focus on the role of axioms, how they differ from theorems, the role of examples, and implications of assuming or not assuming particular axioms.

Chapter 15: Unlike most of the chapters in this volume, Nathaniel Miller's "The Philosophical and Pedagogical Implications of a Computerized Diagrammatic System for Euclidean Geometry" is less about how to use philosophical issues in mathematics to teach mathematics, than about a partial resolution to a philosophical issue that we believe everyone who teaches mathematics, and particularly geometry, should be aware of. This is the issue of whether diagrams alone can provide a rigorous justification of a mathematical statement. He has shown that it is possible to develop a purely diagrammatic system in which all the proofs in Euclid's *Elements* can, in principle, be carried out. His system captures the rigor that underlies the geometric arguments in the form of how one moves from one diagram (array) to another. Further, with his computer proof system, CDEG, he has taken this a step farther: by making it sufficiently formal that it can be written as a computer program, he is showing that there is, in fact, nothing missing, nothing informally assumed. So while a small percent of the readers of this book are likely to learn the details of CDEG, and even fewer to use in in their classes, the chapter aims at giving readers an understanding of how one can argue rigorously using diagrams—so a picture (or rather, a sequence of pictures) can be a proof! So let's stop saying, incorrectly, "that's a nice picture, but pictures aren't proofs."

See also the three chapters in Part X. Morrow's chapter concentrates on geometry, but also applies to a variety of other courses; Becerra and Barnes' chapter discusses material used in both Geometry for Teachers and Differential Geometry; and Belnap and Parrott's chapter discusses geometry in a course for elementary education majors.

Part VII: Other upper-level or capstone courses for mathematics majors

The advantage of discussing issues in the philosophy of mathematics in upper-level or capstone courses for mathematics majors is that students at that stage are sophisticated enough to appreciate many of the issues that arose during the foundational period. They bring their own experiences with mathematical objects to epistemological and ontological discussions. On the other hand, in most upper division courses (except capstone or senior seminar courses) there is so much mathematics one wants to include that it is hard to make time for philosophical considerations. Yet sometimes finding that time improves what students remember long after the course is over.

Chapter 16: Sally Cockburn, in her "Senior Seminar in Philosophical Foundations of Mathematics," discusses a wide variety of topics in the philosophy of mathematics. The course begins with students studying transfinite numbers as a mathematical topic. In the second half of the semester, the discussion turns philosophical. Among issues considered are the ontological status of infinite numbers: are they mathematical objects in the real world or social constructions? Students are led to discover the set-theoretical paradoxes, and then read of the response to them: logicism, formalism (axiomatics) and intuitionism, and Gödel's theorems. They also read about the problems with infinity earlier in the history of mathematics, as well as other earlier topics (such as Kant's classification of knowledge) in the philosophy of mathematics.

Chapter 17: Nathan Moyer, in "Connecting Mathematics Students to Philosophy," reports on a three-week project (one which uses little class time) to provide students with an overview of philosophical questions. He uses it in a real analysis course, but it would work equally well in other upper-level proof courses (such as number theory, abstract algebra, and senior seminars). To consider "what is mathematics?" students learn about logicism, intuitionism, formalism, and platonism. They also consider how they are connected with the four ways of knowing (intuition, empirical senses, innate reason, and authority).

See also the chapters in Part X. The one by Becerra and Barnes has aspects relevant to modern algebra and differential geometry, among other courses. Belnap and Parrott's chapter has examples from linear algebra, geometry, and point-set topology, and Morrow's has examples from linear algebra, abstract algebra, and geometry.

Part VIII: History or philosophy of mathematics courses

Chapter 18: Thomas Drucker, in "History Without Philosophy is Mute," describes how, throughout a history of mathematics course, some consideration of philosophical issues significantly enhances student understanding of the changes that were taking place. This can be especially important when the course is taken by future teachers, for whom an understanding of how mathematics has developed will inform their later teaching. He offers ways to incorporate the philosophy of mathematics no matter whether the instructor is taking a chronological or a thematic approach.

Chapter 19: Reuben Hersh was an early advocate of teaching a course in the philosophy of mathematics that goes beyond the early twentieth century foundational schools. In "Let's Teach Philosophy of Mathematics" reprinted from the *College Mathematics Journal* from 1990, he advocates teaching a philosophy of mathematics course from a humanistic perspective. The course discusses the nature of mathematical reality, the nature and meaning of infinity, and the relation between mathematics and the physical world. It studies the development of mathematics, such as the Dedekind-Cantor construction of the real numbers, from both a historical and philosophical perspective, as a solution of problems going back to the origins of the calculus. The course includes the foundational period, the problems that led to it and the outcomes, but it also includes new trends in the philosophy of mathematics since Gödel's results.

See also Elizabeth de Freitas's chapter about a philosophy of mathematics course for future teachers, described in the next part, and Linda Becerra and Ron Barnes chapter in Part X for discussion of a history of mathematics course (and, more generally, using the historical development of beliefs about certainty in mathematics).

Part IX: Mathematical education of teachers and teacher educators

It is particularly important for future teachers and teacher educators to have an opportunity to consider a range of philosophical questions concerning mathematics, for exactly the same reason that this volume has been written for college-level mathematics faculty. Understanding some of the subtleties involved in our understanding of mathematical objects and processes will help them understand some of their students' confusions. Further, it will give them a deeper understanding of the material they will be presenting.

Chapter 20: Erin Moss discusses "Philosophical Aspects of Teaching Mathematics to Pre-service Elementary Teachers." Her goal is that her students come to the views that all people are capable of developing and verifying the mathematics they need to use, and that mathematics is a collaboratively developed discipline in which reason is paramount and in which multiple approaches are valuable. To acquire these views, students must be active participants in the development of their understanding. The course is inquiry-oriented and problem-based.

Chapter 21: Elizabeth de Freitas, in "Pre-service Teachers Using Core Philosophical Questions to Analyze Mathematical Behavior," discusses a philosophy of mathematics course for pre-service teachers. It covers a broader range of modern philosophy of mathematics topics than any other discussed in this volume. They include realist, conceptualist, and nominalist views, and work by Corfield and Lakatos, which she refers to as descriptive epistemology, work by Lakoff and Núñez and others on embodied cognition, and work by Roth on material phenomenology. She discusses how answers to core philosophical questions are linked to pedagogical approaches.

Chapter 22: Alfinio Flores, Amanda Jansen, Christine Phelps, and Laura Cline describe "A Mathematics Inquiry Course: Teaching Mathematics in a Humanistic Way." The philosophical viewpoint underlying the course starts with Alvin White's humanistic mathematics, a thread that was then taken up by Reuben Hersh and others and called social constructivism. From this viewpoint, mathematical praxis has various facets—it is not just the finished product—and social interactions are crucial to the process. The course, for graduate students in mathematics education, is inquiry-based, but in a very different sense from what is now called "IBL" (or Inquiry-Based Learning). Students gain an understanding of mathematical practice by engaging in mathematical research at a level at which they are capable of making progress. They learn problem posing by varying previously solved problems.

See also the chapter by Jason Belnap and Amy Parrott, described in the next Part, as its examples are primarily from a class for future elementary teachers, although it is applicable to all mathematics students. Also see the chapter in Part VIII by Thomas Drucker about a history of mathematics course primarily taught to future teachers.

Part X: Philosophical issues that can be related to multiple courses

Most of the ideas discussed in this book can be adapted to many mathematics courses, but here we include three that explicitly discuss several courses.

Chapter 23: Jason Belnap and Amy Parrott discuss the importance of creating opportunities for students to make mathematical conjectures in "Mathematical Enculturation through Conjecturing." Their overall philosophical viewpoint is socioconstructivism (a term that arose from the mathematics education world at around the time that social constructivism was developing among mathematicians; they are closely related). Conjecturing is a mathematical practice that is accessible to students at many levels, gives students ownership over and commitment to mathematical concepts and ideas, and helps them learn to ask questions. It leads students to develop their own philosophical viewpoints concerning mathematics.

Chapter 24: Linda Becerra and Ron Barnes focus on a specific topic in the philosophy of mathematics, that of the evolution of belief in the certainty of mathematics in "Consideration of Mathematical Certainty and Its Philosophical Foundations in Undergraduate Mathematics Courses." Topics considered include the relationship between belief, truth, and proof, and how our understanding of the certainty of mathematical statements and proofs has changed over time.

Chapter 25: Margaret Morrow, in "Proofs That Do More than Convince: College Geometry and Beyond," discusses the role of proofs in mathematics: one role is to convince the listener or reader that the statement is true, but a more important role is to give the listener an understanding of why the result is true. A further role of proof is to display the strategies and concepts of mathematics. An important lesson to take away from this chapter is that we should give more thought to the best proof to present, and perhaps to the value of presenting proofs from several different perspectives.

Part XI: Philosophy of mathematics and teaching

The chapters in this section are really more about philosophical issues related to mathematics education rather than specific applications of philosophy to mathematics courses, but involve approaches that we feel are worth sharing with the community. Although the primary focus is on a philosophical issue, both suggest ways of directly applying their conclusions to teaching mathematics.

Chapter 26: Sheila Miller proposes, in "On the Value of Doubt and Discomfort in Education," that there are two tools faculty can use to help students develop the habits of inquiring and persistence that are critically important characteristics of educated adults: doubt and discomfort. Students need to learn to insist that they understand how things work, rather than simply accept conclusions from authorities. She uses the topic of mathematical infinity in a range of classes to move students beyond their comfort level. Students first learn that one property infinite sets have is that they can be put in one-to-one correspondence with proper subsets. Then, via Cantor's proof, they discover that there are nonetheless infinite sets that are of different sizes.

Chapter 27: Alejandro Cuneo and Ruggero Ferro, in "From the Classroom: Towards A New Philosophy of Mathematics," introduce a general theory, coming from the experience of teaching mathematics, of the acquisition of (mathematical) knowledge from our experiences and direct perceptions and from mental operations. The chapter is, in effect, a reverse engineering of the usual approach in this book, which takes a philosophical stance and applies it to teaching. Instead, here the authors look at how questions in the teaching of mathematics lead to a philosophical position about mathematics that is consistent with it.

3.2 Organized by Philosophical Issues Considered

If you are interested in teaching approaches that discuss specific philosophical issues, the following organization of the chapters in the volume should be helpful. The categories are **Foundations** (logic, logicism, intuitionism, formalism); **Ontology** (existence or nature of mathematical objects); **Epistemology** (justification of mathematical knowledge, including proofs); **Applicability** of mathematics; **Aesthetics** of mathematics.

Topic \Chapter:	1	2	3	4	5	6	7	8	9
Foundations				Y					Y
Ontology		Y	Y		Y				
Epistemology	Y	Y	Y	Y				Y	Y
Applicability	Y					Y			
Aesthetics			Y			Y			

Topic \Chapter:	10	11	12	13	14	15	16	17	18
Foundations		Y		Y	Y	Y	Y	Y	Y
Ontology							Y		Y
Epistemology	Y	Y		Y		Y		Y	Y
Applicability	Y		Y						
Aesthetics									

Topic \Chapter:	19	20	21	22	23	24	25	26	27
Foundations	Y					Y		Y	Y
Ontology	Y		Y					Y	Y
Epistemology	Y	Y	Y			Y	Y	Y	Y
Applicability	Y		Y						Y
Aesthetics									

3.3 Organized by Philosophical Stance

Most of the articles in the book do not take a particular philosophical stance (though, from their tone, many are implicitly platonists—but none explicitly—so this category isn't included); rather, they aim for an eclectic introduction to philosophical issues. However, several either use or take a particular approach, and these are listed here. The categories are **Intuitionism**; **Formalism**; Social **constructivism** or humanistic mathematics.

Topic \Chapter:	1	2	3	4	5	6	7	8	9
Intuitionism									
Formalism							Y		
Constructivism				Y					

Topic \Chapter:	10	11	12	13	14	15	16	17	18
Intuitionism		Y							
Formalism					Y				
Constructivism									

Topic \Chapter:	19	20	21	22	23	24	25	26	27
Intuitionism									
Formalism									
Constructivism	Y	Y	Y	Y	Y				

3.4 Organized by type of Instruction Used

If you are looking for articles that fit with a particular instructional style (inquiry-based learning, lecture, etc.), this organization of the articles should help you decide what to read first. The categories are **Lecture** with some discussion; **Films or readings** with discussion; **Whole-class discussions**; small- **group discussions**; **IBL** (Inquiry-based learning and problem-solving); Other types of student **group work**; **Out-of-class** readings and papers.

Topic \Chapter:	1	2	3	4	5	6	7	8	9
Lecture		Y	Y				Y	Y	Y
Films or readings			Y	Y	Y				
Whole-class discussions	Y		Y						
Group discussions									
IBL				Y					
Other group work	Y					Y		Y	
Out-of-class only									

Topic \Chapter:	10	11	12	13	14	15	16	17	18
Lecture	Y	Y			Y				Y
Films or readings									
Whole-class discussions					Y		Y		
Group discussions		Y						Y	
IBL				Y			Y		
Other group work					Y				
Out-of-class only			Y						

Topic \Chapter:	19	20	21	22	23	24	25	26	27
Lecture						Y	Y		
Films or readings									
Whole-class discussions	Y				Y			Y	
Group discussions					Y				
IBL		Y		Y			Y		
Other group work					Y				
Out-of-class only									

3.5 Organized by Types of Assignments Used with the Philosophical Part of the Course

Finally, if you use certain kinds of assignments and are looking for philosophical issues that work well with them, the following classification may be helpful. Categories are **Readings** with responses; **Papers**; **Collaborative writing** (such as wikis or online forums); **Oral presentations**; **Examinations**; Student-designed experiments, interviews, or **projects**.

Topic \Chapter:	1	2	3	4	5	6	7	8	9
Readings	Y				Y				
Papers	Y		Y						
Collaborative writing									
Oral presentations									
Examinations									
Projects	Y					Y			

Topic \Chapter:	10	11	12	13	14	15	16	17	18
Readings			Y	Y			Y	Y	
Papers								Y	Y
Collaborative writing			Y	Y			Y		
Oral presentations		Y		Y					
Examinations			Y		Y				
Projects									

Topic \Chapter:	19	20	21	22	23	24	25	26	27
Readings									
Papers	Y			Y		Y			
Collaborative writing									
Oral presentations	Y		Y			Y			
Examinations									
Projects			Y	Y					

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I

The Big Picture

The chapters in this part of the book are intended to set the tone of the rest of the volume. Both, unlike all the rest of the volume other than Reuben Hersh's chapter, were previously published elsewhere. However, they contain material that we feel is helpful for readers to consider before reading the later chapters.

Chapter 1: Uffe Thomas Jankvist and Steffen Møllegaard Iversen's "Philosophy in Mathematics Education: A Categorization of the 'Whys' and 'Hows'" is a revised version of their article, "'Whys' and 'Hows' of Using Philosophy in Mathematics Education" that they published in *Science and Education* in 2013 (considerably condensed, and a bit added). It sets the stage for the book by offering two "whys" for using the philosophy of mathematics in mathematics education: as a goal in itself, and as a tool to improve mathematics learning. They also suggest three "hows," approaches to incorporating philosophy of mathematics in mathematics courses: illumination approaches, modules approaches, and philosophy-based approaches. Examples of each are given. They are Danish, but the topics and issues mentioned can be used in the United States, though perhaps at a different level. In any case, the examples provide good illustrations of their ideas.

Their final example, involving Bourbaki, structuralism, and the New Math, brings out an important issue in teaching mathematics and the role of philosophy in doing so: teachers' philosophies of how students learn probably influence their teaching a lot more than their beliefs about the nature of mathematics. Beliefs about how people learn mathematics is part of epistemology (the study of how we gain knowledge). So it is part of teachers' philosophies of mathematics. And the particular structuralist approach—doing everything from the point of view of set theory—was part of why the New Math was a disaster. (Of course, one can be a structuralist about mathematical objects without believing that one should teach mathematics via set theory.) The Bourbaki school's influence on mathematicians was in large part behind the adoption of the New Math. To the extent that it is a philosophy of "teach sets first," it is a demonstrably incorrect philosophy of mathematical epistemology.

Chapter 2: Bonnie Gold's "How Your Philosophy of Mathematics Impacts Your Teaching," reprinted from the *College Mathematics Journal* (2011), observes that, without necessarily being aware of it, faculty take stances (platonist, formalist, social constructivist) on philosophical issues when they teach mathematics, including elementary courses. The chapter can also serve as an introduction to philosophy of mathematics for the novice. The selection of topics is broad enough that most people will find some areas discussed of which they don't have full knowledge.

1

Philosophy in Mathematics Education: A Categorization of the "Whys" and "Hows"

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1 Introduction

In this chapter (a condensed version of [20]), we propose a categorization of the possible reasons for using philosophy in mathematics education and different approaches to doing so, the so-called "whys" and "hows." The resulting categorization encompasses two categories of the "whys" and three categories of the "hows" (similar to those in [13]). Each will be presented, explained, and illustrated through examples. But before we get to this, some clarification is needed in terms of what we shall consider to be philosophy of mathematics and what we shall not.

2 What Do We Mean by a "Philosophy of Mathematics?"

Of course, we include everything standardly understood to be a philosophy of mathematics. Examples are platonism, mathematical constructivism and realism (including also their social variants), empiricism (traditional and new), mathematical structuralism, logicism, Brouwer's intuitionism, Hilbert's formalism, etc. In addition to such actual directions of philosophy of mathematics we also include the various discussions of philosophical aspects, e.g., ontological, epistemological, etc., by people knowledgeable and competent in mathematics. Examples could be Hilbert's discussion [10] of the role of mathematical and extra-mathematical problems in the development of mathematics; Lakatos's *Proofs and Refutations* [24]; Wigner's paper [40] on the unreasonable effectiveness of mathematics in the natural sciences; Hamming's [8] follow-up on this about mathematics' effectiveness in the engineering sciences; or Hersh's discussions [9] of the question of discovery or invention of mathematics. Another way of phrasing this is to say that in all cases we insist that the philosophical discussions and discourses are grounded in an informed, if not deep, understanding of mathematics itself. (When we write "philosophy" in the chapter, we take this to mean "philosophy of mathematics," when nothing else is mentioned.)

3 The "Whys" of Using Philosophy in Mathematics Education

The arguments, reasons, or purposes for using the philosophy of mathematics in mathematics education fall into two categories: philosophy as a tool and philosophy as a goal.

The first category concerns the use of *philosophy as a tool* for the teaching and learning of mathematics. This category embraces, among others, arguments stating that philosophy may assist students in their making sense of mathematical concepts, ideas, and constructs such as definitions, theorems, proofs, etc., that is to say the internal issues—or *in-issues*—of mathematics. This of course also includes mathematical argumentation and the notion of mathematical proof itself, i.e., how we argue mathematically and why we prove mathematical statements in the first place. Furthermore, it encompasses arguments regarding motivation as well as arguments stating that philosophy may act as a vehicle for developing students' creative thinking.

Arguments belonging to the category of *philosophy as a goal*, on the other hand, include those stating that it serves a purpose in its own right for students to learn about the philosophy of mathematics or philosophical aspects of the discipline. This philosophical education could include, for instance, knowledge about the epistemology or ontology of mathematical concepts as well as the philosophical foundations of mathematics as a discipline. It could also include questions of why mathematical knowledge is often structured (or at least presented) the way it is, its science-philosophical status, etc.: that is to say metaperspective issues—or *meta-issues* of mathematics as a (scientific) discipline.

Although the distinction between the two types of “whys” in many respects seems clear, one should keep in mind that this is not always the case in concrete teaching-learning situations. If, for example, a philosophy-minded teacher of mathematics is using aspects of the philosophy of mathematics with the purpose of deepening the students' understanding of certain mathematical concepts, objects, or procedures this would, at least at first sight, seem like a clear-cut case of using philosophy as a tool. However, one could argue that in cases like this the teacher is in some sense doing exactly what philosophy is meant to be doing, i.e., deepen understanding and awareness, and therefore in this respect philosophy cannot be regarded merely as a tool. We acknowledge the existence of cases like this, which do not necessarily belong exclusively to one of the two categories of “whys.” Still, we find the tool versus goal distinction meaningful and productive as a way to provide analytic clarity and insight as to why the philosophy of mathematics is—or could be—used in the teaching and learning of mathematics.

In order to illustrate the two different “whys” of using philosophy in mathematics education, we offer two different examples from our own research carried out in upper secondary school and at undergraduate university level: one using philosophy as a tool to teach the notion of mathematical proof and one using philosophy as a goal to have students engage in discussions of “the unreasonable effectiveness of mathematics.”

3.1 Example of Using Philosophy as a Tool: the Notion of Proof

The focal point of the first example is argumentation or, to be precise, how philosophy can be used as a tool to work with the general properties of mathematical proof. As noted by Niss [28], students across the different levels of mathematics education often have substantial difficulties understanding what does or does not constitute a proof, and what purpose and functions proofs have in mathematics—a point that also finds support in the vast amount of literature on proof and proving in mathematics education research. This underlines the importance of addressing the notion of proof both on a general level and through concrete examples in the teaching and learning of mathematics.

The teaching unit sketched below was designed on the basis of a series of individual qualitative research interviews of six experienced Danish upper secondary school[1] mathematics teachers. The purpose of the interviews was to explore the teachers' experiences with the use of interdisciplinary activities involving the subjects of mathematics and philosophy in their daily teaching practice of mathematics [12]. Several of the interviewed teachers had themselves implemented actual teaching units involving both mathematics and philosophy and in two cases the units had focused on the notion of proof. In both cases the mathematics teacher in question considered philosophy as a tool to foster students' understanding of the notion of proof. One recapitulated the outcome of the implemented teaching unit:

> When they [the students] engage in a specific task in mathematics, they now have some concepts, some work habits, some patterns, some ways of thinking which they can use to throw light on what they are actually doing. (Teacher 3 in [12], translated from Danish.)

[1] Danish upper secondary school is three years, roughly equivalent to US high school, i.e., students ages sixteen to eighteen. Danish upper secondary students can choose to study mathematics for one, two, or all three years. Each year the mathematics course covers a range of mathematical topics rather than being devoted to a single topic such as geometry, algebra, or calculus, as is most common in American high schools. While several examples here concern Danish upper secondary school, they could be implemented in general education courses at the US collegiate level.

Using the insight gained by the interviews and drawing on theoretical perspectives of interdisciplinary activities involving mathematics (e.g., [26]), a teaching unit about the notion of proof involving the subjects of mathematics and philosophy was designed. A description of it was later published in an official *Education Manual for Upper Secondary Schools* [4] and implemented in several Danish upper secondary schools ([11], [12]).

In this context, we only sketch the content of the teaching unit, since our purpose is to provide illustrative examples rather than to discuss details of the examples. (Further elaboration and discussion of the teaching unit can be found in [11], [12].) Its fundamental idea was that students through argument analysis should characterize the argumentative properties of a collection of classical proofs, found both in mathematics and philosophy. Examples of proofs from mathematics included different proofs of the Pythagorean theorem as well as proofs of the fact that the sum of the angles in a Euclidean triangle is equal to 2π. The students also worked with proofs from philosophy such as modern versions of proofs of the existence of God, including the ontological proofs put forward by Anselm of Canterbury (1013–1109) and René Descartes (1596–1650), respectively, as well as Thomas Aquinas's (1225–1274) five ways (or proofs) (see, e.g., [3]). Common to this last group of proofs is their obvious inspiration from the realm of mathematics, more precisely, their way of using mathematical proof as a model for perfect argumentation. The students analyzed the proofs on the basis of more general models for argumentation (as for example that of [39]), and were subsequently put in positions where they had to discuss and elaborate on questions such as What constitutes a proof? What are the roles and functions of proofs in and outside the discipline of mathematics? Can everything be proven? And, are proofs necessarily true? The underlying idea was to enable the students to develop their understanding of the general properties of proofs in relation to the teaching and learning of mathematics, and in this endeavor philosophy served as an applicable and useful tool.

3.2 Example of Using Philosophy as a Goal: “the Unreasonable Effectiveness of Mathematics”

As our second example we consider a HAPh-module, an abbreviation referring to a teaching unit dealing with the History, Application, and Philosophy of a certain mathematical topic ([17], [18], [19]). The overall topic of the module was “the unreasonable effectiveness of mathematics” based on a philosophical discussion by Hamming [8] on the role of mathematics in engineering (and computer science). The purpose, in terms of philosophy as a goal, was to get the students to reflect on the role of mathematics, both inside and outside the discipline, as well as the nature of mathematics as both a pure and an applied discipline. In order to make this more concrete for the students, they were provided with a specific mathematical application upon which to base their reflections: the use of Boolean algebra in electric circuit design.

The setup of the module was that students read (parts of) and worked with three original texts (in Danish translation): chapters II and III of George Boole's *An Investigation of the Laws of Thought on which are Founded the Mathematical Theories of Logic and Probabilities* [2]; the first parts of Claude Shannon's article “A Symbolic Analysis of Relay and Switching Circuits” [31]; and the full article by Richard Hamming on “The Unreasonable Effectiveness of Mathematics” [8]. (For more detailed descriptions of the module, see [18], [19].) The final task of the HAPh-module was an essay assignment, in which the students were to work on a series of questions, which they first had to discuss in groups and next answer with small essays. Some of the questions took as their point of departure the title of Hamming's article, asking the students: (1) to account for Hamming's viewpoint on a piece of mathematics being effective; (2) compare the effectiveness of Boole's and Shannon's works, respectively, distinguishing between what might be referred to as effectiveness in terms of philosophy and effectiveness in terms of applications; (3) discuss what Hamming means by the title of his paper, and why the effectiveness of mathematics may be seen as unreasonable; and finally (4) to consider if Shannon's application of the idea of an (Boolean) algebra operating only on the elements 0 and 1 along with the mathematical interpretation of “and” and “or” in electric circuits could be seen as an example of Hamming's “unreasonable effectiveness of mathematics.”

The HAPh-module was originally designed for upper secondary level and was also implemented with a class of students in their final year of Danish upper secondary school in the fall of 2011. (For more details and results, see [17], [18], [19].) However, it has also been used as a mini-project assignment in an introductory course in discrete mathematics at Roskilde University in the spring of 2012, and it is from this implementation that we shall display a student group's answers of the essay questions presented above. The group consisted of four second year Natural Science Bachelor program [27] students, who had chosen mathematics as one of their two subjects to major in (their second subjects varying from physics through computer science to philosophy). The following quote stems from the group’s hand-in essays and exhibits their answers to essay questions 2, 3, and 4:

> [2.] If we look at the effectiveness of Boole’s and Shannon’s works, it is necessary to take into account certain conditions. First, we cannot compare the effectiveness of the two directly, since they seek to clarify different questions. One could say that Shannon’s work is more effective from an applicational point of view, whereas Boole’s work has a higher philosophical value, and hence is more philosophically effective. . . . (By “higher philosophical value” the students are referring to Boole setting out to find the logical basis for our language, what he refers to as “the laws of thought.”)
>
> [3.] Why Hamming’s article is titled “the unreasonable effectiveness of mathematics” is probably a double-sided story. One side concerns the effectiveness, as described above, while the other side concerns the fact that it is unreasonable. This is, according to Hamming, due to mathematics in its nature being so simple—Boole’s algebra does not use very many symbols and signs—while at the same time not making use of anything extreme. Mathematics can describe a lot of things outside mathematics itself, but it can only do so through concepts that already exist in the language of mathematics. One might compare the unreasonableness with the situation where two students are to solve a set of math exercises. Student 1 is allowed to use a calculator (symbolizing mathematics in its widest sense), while student 2 is not. This is downright unfair. And it is the same kind of unfairness or unreasonableness that one might imagine if we tried to solve specific problems either with or without mathematics.
>
> [4.] Shannon’s application of Boolean algebra can be seen as unreasonably effective, since he uses some pure mathematics to solve an entirely extra-mathematical problem. And the mathematical content not only offers the solution, it does so in a clear and (relatively) simple, applicable manner. . . . ([25, pp. 5–7], translated from Danish).

One of the insights that students may develop through the essay assignment is that mathematics can be effective in different ways and on different levels. For example, the students quoted seem to realize that it is not only the case that a piece of mathematics can be applied to solve an extra-mathematical problem, but that it may also provide clarity, or, as in the case of Boole, that it can serve as a means for investigating the more philosophical question regarding “the laws of thought.” As for the matter of “unreasonableness,” students could come to realize that the surprising bit is not necessarily that mathematics can be applied to extra-mathematical problems; i.e., some mathematical constructs are developed with a given extra-mathematical problem in mind, a matter that the undergraduate students here have not necessarily realized. (Cf. the following sentence from the essay answer: “Mathematics can describe a lot of things outside mathematics itself, but it can only do so through concepts which already exist in the language of mathematics.”) However, it *is* surprising (or unreasonable) that mathematical constructs developed in one setting, as for example Boolean algebra, can find clear-cut applications in very different settings, including electric circuit design—a matter potentially serving as a stepping stone for further philosophical reflections on the nature of mathematics as a discipline [17].

4 The “Hows” of Using Philosophy in Mathematics Education

The approaches to using philosophy of mathematics in mathematics education can be split into three categories. They are the illumination approaches, the modules approaches, and the philosophy-based approaches.

In the first category, the *illumination approaches*, the teaching and learning of mathematics is supplemented by philosophical information, whether this be in the classroom teaching or the mathematics textbooks used. It is clear that such supplements may be of varying size and scope. The smallest may concern isolated factual information, which may cover names, dates, famous events, problems, and paradoxes. Also, to some degree the telling of (historical) anecdotes concerning philosophical aspects of mathematics belongs to this category. Such smaller supplements are, we suggest, often used to spice up the mathematics teaching. At the other end of the scale we find the more extensive supplements that may be used to put mathematical topics or concepts into a philosophical perspective. Here, excerpts from original sources might even be used to support or underpin a certain perspective.

Unlike the illumination approaches, the *modules approaches* are actual instructional units devoted to the philosophy of mathematics or philosophical aspects of mathematics—units that may also be devoted to specific mathematical cases. Of course, these also may vary in size and scope. The smaller ones can encompass collections of materials concerning a certain topic or case, probably with strong ties to the curriculum, suitable for no more than two or three lessons, preferably ready-made for teachers to use in the classroom. Longer modules of perhaps ten to twenty lessons need not necessarily be tied to mathematical topics from the curriculum. They can provide the opportunity to study branches of mathematics that are not usually part of the syllabus. The ways to implement both the smaller as well as these larger modules are of course numerous: they may be introduced through textbook readings, studies of original sources, student projects, etc. The most substantial version of this approach involves full courses (or books) on philosophy or philosophical aspects of mathematics within a mathematics program. Such courses can rely on original or secondary sources (or both) depending on the level of philosophical studies intended as well as the students' mathematical background and education. Also the approaches can be implemented in many other ways than just through an actual course or reading of a book. One example is that of extensive student research projects as described by Kjeldsen and Blomhøj [23] and practiced at Roskilde University.

The third category, the *philosophy-based approaches*, covers approaches where the instruction follows strictly and is dictated by a particular philosophical school of thought, as for example that of intuitionism. Such approaches do not necessarily deal with the study of the philosophy in a direct way, as the philosophical positions underlying the approaches need not even be explicated in order to set the agenda for the manner in which mathematical topics are presented, how arguments are given, how theorems are proven, etc. Of course, one could argue that if the underlying philosophical framework of the teaching and learning activities remains in the dark, never to be examined critically by the students, then the approach might not be regarded as philosophy-based from the point of view that philosophy necessarily involves critical examinations of implicit fundamental conceptions. However, we regard this as too strict an interpretation of what should be considered a philosophy-based approach. The way we think about the manifestation of this approach in practice is more along the lines of Toeplitz's [38] distinction between a direct genetic method and an indirect genetic method regarding the use of the history of mathematics. The term "genetic" here refers to the genetic principle of following the historical development of a mathematical topic when presenting it to students, and Toeplitz outlines how this can be done in an either direct or indirect fashion. In the indirect case, the historical development makes up the basis for the order of presentation, but no history is ever brought forth and discussed with the students. (The ideas of Toeplitz are discussed in more detail in [14] and [30].) We realize naturally that the history of mathematics is a different domain than the philosophy of mathematics. What we argue is that, analogous to Toeplitz's idea of a direct and an indirect method, one can reasonably think about a philosophy-based approach as being either direct or indirect with regard to the way in which teachers can make use of specific philosophical frameworks in mathematics education.

4.1 Example of the Illumination Approach: Invention versus Discovery Through a Concrete Case of Error-Correcting Codes

Our example of illumination concerns the long-standing discussion of whether mathematics is discovered or invented (see, e.g., [9]), typically drawing on views of platonism versus various kinds of constructivism. The example stems from the early history of error-correcting codes. We shall outline it briefly, and then relate it to an educational setting afterwards.

After World War II, Richard Hamming, who had previously been part of the Los Alamos project as an applied mathematician, joined the Bell Laboratories, where Claude Shannon also worked. At Bell Labs Hamming did not have first priority to the computers, so many of his calculations were run over the weekends. Computers back then were only using error-detecting codes, meaning that whenever a computer detected an error due to noise, it would drop the current calculation and move on to the next in line. After finding that the computer had dropped his calculations two weekends in a row, Hamming said to himself:

> Damn it, if the machine can detect an error why can't it locate the position of the error and correct it? [34, p. 17].

Thus, Hamming began to develop his error-correcting codes. And by 1948, when Shannon published his “Mathematical Theory for Communication” [32], in which he proved the existence of good error-correcting codes, Hamming was able to provide Shannon with an actual example of a good (efficient) code [32, p. 16], i.e., his (7,4)-code (each codeword in the code consists of seven symbols, four being information symbols). The reason that Shannon needed this example was that his proof only showed the existence of such codes, not how to construct them. Now, when the Bell Labs learned of Hamming’s codes, they wanted to patent them. Hamming himself was indeed quite skeptical:

> I didn’t believe that you could patent a bunch of mathematical formulas. I said they couldn’t. They said, ‘Watch us.’ They were right. And since then I have known that I have a very weak understanding of patent laws because, regularly, things you shouldn’t be able to patent—it’s outrageous—you can patent [34, p. 27].

This matter of patenting lends itself to the discussion of invention versus discovery; because surely you should be able to patent inventions, but what about discoveries? And as can be seen from the quotation, even Hamming himself was rather puzzled by the very circumstances of this.[2] In fact, the patenting of Hamming’s codes led to a long delay of their publication. Not until 1950 was Hamming able to publish the codes [7].[3]

Although given here as an example of illumination for including philosophy in mathematics education, the case of Hamming codes has previously been used within a modules approach for using history as a goal in upper secondary mathematics education ([14], [15]), only in this case much more emphasis was put on the actual mathematics of error-correcting codes. As part of the research study, students were on several occasions asked about the question of discovery versus invention of mathematics, and were asked to relate this to concrete mathematical cases, including that of error-correcting codes. To illustrate how the aspect of patenting played a role in this regard, we offer the following excerpt from an interview session with the Danish upper secondary school student, Andrew:

> Andrew: That is what you asked me the last time as well: if mathematics is invented or discovered? I don’t really know. I think it’s because it’s something that reaches so far back that I really can’t see that you can invent something without it building on something already there. So I’d still see it as if mathematics is something you have and then discover new branches of it.
>
> Interviewer: And what defines “mathematics,” what do you mean by that then?
>
> Andrew: Well, I don’t know, all of mathematical thought or something, far back, in history, right, the logical way of thinking or something. And then there are some branches that are discovered, for example all that with the binary numbers that is a branching of mathematics. Therefore I believe it to be a discovery—or an extension of mathematics, I suppose you could call it. But not an invention!
>
> Interviewer: And what about coding theory?
>
> Andrew: Well that is where I’m actually a little in doubt, because when you can *patent* it, it ought to be an invention, but for me, logically speaking, I still believe it to be a discovery within mathematics ([14, pp. 238–239], translated from Danish).

So, Andrew finds himself in a similar situation as Hamming did, puzzled by the fact that you are able to patent something that he firmly views as and believes to be a discovery. From an educational point of view, it is not important whether students believe in invention or in discovery. What is important is that they reflect about the beliefs they do have, e.g., by comparing them with actual mathematical cases, in order to eventually come to hold their beliefs and viewpoints more evidentially [14].

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[2] For examples of another research study, where students were asked to make up their minds about the question of discovery or invention of mathematics, see [17]. Yet another example may be found in [29].

[3] By this time another mathematician, Marcel Golay, had already generalized the (7,4)-code given in Shannon’s paper to the whole family of Hamming-codes (plus finding a few additional ones himself, see [6])—a matter which gave rise to a long quarrel of who actually was the originator. For the students’ discussion of this aspect, see [15].

4.2 Example of the Modules Approach: the Concept of Infinity in Mathematics and Philosophy, Respectively

Our example of the modules approach is a medium-sized module, taken from a course on interdisciplinarity and modeling for future upper secondary teachers at the University of Southern Denmark. It contains both elements of philosophy as a tool and philosophy as a goal. While it appears to have its main focus on meta-issues of philosophy as a goal, it is clear at the same time that some insights cannot be reached by the students without acquiring some understanding of the involved mathematical in-issues along the way.

As part of the science and mathematics program at the University of Southern Denmark, all students who wish to fulfill the requirements for becoming an upper secondary school teacher take two mathematics and science education courses. The second course focuses on interdisciplinarity and modeling, due to recent requirements in the ministerial order for Danish upper secondary schools [21]. As part of the course, participants were, among other things, to design an interdisciplinary teaching activity and describe how the upper secondary students should be introduced to the interdisciplinary problem area of the activity, how they were to work during an implementation of the activity, and discuss how the students' meta-perspective discussion in relation to aspects of general education could be anchored in their mono-disciplinary content knowledge [16].

One group of student teachers designed a potential activity with the purpose of providing upper secondary school students with a historical insight into the interplay between a series of philosophical considerations about the concept of infinity and the more concrete mathematical handling of the concept. They gave a description of their idea:

> The activity begins with Aristotle's discussion of infinity and Zeno's paradoxes as well as their foreshadowing of the concept of infinitesimals. This is carried on to the historical development of the calculus of infinitesimals focusing on the works of Archimedes, Newton, and Leibniz. Leibniz's use of infinity from a philosophical perspective is also brought in here. The modern conception of infinity is illuminated within mathematics through Cantor's transfinite-ism and within philosophy through “Super Task theory,” which again takes us back to the paradoxes of Zeno[4] ([35, p. 8], translated from Danish).

Aristotle's discussion of infinity is to provide the upper secondary students with the philosophical introduction to the problem, whereas Archimedes' work on the approximation of areas introduces them to the concept of infinity, and eventually the notion of limits, from a historical, mathematical point of view. The paradoxes of Zeno, however, are thought to act as the interdisciplinary glue binding the philosophical and mathematical discourses together. The mathematical walk-through from Archimedes via Newton and Leibniz to Cantor as well as the simultaneous philosophical discussions of Aristotle, Leibniz's *Monadology*,[5] and finally the modern notion of super-tasks[6] should provide the students with the historical insights regarding development and legitimacy. As for the possibility of anchoring the interdisciplinary meta-perspective discussions in the mathematics and philosophy content knowledge, the student teachers provided the following reflections:

> In relation to philosophy, the activity puts focus on the significance of mathematics for the history of philosophy, not only by showing that several great philosophers were practicing mathematicians who had mathematics as an ideal of realization and source of inspiration, but also by applying philosophical texts where mathematical thinking is used in the actual argumentation. . . . Besides the fact that philosophical formation evidently becomes both wider and sharper by including knowledge from mathematics and science, the activity also shows that mathematical overview and understanding actually are preconditions for making sense of a series of philosophical insights. . . . Students should get the idea that you can use the subject knowledge of philosophy to unveil the general education aspects of mathematics, and that you can use subject knowledge of mathematics to strengthen and expand the general education aspects of philosophy. . . . In the actual encounter—the concept of infinity—students should realize that not only is this concept something

[4] Due to the complexity of some of the mathematical and philosophical constructs in play here, the student teachers state that their activity is directed towards the upper secondary advanced mathematics level, which in Denmark means that the students are taught mathematics throughout all three years of upper secondary school.

[5] The group mentions that philosophers such as Descartes and Spinoza may be included in the discussions.

[6] The references the student teachers give here are James F. Thomson ([36], [37]) and Paul Benacerraf ([1]).

> to consider within philosophy, it is actually something that you operate with in mathematics, in a formal and well-defined manner. And that the development of this construct is a result of a historical interplay between philosophy and mathematics, where it has been practically impossible to separate the two ([35, p. 10], translated from Danish).

The group does not give any explicit estimate of the length of the activity, but based on their content description it is fair to say that the approach is one of modules.

4.3 Example of the Philosophy-Based Approach: Nicolas Bourbaki and the New Math Movement

To exemplify the third category of approaches we present an example of a philosophy of mathematics that has influenced mathematics teaching, i.e., making it philosophy-based. The implementation may be regarded as either a direct or indirect philosophy-based approach, depending on whether the philosophical basis is explicated to the students or not. In our example, philosophy is used as a tool.

Perhaps the most well-known example of a philosophy-based approach in mathematics education may be found in the New Math movement, as introduced to numerous classrooms of mathematics around the world from the 1960s, and its relation to the Bourbaki Group, which since the 1930s published several substantial texts on the fundamental structures of analysis (e.g., [22, p. 813]). The Bourbaki group’s strong focus on strict rigor, a formal axiomatic approach, and general structures was the most prominent mathematical inspiration for the New Math movement and its choice of mathematical structures as the foundation of mathematics education around the world. Within the Bourbakist approach, mathematics takes its point of departure in general abstract concepts, leaving the consideration of concrete examples for later. In the Danish context, the New Math movement was implemented in a widely used textbook series called *Hi Mathematics!* (*Hej Matematik!*). These textbooks, which were often used at the primary level of mathematics education, took set theory as the basis of learning mathematics in accordance with the doctrine of the New Math movement.

Despite the fact that the Bourbakists themselves probably would have denied it, we will argue that their paradigm of pure mathematics may be seen as a philosophy of mathematics, both from a foundational point of view and an educational one. Even though the Bourbakists never formulated an explicit philosophy of mathematics they still, at least, worked within a specific framework of mathematics often associated with the name of “structuralism” in the philosophy of mathematics [33, p. 176]. At least one member of that group, Jean Dieudonné [5], explicitly discussed the use of structures in the work of Bourbaki. This leads us to argue that the New Math movement and its inspiration from the Bourbaki Group constitute a very interesting, although of course not uncontroversial, example of a philosophy-based approach to the teaching and learning of mathematics.

5 Discussion and Further Remarks

To sum up, the purpose of this chapter has been to present a potential categorization of the whys and hows of using philosophy, in particular the philosophy of mathematics, in mathematics education, and to illustrate each category carefully with a relevant example. The reason for choosing this approach is a consequence of the small number of studies that report on actual uses of philosophy in the teaching and learning of mathematics. It should be made clear that the categorization does not address the use of philosophy at specific levels of mathematics education: primary school, secondary school, upper secondary school, or university. Instead it is one that, so to speak, cuts across the various levels of mathematics education.

The given categorization was originally suggested in [19]. However, in [19, p. 641], a fourth “how” was also debated; one of philosophical discussions or dialogues, where (groups of) students representing different philosophical viewpoints or directions enter into discussion about philosophical questions, for example platonism versus one of the non-platonist directions discussing questions such as mathematics being discovered or invented (cf. the previous example from the illumination approach). Nevertheless, when working more intensively with the categorization and the examples in this article ([20]), we came to abandon such philosophical discussions as an actual approach in itself, which as a consequence aligns the categorization here presented with that for history presented in [13]. Rather we now consider it more as an important pedagogical tool that may be employed in any of the other categories. Of course other tools may be imagined; we have already heard of essays in relation to using philosophy as a goal, but also short lectures, e.g., representing different philosophical viewpoints, or even concept maps, come to mind as worth exploring in the context of using the philosophy of mathematics in mathematics education. Still, we find the idea of an actual philosophical discussion or dialogue central in regard to using the philosophy in mathematics education.

This being said, what can the suggested categorization consisting of the two "whys" and the three "hows" then provide? Besides serving as a source of inspiration for teachers, textbook authors, curriculum designers, etc., who are wanting to use philosophy in mathematics education, it can function as a way for them to formulate clearly their purpose for resorting to philosophy as well as a way of choosing and structuring their approach to doing this. Clearly, some approaches (hows) are more suitable for some purposes (whys) than others. For example, if one is using philosophy as a motivating tool, an illumination approach might be sufficient, but if one wants to use it as a tool for enhancing students' understanding of the notion of proof, a modules approach seems to be a better choice. Similarly, to really have students engage in a deep discussion of philosophical aspects, in terms of using philosophy as a goal, an illumination approach appears insufficient. (For an equivalent discussion regarding the use of history in mathematics education, see [13].) Hence, it is no coincidence that the rather extensive example of the concept of infinity in mathematics and philosophy, respectively, finds its implementation through a modules approach.

It should be emphasized that even though one's main focus is on, for example, philosophy as a goal, this does not mean that various crops regarding philosophy as a tool cannot be harvested along the way, if the mathematical case in question naturally leads to this, and vice versa. In a number of cases it may even be that in order to be able to discuss some of the meta-issues, an understanding of the related issues internal to mathematics is required. Something similar may be the case for the illumination approach. Although the main emphasis of this may be to use philosophy as a tool, it is certainly possible that students pick up on aspects related to philosophy as a goal along the way as well. Perhaps the one exception is that of the philosophy-based approach, since here the use of philosophy as a tool can become so indirect or implicit that philosophy and its role is never articulated. At any rate, from a design or an implementation point of view, the fact that a given use of philosophy in mathematics education is likely to have its main focus on either a use of philosophy as a tool or on one of philosophy as a goal, means that once this emphasis of the "why" has been made clear, it will, necessarily, be easier to choose a suitable "how."

Finally, we would like to suggest that the categorization proposed here can be useful in research on using philosophy in mathematics education by allowing researchers to make clear their motives, namely, whether they are concerned with philosophy as a tool or philosophy as a goal, and in what sense.

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2

How Your Philosophy of Mathematics Impacts Your Teaching

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> *My* philosophy of mathematics? I don't have one! I'm a mathematician, not a philosopher. I leave philosophical questions to the philosophers.

Maybe. Or perhaps you are among those mathematicians who are interested in the philosophy of mathematics. Whatever your attitude toward the philosophy of mathematics, when you teach mathematics, you do in fact take, and teach your students, positions on philosophical issues concerning mathematics. If you do not think about them, then you probably acquired your positions from your teachers when they imposed them on you without discussion. Further, you may find, if you do examine the positions you are taking, that they contradict each other, or disagree with positions you would say are obviously true.

Many mathematicians' distaste for the philosophy of mathematics comes from one of two sources. One is our preference for questions that, if worked on seriously, eventually receive definitive answers, as opposed to questions (as in philosophy) that, at best, clarify what the issues and alternatives are. The other is the detour the philosophy of mathematics took into foundational issues starting in the second half of the nineteenth century. This lasted until about 1975, when work on questions beyond foundations resumed. The detour led to a substantial growth in the area of mathematical logic, and some consensus on what was not going to be solved. (For example, we are not going to be able to show in a finitistic manner that mathematics is consistent.) However, most mainstream mathematicians lost interest after Gödel's work.

I use a rather broad definition of philosophy in this chapter: it includes anything involving our attitudes toward mathematics. I am not trying here to convince you to start working in the philosophy of mathematics. But I do hope to make you aware of a range of issues that we end up taking a stand on, when we teach a relatively standard calculus or introduction to proof course. Many of them are closely related to issues that beginning students are confused by. Perhaps a lack of clarity on philosophical matters adds to students' confusion.

1 The Nature of Mathematical Objects (Ontology)

This is one of the hottest issues among philosophers of mathematics. But most mathematicians simply do not care whether mathematical objects are abstract or concrete, fictions, communally created, or real. The old saw that the mathematician is a Platonist when working on mathematics and a formalist when discussing philosophy is often true. As Errett Bishop replied, when a philosopher questioned him about the distinction between numbers and numerals, "I identify a number with its numeral." (The philosopher was foolish enough to follow up with, "So how do you distinguish the numeral 2 from the numeral 3: they are both curly?" to which Bishop replied, "The numeral 2 is divisible by 2; the numeral 3 isn't." This interchange occurred at Cornell; I was in the audience.) We do not care what the number 2 is; what we care about is that it has certain properties. So most mathematicians do not care if you identify 2 with $\{\emptyset, \{\emptyset\}\}$ or with $\{\{\emptyset\}\}$ or don't identify it with a set at all. What is important to us is that there be no ambiguity about how mathematical objects behave. Whether you identify 2 with a set, or make no such identification, $2 + 3$ still equals 5, and 2 is still a prime number. This is what matters to us.

1.1 Numbers

Our students, however, have difficulties with some mathematical objects. Virtually all of them agree that $1/3 = .3333\ldots$, but many are uncomfortable with $1 = .9999\ldots$. They want $.9999\ldots$ to be the number "just to the left of 1." And we have made a choice when we tell them that $1 = .9999\ldots$: we have decided that there are no infinitesimals in the real numbers. (On the other hand, if there are infinitesimals in the real numbers, then $.9999\ldots$ is ambiguous: how far into the infinite does "..." go?) For that matter, students often believe that $1/3 = .3333333333$ (without the "..."), because that is what they see on their calculators when they punch $1 \div 3$. To add to the confusion, when they then multiply this result by 3, their calculators give them 1, although if they punch .3333333333 in directly and multiply by 3, they get .9999999999. These confusions present an excellent opportunity to discuss a range of issues such as how the real numbers were developed, how mathematics develops in general, a range of paradoxes, and several questions concerning approximation.

Sometimes the difficulty is due to our taking a word that has a meaning in everyday speech and changing its meaning slightly for use in mathematics. For example, calculus students often have difficulty with the idea that a function can achieve its limit. This difficulty may be a combination of the meaning, in English, of "limit" (though many of them, when they drive, not only achieve, but exceed, the speed limit) together with the fact that, in our definition of $\lim_{x \to c} f(x)$, we don't let $x = c$.

Many of Zeno's paradoxes relate to limits or to what are called completed infinities. It can enhance students' introduction to infinite series to consider the question of whether Achilles could ever beat the tortoise in his race. After all, the tortoise had a head start. By the time Achilles got to where the tortoise was when Achilles started, the tortoise had moved a bit further on. Another of Zeno's paradoxes related to calculus: if an arrow is always at some place at every time, then how can it be moving? They all relate to issues of limits and the nature of real numbers.

1.2 Equality

There are often subtle questions, both philosophical and mathematical, about when two mathematical objects are the same. For example, is $10^{\log(a)} = a$? Yes, when a is positive, but the left side is meaningless when a is negative or zero. What kinds of mathematical objects are under consideration at a given time has a lot to do with whether or not they are equal. Also, two mathematical objects that arose in different contexts often turn out to have the same properties and are eventually identified as being the same object. There is an interesting discussion of this issue in Mazur [7].

Functions play a very important role in mathematics. As students begin their university studies, their notion of function is usually a formula: $x^2 + 5x$, $2x/(x-2)$, $\sin x$—what our computer algebra systems insist is an expression. Most calculus courses (starting with the calculus reform movement) work on broadening the concept. Many of us choose to teach undergraduates that a function is like an input-output machine that, given an element of the input set, produces a unique element of the output set. An exercise I use, borrowed from Ed Dubinsky, is to give students in my introduction to mathematical reasoning class a range of situations (such as a statement like "Monmouth has a very good basketball team" or a graph that does not pass the so-called vertical line test). I then ask them, wherever possible, to come up with one or more functions relevant to that situation. This is quite a different approach from giving students the standard formal definition of a function (a subset f of the Cartesian product of two sets A and B, such that

$$(\forall x \in A)(\exists y \in B)((x, y) \in f) \wedge (\forall x, y, y')(((x, y) \in f \wedge (x, y') \in f) \Rightarrow y = y').$$

The first way of working with students fits well with a platonist or social constructivist view of mathematics. Platonists view mathematical objects as real objects that we can explore as if we were exploring a new country. Social constructivists view mathematical objects as constructed by the mathematical community. For them, engaging students in an exploration of functions is part of bringing them into this community. The formal definition approach fits better with formalism than with either of these more popular views. Certainly eventually we want all budding mathematicians to be comfortable with the formal definition for efficiency of communication. But only people who are formalists about pedagogy (and not even all of them) believe it to be the best way to introduce the concept.

Formalists and nominalists often find themselves speaking in a way that appears inconsistent with their beliefs. You may believe that, since the only things that exist are physical, there are not really any mathematical objects. But watch out! In class, when discussing continuity versus differentiability, you may find yourself saying "now let's construct a continuous, nowhere differentiable function."

In general, our understanding of a mathematical object grows as we study more mathematics. Our concept of function, as a map from real numbers to real numbers, is extended to functionals such as the derivative, to homomorphisms of assorted types, and so on, sometimes even to objects, in category theory, that no longer meet the formal definition of function because they are no longer defined on sets.

2 Mathematical Truth

As a student, one of the most attractive features of mathematics for me was that a mathematical problem had just one right answer. Further, that answer was as open to inspection and verification by a beginning student as by a teacher. However, as I grew a bit more sophisticated, I learned that there are mathematical questions with several correct answers (not inconsistent ones, but of a range of depth), depending on the context in which the question is set. Also, often when it seems that a question has been answered, it is later possible to relate it to other mathematical areas and, in doing so, raise variations on the question that have not yet been answered. (There is a nice article by Phillip Davis [4] on this.)

2.1 Theories of Truth

When we teach certain standard theorems of calculus (such as the extreme value theorem or the intermediate value theorem), we take a stand against intuitionism and constructivism. In particular, the theorems have constructive versions (slightly more complicated to state) that, classically, are equivalent to the theorems we teach. However, the standard versions found in most calculus or real analysis books are not theorems constructively. For example, a constructively true form of the intermediate value theorem is that, if $f : [0, 1] \to \mathbb{R}$ is continuous, with $f(0) < 0$ and $f(1) > 0$, then for each $\epsilon > 0$ there is an x in $[0, 1]$ such that $|f(x)| < \epsilon$. Or, if you add to the standard hypotheses that for every a and b in $[0, 1]$ there is an x in $[a, b]$ with $f(x) \neq 0$, the standard conclusion becomes constructively true. See [3, p. 59], for more variations.

Simply to say a certain mathematical statement is true involves taking a philosophical position. If you are a formalist, you say, rather than that "the theorem is true," that "it is a theorem within a given axiom system." For a substantial collection of philosophers of mathematics (called nominalists, or a subcategory, fictionalists: see the article by Mark Balaguer [1]), there are no mathematical truths, because there are no mathematical objects for them to be true about. The most you can say is that an assertion is "true in the story of mathematics," a story that the community of mathematicians builds, just as Conan Doyle built the story of Sherlock Holmes.

I've always been some kind of *realist* about mathematical objects and mathematical truth (though partly a social constructivist about our knowledge of them). Mathematical statements are true because they accurately describe mathematical objects. Yet as a young faculty member, when asked why $0! = 1$, I tended to give a formalist (or conventionalist) answer: "Because we can make it anything we want, and we define it that way." Yet I really don't like this answer either pedagogically—"because I say so" is not an attitude I want to teach—or philosophically, since there are far better, if longer, answers that are consistent with my philosophy.

2.2 The Persistence of Truth

Mathematics appears to be almost unique in that, once a problem is solved, the original solution remains true forever (except, of course, in the relatively rare cases where an error is found). This is a phenomenon that is a challenge for some (nominalist, fictionalist, and some social constructivist) philosophies of mathematics. It is certainly a property students should be made aware of. In fact, students often fail to appreciate that a mathematical property they learned in one mathematics course remains true in the next one. As my Introduction to Mathematical Reasoning students are constructing their first baby number theory proofs about even and odd integers, they often forget basic algebraic properties. So I get many proofs in which, for example, since a and b are even and they are trying to prove that ab is even, they let $a = 2x$, $b = 2y$, and then conclude that $ab = (2x)(2y) = 2xy$.

2.3 Verification of Truth

For most beginning students, however, the main problem with mathematical truth is in our rather peculiar method of determining it, completely different from anything in their previous experience. In many real-world situations, the example "proves" the rule. That is, giving one example may make it clear that the statement is, in fact, generally true, and when this is not the case, some kind of statistical inference from data is often used to establish truth. Not in mathematics: a general statement is only agreed to be true if we can give a proof. Although we insist on this in our proof courses, in lower-level classes we often imply exactly the opposite. Rather than giving a proof, we may give just one or two examples illustrating a theorem, perhaps along with a counterexample when one of the hypotheses fails. I am not arguing against doing this. I do believe, at least in beginning courses, that proofs are appropriate when the result is counterintuitive (such as the product rule for derivatives), but not when a picture accurately illustrates the general situation. What I advocate is being more explicit about the role of proof and the role of examples in determining truth in mathematics. And there are quite a few roles for examples. First, examples are used to show that a general statement is false. Second, a good (fairly generic) example often leads to a proof. Examples are also extremely helpful for understanding what a theorem is saying. They often also lead to an understanding of what is happening in a mathematical situation, which helps us generate a conjecture. Far too often, once our students have become convinced that an example cannot replace a proof, they stop using examples altogether.

The issue of examples also comes up in proof by cases. For students, doing a proof by cases seems a lot like doing a proof by giving a few examples, and the distinction needs to be made clear.

3 Mathematics and the Real World (Epistemology)

3.1 Pictures, Graphs, and Doodles

The role of drawings in mathematics is getting increasing attention recently among philosophers of mathematics. Several books and articles (e.g., [2], [6]) have been written on the topic. Most of us enjoy the proofs without words that sometimes appear in assorted MAA journals. But I think most of us feel that these are not quite proofs. To turn them into what we would consider proper proofs, one needs to add statements such as "Given any triangle, with . . . " and turn the picture information into algebraic information to make sure that it is indeed sufficiently general.

On the other hand, pictures are efficient ways to communicate the idea of what is happening mathematically—and, depending on your philosophical viewpoint, that may be exactly what mathematics is about. Some of our students are not visual learners, but for most of our students, accompanying algebra with a picture makes the situation clearer. In any case, whatever you say to your students about pictures and their use in mathematics implies a philosophical standpoint. It thus is worthwhile to give some thought to what you believe pictures have to do with mathematics. If you are a formalist, until we have a formal system for using pictures, they are simply a device to help people, who are visual, develop formal proofs. For a platonist, pictures may play an important role in how the physical beings that we are can have contact with the mind-, space-, and time-independent realm of mathematical objects. As a fictionalist, pictures may be part of "our story of mathematics"—the illustrations?

3.2 The Unreasonable Applicability of Mathematics

Another issue is the question of why mathematics is so applicable, particularly to problems other than those it was developed to solve. Again, this poses problems for various philosophies of mathematics (some platonist views and formalism, for example). Why should abstract objects, or formal deductions, have anything to do with the physical world? Students also need to be made aware of the dangers of applying mathematics to the world without a careful model-building process, and of the dangers of extrapolating mathematical conclusions beyond the limits of the model.

This gets back to the very basic philosophical issue of being able to distinguish what we know from what we do not know.

4 Language and Logic (Semantics)

As we move through school as budding mathematicians, we learn the language of mathematics, and by the time we start teaching, it is second nature. But there are a number of ways that mathematicians' use of language differs from our students' pre-college experience.

4.1 Logical Connectives

Susanna Epp has written eloquently about differences between everyday speech and mathematical use of logical terms such as "if . . . then . . . ," "or," and "not" when combined with a quantifier (see, e.g., [5]). In everyday usage, "or" can be inclusive ("Would you like sugar or cream with your coffee?") or exclusive ("Would you like soup or a salad?"). It is most often used exclusively, whereas in mathematics it is almost always used inclusively. In everyday speech we very frequently use the "if . . . then . . . " structure when we mean "if and only if." ("You may watch television if you finish your homework.") In mathematics, this distinction is essential: a statement is equivalent to its contrapositive, which is important in several types of proofs, but not equivalent to its converse or inverse. Also, in mathematics, we say that $P \Rightarrow Q$ is true as long as P is false or Q is true. This is very counterintuitive to most students. They usually believe that "if pigs fly, all even integers greater than 2 are composite" is false, because pigs do not fly. Negations are also used differently: "Mom, everyone is going to the after-prom." "No, Jenny, everyone is *not* going to the after-prom—I just talked with Cindy's mom, and Cindy isn't going." Mathematically, the "not" belongs before the "everyone": "No, Jenny, *not* everyone is going to the after-prom. . . ." But that is not how we speak. We are also quite insensitive, in everyday speech, to the order of quantifiers, yet in mathematics $(\forall x)(\exists y)$ is very different from $(\exists y)(\forall x)$.

On most of these issues, mathematicians are united, but pedagogically we need to be aware of the notational issues when we are trying to initiate our beginning students. That is, we need to be sure we are all using the same *semantics*. There are other logical issues on which we are not completely united, where you must take a stand.

4.2 Other Logics

First, of course, is where you stand on intuitionism (and constructivism) versus standard logic. Intuitionists only accept the law of the excluded middle in restricted circumstances, and certainly not for proving that something exists by showing that its non-existence leads to a contradiction. You may feel that the law of the excluded middle—that for every statement P, either P or its negation (symbolized here by $\neg P$) must be true—is obvious. For an intuitionist or constructivist, however, you do not know this until you know which one is true. Fermat's last theorem became true when Wiles proved it. Before then it was neither true nor false, simply unknown. For very simple statements P, students will usually agree that either P or $\neg P$ must be true, but once a statement gets complicated with quantifiers and negations, this is less clear to them. Certainly in everyday language, it can be the case that neither P nor $\neg P$ is true: for example, one or both may be meaningless. (E.g., "Circles are green.")

Students are often uncomfortable with the use of the law of the excluded middle in proofs by contradiction. It is bad enough that we tell them they cannot assume, in their proofs, the statement they are trying to prove. Now we tell them that one method of proof is to assume the negation of the statement they are trying to prove! It may help somewhat to tell them that, while most mathematicians currently agree that this is a valid method of proof, in some situations there are mathematicians who reject it.

More generally, there are more logics that people use than simply the two-valued logic of most mathematics textbooks: tense logics, modal logic, and so on. You certainly are welcome to stick with two-valued logic, but you thereby make a philosophical choice.

4.3 Definitions

Definitions have a different status in mathematics than in everyday language. Usually one learns words by ostension—that is, by seeing an example or having one described (say, a table, or the disease called shingles) and the person who is teaching the word attaches it to the example or situation. One usually uses definitions to distinguish between two words that have similar (but not identical) usages. An object may satisfy a particular definition more or less well—is a stool a chair? In most situations a tree stump is not a chair, but it may function as one. In mathematics, definitions are used to carve out, with no ambiguity, certain collections of mathematical objects. Any statement about one of the objects generally can be replaced with the definition in proofs or examples. So, in some sense, giving a definition for a mathematical object says everything there is to be said about it. Of course, in other ways it certainly does not: one actually learns the meaning of a mathematical concept by working with examples as well as theorems involving it.

From a formal standpoint, however, the definition is all there is to the concept. Thus, if you teach a course in the traditional definition-theorem-proof mode, you essentially endorse formalism. (Of course, one can be a formalist about how one learns mathematics without being a formalist about what mathematics is: more on this distinction later.) In any case, some discussion with students about the role of definitions in mathematics (at least in courses for mathematics majors, and courses for teachers at any level) is helpful for them.

4.4 Notation and Its Abuse

A related issue is symbolism. Much mathematics (for example, certain solutions of differential equations) was discovered by saying, "let's assume that this notation works in this new situation, and see if it leads us to a correct solution." (Some theoretical physicists similarly seem to use mathematical symbolism as magic.) Further, much of students' high school mathematics consists of manipulating a long chain of symbols until it turns into what is required. Perhaps from this, our students often use symbols as if certain properties automatically come with them, particularly distributivity and commutativity. Hence they will tend to move without thinking from $\sin(x + y)$ to $\sin(x) + \sin(y)$ (and similarly with most functions, whether linear or not).

This is not entirely our students' fault. Mathematicians are notorious for abuse of notation. We often use the same symbols in several different contexts (overloading the notation). For example, (a, b) can be a point in the plane, an interval on the real line, the greatest common divisor of a and b, a member of any Cartesian product, and so on. We use variables in a wide range of ways, expecting the student to figure out how it is being used from the context. Does a represent a range of values of an input variable for a function, the (finite number of) solutions to a given equation, an arbitrary real number that can be substituted in an identity, a parameter that we are keeping fixed for the time being while x acts as an input variable, a bound variable within some specific range, or a free variable? Eventually, mathematicians get used to determining the meaning from the context, but in our freshman courses some mention of context often clarifies a statement that otherwise mystifies much of the class. A discussion of mathematical notation and mathematical symbolism requires some thought about these issues.

5 Teaching and Learning Mathematics (Pedagogical Epistemology)

How you introduce new mathematical concepts—whether by giving a few examples first, or by asking questions or working on problems that bring out the need for the concept, or by giving the definition and a few theorems—is strongly related to your beliefs about how mathematical knowledge is acquired. And your belief about how mathematical knowledge is acquired is often related to what you believe mathematics is about: objects independent of us, socially constructed objects, formal deductions from axioms, etc. A formalist, for example, is likely to give axioms and definitions first.

5.1 How do We Learn Mathematics?

The study of how we acquire mathematical knowledge is still in its infancy. We are learning that certain approaches work better than others, but there is certainly no coherent theory that is widely agreed on yet. We learned mathematics ourselves via a range of approaches and activities. Further, what works well for those of us who go on to become mathematicians often does not work for the majority of students in our classes.

Usually the route by which a mathematical concept was originally developed is not how we teach it, once the particular area is well enough understood to be taught to undergraduates. Certainly, it's rare for someone to assert that students should learn a concept in exactly the same way it was originally discovered. On the other hand, the mathematics curriculum broadly recapitulates the history of mathematics. There is a fairly popular view among educators that students should construct their understandings of mathematical concepts. However, if one teaches this way, one carefully guides students' explorations in the process of making the discoveries so that they do not have to take the centuries it took the human community to develop the mathematics.

5.2 Playing One Hand Against the Other

Since many questions about how students learn mathematics remain unanswered, most of us—I certainly include myself—are conflicted in our beliefs about how mathematical knowledge is acquired. Without reflection, most people teach in the manner they were taught, rather than in a manner reflecting their belief of how knowledge is acquired. It is quite possible to be a formalist about mathematics—to believe that mathematics is just a formal game played with symbols from a given set of axioms—and yet believe that, to get students to learn to play the game effectively one should have them make certain kinds of constructions. Or one could be a social constructivist about how mathematical knowledge is developed by the community of mathematicians and still have students learn it formally as their entry ticket into the community. But generally it does seem to make sense that, if you believe that mathematical knowledge is socially constructed, you would have your students socially constructing their own mathematical knowledge, at least to some extent. Similarly, if you believe that mathematics is, in a broad sense, about phenomena that are part of the world we live in (both physical and mental, say), then you are likely to have students approaching mathematics at least somewhat as they approach other sciences, by a certain amount of (guided) discovery. (You can hold both of these beliefs, by the way, as I do. I believe mathematics is in the world around us, waiting to be discovered, but that our mathematical knowledge, just like our knowledge of any other science, is constructed by the community of scholars.)

6 Conclusion

I hope I have convinced you that there is quite a range of philosophical questions on which you take a position when you teach. By what we say in class we take positions on what mathematical objects are, the role of definitions in mathematics, the kind of logical rules that are to be followed, and how we determine truth in mathematics. Your students will benefit if you give some thought to these issues before you take, and teach, a position.

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II

First-year Seminar and General Education Courses

An increasing number of colleges and universities offer some kind of first-year seminar course, and all offer general education (quantitative reasoning) courses for non-science majors. The audience in these courses is less sophisticated than most of our mathematics majors, but many courses allow faculty great freedom to focus on topics that they consider important. They thus offer us an opportunity broaden students' perspective on mathematics by examining various philosophical aspects of mathematics: the nature of mathematics and the mathematical enterprise and the role of mathematics in society, broadly interpreted. Further, helping young citizens understand what mathematics is about and how it contributes to society can make an enormous difference in society's attitude toward mathematics, science in general, and the value of understanding an issue before taking a stance on it.

Chapter 3: Susan Jane Colley, in "What Is Mathematics and Why Won't It Go Away? Philosophy of Mathematics in a First-Year Seminar," uses fiction and non-fiction readings about mathematics, mathematicians, and culture. Her course involves a mixture of some mathematics (some via lectures, some via problem-solving, ideas from propositional logic, elementary number theory, mathematical induction, infinity, topology, geometry, and group theory) and humanistic issues (discussed after students have done some reading and posting to the course website) such as whether mathematics is discovered or invented, what constitutes a proof, the culture of mathematics and how social forces influence its development, and the aesthetics of mathematics. While some humanistic issues are more sociology than philosophy, several are tied to deep philosophical questions. Whether mathematics is discovered or invented is centrally linked to the ontological question of whether mathematical objects are part of the world, independent of human beings. The nature of proof is a central question to mathematical epistemology, as proof is generally considered a primarly component of mathematical knowledge. And the relation between aesthetics and mathematics is another central philosophical issue. Although her course is a first-year seminar, Colley has suggestions for how to include some of this material in general education courses, or even capstone courses for our majors.

Chapter 4: Mike Pinter, in "Helping Students See Philosophical Elements in a Mathematics Course," describes a course he has developed, Analytics: Math Models, that fulfills the quantitative reasoning requirements for honors program sophomores and juniors. The metaphor for the course is "seeing the unseen." He takes a humanistic and social constructivist bent. Philosophical elements are interspersed throughout the course, intermixed with doing mathematics (problem solving, logic, and discrete mathematics topics). He uses readings and films about mathematicians (Erdös, Wiles, and Robert Moses) to show the humanity of mathematics and introduce sociocultural aspects of the development of mathematical knowledge, foundational questions, Gödel's theorems (via *Logicomix*) and elements of uncertainty in mathematics, and other topics. The style of the course is inquiry-guided learning. Although such an honors course for non-majors is relatively uncommon, it could well be adapted to a capstone course for mathematics majors, or portions of it could be adapted to quantitative reasoning courses for more general students.

Chapter 5: Chuck Rocca's "An Exercise in the Philosophy of Mathematics" involves three general education courses he teaches (Great Ideas in Mathematics, History of Mathematics, and Mathematics in Literature). The exercise is a four-week project at the beginning of each course, aimed at broadening students' understanding of the nature of mathematics. Via videos, readings, and class discussions, it concentrates on three questions: what is mathematics, what does it mean to think mathematically, and do we invent or discover mathematics? The project culminates in an essay students write in response to them (and he includes his scoring rubric for the essay). Rocca has also investigated, via an activity on the last day of class, how much of this material is retained. One of the strengths of this chapter is the wide variety of quantitative reasoning courses in which the project has been used.

Chapter 6: Kayla Dwelle's "Evangelizing for Mathematics" describes her work in Mathematics for the Liberal Arts, a general education alternative to college algebra, on changing students' attitudes toward mathematics. She asks, "How do we evangelize mathematics to mathematically agnostic students?" She wants her students to understand that mathematics is intimately connected with the world, is accessible to everyone, and has aesthetic value. To help her students develop these understandings, she has them work in groups on problems requiring modeling or visualization, investigate Fibonacci numbers' occurrence in nature, do an art project and a community volunteering project, both of which must use mathematics, and several other activities. One noteworthy aspect of this chapter is Dwelle's ideas on getting to know the students and their mathematical viewpoints, and then working with what she learns.

3

What Is Mathematics and Why Won't It Go Away? Philosophy of Mathematics in a First-Year Seminar

Susan Jane Colley
Oberlin College

1 Introduction

Mathematics is much more than calculation and procedure. It gives rise to fundamental epistemological and ontological questions, and it has a vital place among all intellectual discourses. The first-year seminar—an increasingly popular way of providing beginning undergraduates with a small class experience, closer contact with faculty, and plenty of give-and-take discussion—is an excellent vehicle for introducing students to metamathematical and philosophical questions. In this chapter I describe a seminar that I developed and taught four times to first-year students at Oberlin College. Students explore mathematical proof and problem-solving and participate in discussions of related philosophical, cultural, and aesthetic issues. My goal for them is that they consider a familiar subject—mathematics, or at least what they think of as mathematics—in new ways. And although the intimate format of a seminar conveniently allows for contemplation of philosophical and metamathematical questions, I believe that some of the approaches I take can be transferred to other class settings.

2 Background

Oberlin College is a highly selective liberal arts institution consisting of two divisions: the College of Arts and Sciences and the Conservatory of Music. Total enrollment in both divisions is approximately 2900 students; of these, some 2300 are pursuing the B.A. degree in Arts and Sciences and another 175 are pursuing both the B.A. and B.M. degrees in a double degree program. The course I discuss here is actually offered not, technically, in the mathematics department, but in Oberlin's First Year Seminar Program for Arts and Sciences students. This program began in 2001 with the general goal of providing new students with an introduction to learning in the liberal arts environment. More specifically, each seminar should engage students in critical and creative thinking, discussion, writing and, when appropriate, quantitative work. Hence each seminar enables students to satisfy a portion of Oberlin's writing or quantitative and formal reasoning requirements for the B.A. degree.[1] In order to facilitate active student participation, each seminar has enrollment limited to fourteen. Here is the official catalog description of my seminar:

> **FYSP 177—What is Mathematics and Why Won't It Go Away?**
>
> This seminar will provide opportunities to engage in various activities (problem-solving, conjecture, and proof) and to explore the nature of mathematical thinking and discourse. Works of both non-fiction and fiction will be discussed and issues such as problem-solving vs. theory-building, the nature of mathematical truth and proof, aesthetic qualities in mathematics, mathematics and madness, cognition and mathematics will be considered. Intended for students without extensive background beyond high school mathematics.

The minimal prerequisites for the seminar mean that students are not presumed to have studied calculus, for example. However, I do alert students that I expect them to have good preparation in mathematics below the level of calculus, meaning algebra, geometry, and some precalculus. I have taught the seminar in the fall semesters of 2004, 2006, 2007, and 2011. The first three times the audience included many students who had a background in calculus and professed interest in majoring in the natural sciences. The last time I offered the seminar (in the fall of 2011), the students were more heavily oriented toward the humanities and social sciences and several students had unexpectedly weak mathematical preparation and were, relative to students in prior years, more math-averse.

3 The Seminar and the Philosophy of Mathematics

As the brief description suggests, the seminar involved a mix of some mathematics together with discussions about humanistic aspects of the discipline. In addition to addressing the general objectives of the First Year Seminar Program, I used the course to give students an opportunity to think about what mathematics really is, to consider its significance as an intellectual discipline and the broad cultural context for it, and to confront a variety of affect issues (e.g., frustration, anxiety, as well as satisfaction and joy) related to doing mathematical work. My chief personal goal was to expose students to some of the ideas that I found compelling and that helped to draw me into mathematics. As a result, the reading and topics were eclectic and only parts of the course took up philosophical questions.

Over the course of the thirteen-week semester, students were expected to read (in order):

A. Doxiadis, *Uncle Petros and Goldbach's Conjecture* [5]

D. Auburn, *Proof* [2]

P. Davis and R. Hersh, *The Mathematical Experience* [4]

C. Henrion, *Women in Mathematics: The Addition of Difference* [9]

S. Nasar, *A Beautiful Mind* [11]

J. Weeks, *The Shape of Space* [17] (especially Parts I and II)

T. Stoppard, *Arcadia* [16]

H. M. Enzensburger, *The Number Devil* [6] (prior to Fall 2011)

Y. Ogawa, *The Housekeeper and the Professor* [12] (Fall 2011 only)

C. P. Snow, *The Two Cultures* [15]

G. H. Hardy, *A Mathematician's Apology* [8].

In addition, we also viewed a variety of films and documentaries, both in and out of class. I subdivided the course into four main units:

Doing Mathematics
About Mathematicians
Mathematical Theories: Two Examples
Mathematics in Culture and Society.

[1] Oberlin's writing and quantitative and formal reasoning requirements were revised beginning fall 2013, but the seminar program's overall goals remained unchanged.

During each unit, there was a mix of mathematics and discussion concerning the reading and videos. Student grades were based on six mathematical problem sets (45% of final grade), three five-page essays (45%), plus class discussion and other small assignments (10%). For further details regarding the construction of the seminar, see Colley [3].

The main mathematical objectives revolved around proof and rigorous argumentation and abstraction. After warming up with some general problem-solving, we touched on ideas from propositional logic, elementary number theory, mathematical induction, infinity, topology, geometry, and group theory. The subjects were not given axiomatic treatments, of course, and were chosen in part to coordinate with the readings. Thus, for example, while reading *Uncle Petros and Goldbach's Conjecture* [5] and *Proof* [2], we talked about number theory, which was pertinent to the stories, and during and after reading *The Mathematical Experience* [4], we discussed infinity: infinite sums and the hierarchies of infinity, including Cantor's diagonal argument. The presentation of the primary mathematical content was largely traditional, meaning that I spoke at the blackboard, although I regularly solicited questions and comments from the students while I talked. Nonetheless, on occasion I had students work on problems that we subsequently came together to discuss.

Issues in the philosophy of mathematics were mostly taken up while students were reading *The Mathematical Experience* [4] during the first part of the semester, and then again—to a somewhat lesser extent—in the last part of the course. We entertained questions regarding the nature of mathematical reality:

Does mathematics have a platonic existence, or is it a human construct?

Is it discovered or invented?

How is it shaped by social forces?

Is the universe structured so that mathematics would inevitably develop in the way that it has, or do we see and describe nature through the lens of the mathematics we have developed?

What constitutes a proof?

We spent some time talking about the three foundational schools of the philosophy of mathematics of the early twentieth century, i.e., platonism, formalism, and intuitionism (or its variant, constructivism) (Chapter 7 of *The Mathematical Experience* [4]). Although I did not assign R. Hersh's thought-provoking *What Is Mathematics, Really?* [10], we did read and discuss "Dialogue with Laura" [10, pp. xxi–xxiv] in class as a way to encapsulate some of the ontological questions.

Given that students had previously read *Uncle Petros and Goldbach's Conjecture* [5] and *Proof* [2] and had attempted to construct their own proofs (both in and out of class), we also talked about the act of putting together a formal argument, e.g., how one determines if a proof is correct or complete, as well as the darker side of frustration when searching for a proof or problem solution. As part of this discussion, students had to consider the nature of mathematical truth and knowledge and how it might differ from other forms of knowledge. To highlight the particular nature of rigorous proof, the contemporary use of computers in mathematics offered interesting discussion points. We could contrast machine computation as an exploratory aid for providing examples and counterexamples to formulate or refute conjectures with its more controversial use in mathematical proof itself (e.g., the proofs of the four color theorem or the Kepler conjecture). This helped some students understand the difference between inductive and deductive reasoning. Suggested supplementary reading concerning issues about work in mathematics included Henri Poincaré's famous essay "Mathematical Creation" [13] and Alfred Adler's "Mathematics and Creativity" [1].

We approached the ideas in *The Mathematical Experience* [4] first through student posting to the class website followed by class discussion. To a great extent, I let students control the in-class agenda by asking each student who spoke to recognize the next speaker, including me. (We used this format whenever we discussed readings and videos.) While we were still reading *The Mathematical Experience*, I asked students to write the first of the three main papers for the course: an essay reflecting on mathematics as subject and on doing mathematics. Students were free to respond to this topic in whatever manner they wished, although I did offer the following prompts:

What is the content of mathematics?

How does mathematics operate? Does it work like a science? A philosophy? In what way(s) is it different from these disciplines?

Given the nature and content of mathematics, what characteristics does it take to work on mathematical problems? What habits of mind are needed?

What processes of thinking are undertaken?

Can mathematics be a social activity?

Is mathematics a painful or pleasurable activity (or both)? When and why?

Students took the prompts in various directions. A few wrote prosaically about mathematics, continuing to think of the subject as concentrated on specific algorithms and techniques to be implemented, rather than as a system for abstract thought. But even during the early part of the semester, some students began to make more sophisticated connections between mathematics and other modes of thought. Some found it interesting to compare mathematics and religion (see Chapter 3 of *The Mathematical Experience* [4], especially pp. 108–112): to what degree they are distinct, to what degree they share a bond in the emphasis on regularity (i.e., pattern) and ritual (i.e., algorithm). Here again students had to consider how the subject of mathematics related to its practitioners.

The second essay concerned mathematicians: myths about them, portrayals of them, and the social community of mathematicians. As this topic was the most removed from the philosophical issues we encountered, I will not discuss it further.

Late in the semester, after we had considered essentially all the mathematical content, we focused on notions of beauty and elegance in mathematics. In class we critiqued different arguments and proofs in an attempt to determine what was particularly useful or compelling about them. For example, we compared two proofs of the irrationality of $\sqrt{2}$ that both begin, of course, by assuming that $\sqrt{2} = a/b$ with a and b positive integers. The first proof also assumed that b is the minimal positive denominator; the second made use of prime factorizations of a and b. Students considered in what ways the extreme economy of the first proof might be preferable to the second which relies on the fundamental theorem of arithmetic—which proof gives more insight and why? Discussions such as these took place throughout the course, but especially near the end of the term as students were reading *The Two Cultures* [15] and *A Mathematician's Apology* [8].

The final essay was about aesthetics and mathematics. To prepare students for writing it, I asked for a brief (1–2 page) position paper on beautiful proofs:

> Over the course of the semester, we have seen many examples of mathematical arguments and proofs. Some of them may strike you as especially compelling, others much less so. To help you begin thinking about your final paper, please respond to the questions
>
> Describe a mathematical proof or argument that you especially like. Why does it appeal to you?
>
> Describe a proof or argument that you do not like. Why do you feel the way you do?

For the final essay, students were given the following prompts:

Is mathematics beautiful? In what ways?

Discuss aesthetic qualities in mathematics that are wholly internal to mathematics. For instance, you might describe some especially compelling proofs or theorems. What about your examples are beautiful and why? (You might also consider contrasting mathematical arguments you find particularly pleasing with some that you do not find especially aesthetic.)

Provide examples of aesthetic qualities of mathematics that are related to other modes of thought or expression. Again, how does the particular elegance or beauty arise and why?

Describe any analogies (or the lack thereof) between elegance in mathematics and elegance in other intellectual activities.

Is the beauty one finds in mathematics relative to one's expertise in the field?

The best essays developed good general criteria for assessing mathematical beauty, discussed them by providing clear examples, and related ideas to the readings. For instance, one student wrote

> . . . The Professor in the novel *The Housekeeper and the Professor* explains that “a truly correct proof is one that strikes a harmonious balance between strength and flexibility.” (Ogawa [12, p. 16]). Thus, beauty is required for a proof to be considered truly correct. Being technically correct does not matter if the proof is messy, inelegant, or counterintuitive.
>
> . . . The beauty of an individual proof is enhanced when the mathematical idea “can be connected, in a natural and illuminating way, with a large complex of other mathematical ideas.” (Hardy [8, p. 89]). This integration of a beautiful proof in the larger body of mathematical knowledge is what mathematicians strive for, because the true beauty of mathematics is its lasting ability. As Philip Davis and Reuben Hersh explain in *The Mathematical Experience*, “the highest aspiration in mathematics is the aspiration to achieve a lasting work of art.” [4, p. 86].

Another offered the view

> In examining mathematics from an aesthetic perspective, I find myself coming back to considering the similarities between the mathematics I most love and the poetry I most love.. . . I am similarly drawn to those proofs and mathematical concepts that possess a simple, brief elegance—for instance, Euclid’s proof that there exists an infinite number of primes. The concept of infinity itself might bother or frustrate some, but I think of it the same way I might think of a favorite poem: infinity must begin somewhere, must be grounded in something (in this case, the smallest primes: 2, 3, 5, 7, 11, 13, and so on), just as an accessible poem begins in an easily understandable place. Euclid’s proof contains an awe-inspiring largeness in a physically brief space; he has taken a vast concept—perhaps the vastest in all of mathematics—and made it countable, reachable, and accessible. My favorite poetry and my favorite mathematics strive for similar goals: to bring the massive ideas of the universe to an understandable level.

4 Student Learning and Student Reactions

Overall, I was pleased with the progress most students made throughout the semester, in terms of what they were able to do mathematically and the way in which they approached their arguments, both in class and in their writing. Students had to engage some sophisticated material, particularly with respect to the mathematics. This posed a considerable challenge for those with weaker preparation. Mathematical proof is hard for students at every level (mathematicians, too) and—no surprise—not every student was successful. Abstraction and precise argumentation and communication, whether mathematical or philosophical, are difficult skills to master. Nonetheless, my students made some significant progress and began to see mathematics as an effective mechanism for working with abstraction.

Despite what I see as a generally successful seminar, the relatively large amount of reading I assigned did at times work against depth of discussion and close analysis. In particular, *The Mathematical Experience* [4] was a complicated work for students and, in subsequent iterations of the seminar, I recognized the need to focus student reading on selected chapters. Going forward, I would try to focus the reading of *The Mathematical Experience* still more. Another possibility could be to have students also read selections from *Proof and Other Dilemmas* [7] or *Meaning in Mathematics* [14], although I would not want to lengthen the required reading very much and I would need to choose the selections carefully to ensure their accessibility by first-year students. I did find that introducing *The Mathematical Experience* early in the course was valuable and made for richer discussions, as some of the basic philosophical issues, such as mathematics having a platonic reality, were reprised in subsequent readings and videos.

I cannot conclude that considering philosophical questions about mathematics enabled students to grasp the subtle points of mathematical arguments any better than more direct mathematical practice. However, I note that consideration of philosophical issues about mathematics served to justify to the students the trouble that mathematicians take with questions of rigor. Moreover, connecting mathematics to humanistic areas and stressing the degree to which mathematics is about ideas, rather than just symbol manipulation, served to elevate the discipline for students and give it greater meaning and significance.

All these observations appear to be borne out by student reactions to and comments about the seminar. For some, the mathematics was indeed too hard:

> The handouts and books read in class were great, but the problem sets were way too difficult. (Fall 2006)

However, for others, the course imparted some of the critical thinking skills I desired for them:

> I think I have learned more about writing real proofs and gotten a good basic introduction into fields I never knew existed (group theory, number theory, etc.). (Fall 2007)

> I feel that I learned how to effectively argue a point, both through proof and essay writing. (Fall 2011)

More generally, it was a revelation to some that mathematics was much more than calculation, or finding solutions to equations, or even setting up quantitative models:

> I enjoyed every aspect of this class. I hope it will be offered next year because it truly opened my eyes to the beautiful world of mathematics. (Fall 2004)

> I came in with very little appreciation for math. I learned a *huge* amount of stuff that I hadn't even thought of before. I feel like I can competently talk about a wide range of mathematics, even though I still don't consider myself a "math person." (Fall 2011)

For me, the seminar has been a challenge to teach, since talking about non-mathematical subjects, leading discussions, and critiquing student writing all lie outside my pedagogical comfort zone. It has also been a rewarding experience, as it has provided me academic contact with many students I would not encounter in my other classes and an opportunity to have meaningful impact:

> I just wanted to email you an interesting way that our freshman seminar popped up in my psychology major! . . . [W]e're studying multiple regression right now. We just started talking about how to graphically describe data with more than one predictor variable, and how you would need to be able to draw in hyperspace if you wanted to show three or more predictors. I'm really glad I took your freshman seminar, I feel like it often pops up in interesting and unexpected ways, and really broadened my mind to a different way of thinking. (From a sophomore who took the seminar in Fall 2011.)

I am gratified to know that aspects of the course have remained with some of my students.

5 Some Possible Adaptations

The structure of my seminar made it straightforward to engage students with questions concerning the philosophy of mathematics; it would require care to incorporate some of the discussions in a traditional mathematics course given typical technical content goals. Nonetheless, ideas surrounding mathematical reality and the nature of proof are worthwhile subjects and, I believe, can be included in a variety of courses to good advantage. Although first-year seminars exist at a number of institutions, they are not commonplace. However, many colleges and universities have a mathematics for the liberal arts course that presents a selection of topics, often showcasing the applicability of mathematics to other disciplines. It would be both worthwhile and intellectually honest to include a unit on mathematics as considered by (pure) mathematicians. One might assign Poincaré's "Mathematical Creation" [13] or Adler's "Mathematics and Creativity" [1]. Mathematical ontology and epistemology make appropriate topics for a transition to proof course for mathematics majors. Thus, assigning (carefully) selected readings from *Proofs and Other Dilemmas* [7], *Meaning in Mathematics* [14], or *The Mathematical Experience* [4], together with class discussion plus possibly a short reflection paper would provide valuable enrichment for students, particularly those for whom mathematical proof sits in jarring contrast to all the computations to which they have become accustomed. At institutions where the mathematics major includes a capstone course, students would also be well served by taking on some of these readings and ideas. In fact, I developed my seminar after a trial run with upper-level mathematics majors during Oberlin's January Winter Term. That experience convinced me that the seminar could be designed so as to have broad appeal, even to beginning college students. I encourage faculty to think about ways to introduce their students to the philosophical side of mathematics.

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4

Helping Students See Philosophical Elements in a Mathematics Course

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1 Introduction

In this chapter, I briefly develop some ways I arrange for students to explore philosophical issues they encounter in my Analytics: Math Models course at Belmont University. By combining reading and writing assignments with both specific topics and broad concepts from mathematics and closely related areas, students are invited throughout the course to consider our work and conversations through multiple lenses. On the first day of class, I indicate to students that my broad course learning outcome for them is that they will come to “see the unseen” as a result of their efforts. As we move through the course, with a wide array of mathematical topics in the mix and several classes devoted to discussion of readings associated with ideas that typically connect mathematical and humanistic realms, students have multiple opportunities to see that mathematics and philosophy have a rich and meaningful overlap.

At Belmont University (a comprehensive masters level institution with a total of 7000 students, 5000 of whom are undergraduates), we provide a set of core offerings for our undergraduates that includes a quantitative reasoning component fulfilled by special focused mathematics courses. These typically involve discrete mathematics topics and general problem-solving approaches that neither rely on nor build from algebraic language and models. In addition to students in arts and sciences programs, we have many undergraduates in professional programs such as nursing, music, and music business.

Although I have used some of the elements described in this chapter for the Introduction to Mathematical Reasoning course I teach for our general student population, for the narrative that follows I focus primarily on topics, ideas, and applications that I developed for the Analytics: Math Models course that I have taught for twenty years for students in Belmont’s honors program. Analytics fulfills the quantitative reasoning requirement for the honors program students and, in some sense, is an enriched and accelerated version of the Introduction to Mathematical Reasoning course. The Analytics course typically has eighteen to twenty students, mostly sophomores and juniors, whereas the Mathematical Reasoning course has around twenty-five students who are nearly all freshmen and sophomores. Because the honors program students are usually very interested in big ideas and are willing to tackle difficult problems and challenging readings, I am better able to effectively incorporate philosophical questions in the Analytics course. Since Analytics is the core mathematics general education course for the Honors Program, the student population for the course represents the broad university undergraduate population and thus includes pre-professional majors such as business and nursing in addition to more traditional arts and sciences majors.

2 General Course Structure and Philosophical Considerations

Philosophical elements are introduced throughout the Analytics course. Early in the course, while studying general problem-solving strategies, students engage in readings and conversations about people who do mathematics. I indicate to the students that I am having them explore the humanity of doing mathematics—as humans, I suggest to them, it seems that we have always included a consideration of mathematical issues among the big questions we pursue. They encounter Paul Erdős and many other mathematicians in *The Man Who Loved Only Numbers* [12], a required reading at the beginning of the course. In *The Proof* video [17], they see Andrew Wiles as he and others describe his efforts to prove Fermat's last theorem. Robert Moses and his work as a Freedom Rider in the 1960s and later with The Algebra Project [22] offer a different perspective on mathematics as a potential social good.

For a required response essay assignment early in the course, students choose from a variety of options that allow them to consider additional mathematical characters, real and fictitious, and the interesting questions, challenges, and issues these mathematical characters encounter. Students are instructed to select at least two options from the list that I provide for them and to prepare a short essay (approximately 750–1000 words) in response to the reading(s) or film(s). I indicate to the students that a strong essay will develop connections of some sort (or possibly a comparison and contrast) between the reading(s) or film(s) as well as links to ideas or people from other honors courses. I encourage students to talk with each other about what they read or view and write. Students are welcome to use a first-person narrative and are expected to observe the appropriate conventions of writing such as sentence structure, verb agreement, and mechanics. Choices for the essay assignment include Danica McKellar [21], John Nash [1], Max Cohen [20], Will Hunting [10], Christopher Boone [11], Charlie Epps [18], and The Professor [19]. The readings and films typically raise philosophical questions of some kind as they develop their stories. Byers [3] provides an interesting perspective in his explanation of mathematical thinking, including a constructive role for paradox and ambiguity. Gladwell [9] makes a case connecting success at mathematics to cultural and linguistic features. Frequent references to Keith Devlin are sprinkled throughout the course, including his notion that mathematics "makes the invisible visible" [7]. Borrowing from Devlin's idea and the title of an exhibit of Harold Edgerton's work [2], I use "seeing the unseen" as a metaphor for the course. I occasionally remind students of this guiding metaphor as I develop the course and invite them to consider their own metaphors.

During the early weeks of the course as students read about people who do mathematics they also explore several questions. These include how mathematical knowledge is developed within a set of rules that governs the sociocultural element of the acceptance of new mathematical results and ideas. Andrew Wiles' efforts to prove Fermat's last theorem [17] offer an interesting example for students because of the ideas from many other mathematicians on which Wiles relied. With Wiles' proof requiring nearly two hundred pages, the question of verifying its correctness arises. Having students consider a problem or task that is impossible yields interesting questions and observations. For example, in addition to Wiles' work with Fermat's last theorem, students encounter the well-known party problem from graph theory (more specifically, Ramsey theory) very early in the course. For the party problem, I ask students if it is possible to avoid triangles with six people. Typically they are reluctant to decide that what they seek is not possible (they are honors students, after all!) and they have some cognitive dissonance when they must confront the issue of proving that something is impossible. This example helps to raise questions about determining truth in mathematical pursuits. Philosophical questions about the origins of mathematics inevitably also arise. Is it discovered? Is it created? Must it be constructed? They stay with us as a backdrop to the course as we become more focused on specific topics, which are chosen mostly from discrete mathematics.

After the general course introduction that is developed through problem-solving topics and the humanity of doing mathematics, we shift to studying symbolic logic as a mathematical system. Logic's emphasis on deductive reasoning offers a contrast to the inductive reasoning of our problem-solving explorations. While working with truth tables and logical arguments, students simultaneously read, discuss, and write about the graphic novel *Logicomix* [8]. In addition to giving students some historical perspective on the development of symbolic logic, *Logicomix* pulls us into significant philosophical questions about the nature and foundation of mathematics. Within that context, Gödel's incompleteness theorems appear in our classroom conversation and reading [6] as an important component of early twentieth century thinking about mathematics and logic. This work in logic intersected with revolutionary thinking from other academic and artistic domains (for example, the uncertainty principle in physics and avant garde ideas in art). Because students are previously exposed in the course to the challenge (and potential importance) of proving that something is impossible, I believe they are better prepared to engage Gödel's work. *Logicomix* and Gödel's incompleteness theorems then allow for a fairly smooth transition to a discussion of Thomas Kuhn's ideas about the philosophy of science [14], which we consider as they relate to the history and development of mathematical ideas (for example, imaginary numbers and non-Euclidean geometry).

With Gödel's theorems as a launching point, elements of uncertainty within mathematics are interwoven with mathematical content throughout the course. Arrow's impossibility theorem, issues and challenges associated with extremely large numbers (including readings from Cole ([4], "Exponential Amplification," pp. 17–26) and Knuth [13] about the limits of computing and the vastness of possibilities), and dynamical systems that may result in mathematical chaos are three additional course topics where fundamental limitations or uncertainties enter the conversation with students. We end the course with a discussion of Michael Crichton's novel *Prey* [5], which addresses a number of theoretical, practical, and philosophical points we encounter during the course. Examples from *Prey*, including connections to the course topics mentioned above, and Kuhn's observations about the nature of how science is actually done, come together to help students consider ideas of uncertainty.

The classroom setting for the course is inquiry-guided learning. Short lectures occur during most class periods to offer examples or some basic terminology (which occurs regularly as we transition from one topic to another). Students often work collaboratively in class on problems associated with a topic, whether that means developing good problem-solving strategies, analyzing an argument via a truth table, or attempting to draw graphs without edges crossing. Analytics is a four credit hour course, and thus in addition to three 50-minute sessions per week it includes one 75-minute period that allows expanded time for exploratory and discovery-based learning.

With the four credit hours there is sufficient flexibility to include several class periods for discussion. For example, early in the course a class period is devoted to a student-led discussion of *The Man Who Loved Only Numbers* and Robert Moses' work as a Freedom Rider and with the Algebra Project. A few weeks later, we spend another class period discussing *Logicomix*, Gödel's theorems, and associated historical, philosophical, and cultural elements from the late 1800s to the first part of the twentieth century. On other occasions, after viewing a video clip or film (such as [16] and [17]) we have brief discussions that often include sociological or philosophical considerations. For example, Andrew Wiles' proof of Fermat's last theorem leads to comments about the process within mathematics for accepting ideas as new knowledge, and students connect Benoit Mandelbrot's early work with fractals to Thomas Kuhn's thoughts regarding the difficulty of getting new ideas accepted. I generally vary the discussion methods used, and nearly always ask students to bring brief written discussion points to class (which I simply check for participation). As the course ends, we spend two or three days on class discussions of Crichton's *Prey*. We note that many of its themes and issues (including some philosophical points) connect to our course topics. Student responses to *Prey* suggest that it offers an interesting, entertaining, and effective exclamation point to end the course.

3 Problem Sets and Exams

Problems sets are regularly assigned as a means of assessing student understanding and mastery of course concepts. Typically a problem set requires problem-solving. Some of them ask students to respond in two or three paragraphs to a prompt or task. For example, an early problem set requires students to explore the Monty Hall Problem [15] from the *Let's Make a Deal* television game show and to comment on how the solution to the problem lines up with their intuition. Toward the end of the course, students are required to teach one of the course ideas that was novel to them (examples might include drawing graphs on the surface of a torus, proving that something is impossible, Gödel's theorem, or encryption techniques) to a friend or family member. For this assignment, which is incorporated into the last problem set for the semester, my Analytics students write a narrative reflection on the experience. It includes what they did to prepare to teach a topic and to whom their teaching was directed, how the recipient responded in terms of learning and attitude, and what my students learned from the experience.

Overall, the problem sets focus on students doing mathematics and, to a lesser extent, a metacognitive element designed to reinforce classroom activities and discussions in which students are encouraged to ponder their thinking and learning. For example, after a classroom conversation regarding current ideas from cognitive science about human learning, students are asked to reflect on how the ideas are associated with their learning. The metacognitive aspect works well in conjunction with our ongoing interface with philosophical, social scientific, and humanistic readings and discussions.

Course exams primarily focus on assessment of student problem-solving, logic, and quantitative reasoning skills as developed with our specific set of topics. Each exam also has a take-home portion that includes prompts or questions associated with readings and discussions. The following questions and prompts are examples I have used on take-home exams, the first two during the first half of the course and the second two as part of the course final exam.

> Consider *Logicomix*, including Gödel's theorem specifically, in the context of this course (and other courses you've taken, if applicable). Briefly discuss connections between Gödel's theorem and *Logicomix* with at least two of the following: (i) other artistic and intellectual mindsets leading up to the time of Gödel's theorem; (ii) the notions of proof and truth; (iii) Thomas Kuhn's ideas about scientific revolutions.
>
> Respond to one of the following:
>
> (i) In what ways does Thomas Kuhn call into question the scientific realist position? With which one (Kuhn or the SR) do you find yourself in greatest agreement? Why?
>
> (ii) Discuss how Kuhn's ideas mesh with elements in your major discipline or another discipline of interest to you.
>
> (iii) Discuss how Kuhn's ideas relate to the segment from the video *A Private Universe* (set at Harvard and in a ninth-grade science classroom).
>
> Several times during the course there have been concepts or problems for which we've encountered difficulties, uncertainties, or impossibilities within mathematics (mathematical chaos and Gödel's theorem are two examples). Comment on at least four such problems or concepts and include what we seem to know and what limitations are encountered. What do you see as implications, broadly speaking, that follow from the limitations of mathematics?
>
> At the beginning of the course, I suggested seeing the unseen as a metaphor for what I hoped to have as a course outcome for you. Develop a several paragraph response that explores seeing the unseen as you've observed it develop during the course. Alternatively, offer your own metaphor for the course and provide support for it.

Students are asked to respond to each prompt with a few paragraphs. They seem to enjoy the opportunity to express their ideas as a complement to their work with solving problems.

4 Impact on Student Learning and Understanding

Because the philosophical components are embedded throughout the course, students gradually come to expect that most of our topics will include something at least tangentially philosophical. Nearly all of the students take Analytics concurrently with one of the required core honors program humanities courses that incorporates philosophical elements from some specific historical period or intellectual or cultural tradition. The humanities courses apparently only rarely include a connection to mathematical ideas, according to student comments to me. Consequently, students seem to welcome a placement of mathematics (and other Analytics course topics) within the larger historical, philosophical, and intellectual framework they are developing.

From essays (including one early in the course), several class discussions, and written responses to take-home exam questions, students develop the ability to explore their thinking about mathematics, from the inside and the outside. By finding their own voice, possibly a refinement of previous thinking but often a radical shift, students generally seem to embrace the big questions that arise as we work. Nearly all students find a connection from the philosophical components of Analytics to their academic area of interest. For some students, the idea of knowledge construction from Thomas Kuhn or other readings or films is a connecting point; for others the limitations resulting from Gödel's theorems matches some ideas from their disciplines; and others still find important pathways from mathematical chaos to their chosen fields. A sample of connections made by students is in the following comments from recent Analytics students.

> It is perhaps the [Gödel] incompleteness theorem that most precisely captures this sense of uncertainty in mathematics. To summarize, the theorem states that truth cannot be known in its entirety, but rather

> that mathematics is limited to discerning a smaller subset of truth, founded on the methods of deductive reasoning. . . . Assuming mathematics is the key to *all* answers in the universe is absurd, and man ought to be able to cope with the unanswered, for it will remain. [from a physics major who was taking quantum physics alongside Analytics.]

> [*The Housekeeper and the Professor*] meant a lot to me as someone outside looking into the minds of mathematicians, and it gave me really important insight into the humanity and feeling behind math. I have always thought of math as different from writing, but when I read this novel I found myself looking at numbers poetically and finding the beauty in them. This [a response essay] was my best work this year because writing it freed me from the conventions I was harboring. [from an art major.]

> The biggest "aha!" moment for me was the section of *Prey* when Crichton casually commented that humans are nothing more than collections of swarms. My mind went crazy at such a statement because it plays into philosophy's more recent conversations about whether things in themselves or relationships between things are more ontologically primal. [from a religion in the arts major.]

With these connections made, I see the students as generally more interested in the mathematical ideas that accompany the philosophical and humanistic considerations.

From conversations I have with individual students outside of class, midcourse feedback about all aspects of the course that I solicit from the students, and end of course student evaluation comments, I have the impression that students feel that they can thrive intellectually in the course because the mathematical concepts we address are embedded in a larger framework that includes a mix of philosophical and practical considerations. For example, our class time spent with voting methods uses simple mathematics that connects directly to real examples but also includes Arrow's impossibility theorem that raises very interesting questions that cannot be answered only with mathematics.

The most useful and direct feedback I receive in writing from students comes through end of course reflections that I ask them to write. While they are not anonymous (students receive a few points toward their course grade for completing them), they are typically consistent with the anonymous feedback I receive through other avenues. Students respond to the prompts:

(i) What item (or two) from your work during the course most effectively demonstrates an accomplishment for you in this course? Give some reasons why.
(ii) Describe one or two things about the course that were most helpful to your learning.
(iii) Has your perception of mathematics or science and their place in your world changed as a result of this course? Has the course affected your views about mathematics, science, education and learning, critical thinking, yourself?
(iv) Explain a specific connection or two that you have made between your work for this course and some other part of your life (in other honors courses, academic otherwise, personal, spiritual, or any other dimension of your life).
(v) Identify an interesting topic, issue, or problem that we studied this semester. Why is it interesting to you?

Some student responses to the reflection prompts from a recent time I taught Analytics include the following comments on their perception of mathematics.

> My view of math has completely changed. It can be applicable to everyday life. It does not have to be a painful experience. It can be fun and interesting and engaging. There are philosophical, psychological, and scientific elements to math I did not see before. . . . Watching *Numb3rs* and reading *Prey* showed me math in a whole new way. I think this class made me much more optimistic about math in general, gave me a confidence boost, and captured my interest in the subject once again for the first time since I was in third grade. . . . Math became "deeper" after this course.

> It was interesting to get some insight into not only different types of problems that do not have one clean-cut solution, but also to learn a little bit about the mathematicians and the history behind mathematics through readings such as *Logicomix*. This experience provides the students with a well-rounded approach to the never-ending realm of mathematics whereas other math classes provide little more than formulas and crunching numbers.

> The semester was a weird juxtaposition of finally discovering applications for math that I was learning to do while learning to respect math in an aesthetic way that ultimately led to me not really caring what the specific applications were. I think a lot of the subjects we tackle have greater significance outside of the classroom setting, too, like the limitations of human certainty and the idea of mathematical chaos.

> My perception regarding mathematics has drastically changed having taken this course. . . . I didn't realize how intimately connected mathematics is to every other subject and the universe as a whole. I always thought that mathematics was a way to predict the future and calculate different quantities. Indeed, it is a language that describes the universe.

Other student comments from their course reflections suggest how the course framework and approach has influenced them very broadly in ways they would not have previously imagined.

> In general, my college experience is teaching me a few things about science in general: that we do not have the answers to everything and that all aspects of science are ever evolving and developing. I think that accepting that not everything is understood is a part of growing up and an experience that I have begun in Analytics as well as other science classes.

> The basic idea behind this class is seeing things you have not seen before, being aware of things you were not aware of. . . . I took a costume design class this semester and in some strange way I kept using things I learned in math class to dissect the characters in the shows we were designing.

Consequently, my general observations about student reactions are that most students like the combination of a course focused on analytical topics that are intentionally connected to broad intellectual pursuits. In my twenty years of teaching the course, students from all academic areas typically respond in a similar fashion.

5 Course Challenges and Modifications

Even though I have taught this course many times, a challenge is holding the course together without a primary textbook. I have not found a textbook that works the way I need for the course. Instead of a single textbook, I have the students purchase several small inexpensive modules along with many materials that I have developed and compiled as a coursepack for the course. Although this works well, I nevertheless often feel as though we are working without a safety net that a textbook affords.

For many years, a philosophy faculty member was a guest for about two weeks in the course and a computer science faculty member was a guest for about one week. For those years, my task included effectively weaving into the rest of the course their contributions about critical thinking, computer science (including connections between ideas of Alan Turing and Gödel), and the philosophy of science. Over the years I had the opportunity to learn a great deal about some key basic concepts of both computer science and the philosophy of science as well as a general approach to critical thinking as my students were exploring these ideas. I also gradually learned how to more effectively incorporate these topics with the more typical mathematical topics. When my two faculty colleagues were no longer able to be guests, I was able to keep what I thought were the most important and relevant of their ideas and topics in the course because I had come to understand how to teach them myself from observing my colleagues. For those interested in adapting the ideas for their own use in a course but without the luxury of observing colleagues from philosophy or computer science, I believe that a broad general understanding of concepts, ideas, and historical frameworks presented in Byers [3], Davis [6], and Kuhn [14], along with a careful reading of *Logicomix* [8], would suffice initially. This would provide an instructor with enough background to invite students into related explorations; several iterations with the material in a course would then help to determine a good match with the instructor's interests and the desired course learning outcomes.

After teaching the course a number of times, I realized that I needed to emphasize some connecting threads that are woven into the course overall. (This is also true of our general education core quantitative reasoning courses that students not in the honors program are required to take.) From that point on, I learned more and more ways to emphasize and reiterate that problem solving and logic are the two primary connecting threads, with mathematics as a human endeavor serving as a third. Aspects of logic needed include inductive and deductive reasoning in general problem solving, symbolic logic and its connections to argumentation and binary circuits (and binary notions more generally), and careful work (using graph theory topics such as drawing graphs in the plane and on a torus) that is designed to make clear the difference between a necessary condition and a sufficient condition. After the general introduction to problem solving strategies at the beginning of the course, I remind students how they can be useful as we work through our specific topics (counting methods, voting, coding and encryption, and graph theory). For example, starting small serves many students very well at different points in the course when they feel stuck and don't quite know where to begin in a large problem.

To help make the philosophical elements of the course seem more naturally intertwined with its broad themes, in recent years I have increased my effort to build in multiple references to the humanity of mathematics in the early stages of the course. This seems useful for students as they grapple with some of the (typically unexpected for them) messiness of how mathematical ideas come to be accepted as truth, political and sociological aspects included. Toward the end of the course, as we briefly explore the ideas and implications of mathematical chaos and read *Prey*, I'm able to refer again to the humanity of mathematics (and science), reinforcing a unifying theme for the entire course.

The course framework and general set of topics has been quite stable for a number of years. Nonetheless, I routinely add updated readings so that the course will feel fresh and relevant to each group of students. For example, when discussing voting methods there are often real world recent votes of some sort that are of interest to the students. To connect the study of mathematical chaos and our reading of *Prey* to the most current ideas, I incorporate news from science for the general public that relates in some way to either the science ideas or to the work of scientists.

6 Conclusion

The Analytics: Math Models course continues to challenge and excite me every time that I teach it. Philosophical and humanistic perspectives are vitally important to my conception and delivery of the course. In addition to enriching the course atmosphere and experience for students, such perspectives push me out of my mathematical comfort zone and into a space filled with creative tension. After twenty years' experience with the course, the creative tension serves to make each encounter with a new group of students an opportunity for me to broaden my understanding of philosophical connections to mathematics.

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5

An Exercise in the Philosophy of Mathematics

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1 Introduction

1.1 Overview

In the fall of 2013 I gave students in three different math courses designed for non-majors an assignment to help them to understand the nature of mathematics and to appreciate mathematics as something more than procedural routines. In this chapter we discuss that assignment and the philosophical questions that it explored. We will see that through videos, in-class and out of class readings, and class discussions, students in the courses developed their own answers to three questions: What is mathematics? What does it mean to do mathematics or to think mathematically? Do we as humans invent mathematics or do we discover it? Along the way we will discuss the students who participated in these classes, how their views evolved, and their responses to the assignment.

1.2 Objectives

As stated above, in the assignment we will be discussing, students needed to try to answer the questions:

1. What is mathematics?
2. What does it mean to do mathematics or to think mathematically?
3. Does mathematics exist independently in nature or is it a creation of human thought?

They initially answered these questions on the first day of class when we had done nothing more than read the syllabus. I gave students five to ten minutes to write their answers. I placed them in a folder and did not look at them until the end of the semester after they had conducted research and answered the questions twice more.

The point of asking the students the questions, and asking them more than once, was to examine and hopefully broaden the students' understanding of the nature of mathematics. Overall the goals were to

investigate student perceptions of mathematics,
give students specific work geared to expand those perceptions,
assess how well students retained any changes in perceptions, and
look at the effect of course content and delivery on that retention.

The first three goals are appropriate for any class of this type. I was only able to examine the last goal because I was teaching three classes that were taught at a similar level and to similar audiences, but which covered different material in different ways.

2 Audience

Western Connecticut State University is a liberal arts university in the Connecticut State College and University system. There are 6,025 students enrolled in one of four schools with 38 bachelor's degree programs, fifteen masters's programs, and two doctoral programs. Most of the students, 94%, come from Connecticut; many are the first in their families to attend university and must work part-time, if not full-time, in order to afford college.

Currently WCSU requires all students to complete ten credit hours of mathematics, natural sciences, or computer science; they cannot complete all ten credits from the same area. As a result, almost all students take at least one mathematics class. Many programs specify a mathematics class that students are required to take, but this is not universally the case. Students majoring in philosophy, English, writing, theater, music, art, etc. have no specific mathematics requirements. To serve this portion of the student population the mathematics department has three classes geared toward them and has piloted a couple more.

The most common class that students take, and in which I used this assignment in the fall of 2013, is MAT 110: Great Ideas in Mathematics. This is a typical mathematics for liberal arts majors course, frequently taught using books like *The Heart of Mathematics* [3] or *Excursions in Modern Mathematics* [12] if a textbook is used at all. The course as a whole has no particular focus and is intended to give students a big picture view of the world of mathematics beyond arithmetic and algebra. I primarily focused on reasoning and how we know the things we know. I used *Discovering the Art of Mathematics: Truth, Reasoning, Certainty, and Proof* [7] as my primary text, with supplements from a number of other sources. This text, as with all the materials in the same series, is written for an inquiry-based classroom and is available free online.

The second course in which I used this assignment was MAT 113: Introduction to the History of Mathematics. Taught to the same level of preparation as MAT 110, it can cover a wide variety of topics, but the focus is on discussing the place of the topics in the historical development of mathematics. When I teach it, I start with a general overview of the history of mathematics, and then examine a number of topics in more depth. The intended audience for the course is history, English, and writing majors; so there is a heavy emphasis on writing and historical research. The actual make-up of the students in the class is far more varied. As with MAT 110 there were significant supplements; however, the primary text was *Math Through the Ages: A Gentle History for Teachers and Others* [2].

The third class in which I had students complete this project was a faculty-developed study called MAT 198: Mathematics in Literature. The intended audience for this course was English, writing, and performing arts majors; as with the history course the audience was more varied. The potential content covered varied widely, but it all centered on reading works of literature that contain or refer to mathematics. There were three texts: the play *Proof* [1], the novel *Uncle Petros and Goldbach's Conjecture: A Novel of Mathematical Obsession* [5], and the collection of short stories *Fantasia Mathematica* [6].

3 A Philosophical Assignment

3.1 Guided Student Research

After they had given their initial responses on the first day, the students spent the next four weeks on a guided research project geared toward helping them develop more extensive answers to the questions mentioned above. In that time they watched and discussed two documentaries during class time and completed five readings, one of which was done during class time and all of which were discussed as a class.

The videos we watched were *N is a Number: A Portrait of Paul Erdős* [4] and *The Proof (Nova)* [11]. The Erdős video conveyed his general dedication to mathematics and *The Proof (Nova)*, about Wiles, demonstrated the dedication required to solve a really hard problem. I chose these documentaries partly because their subjects should be familiar. Also, they gave the students a glimpse of how mathematicians and the mathematical research community works.

For each video the students were given a set of questions to answer, to keep them focused on the video and to remind them, not so subtly, of the questions we were really interested in answering. For the video on Erdős I asked:

1. How would Erdős answer our three questions?
2. What does Erdős mean by "the book?"
3. In what ways did Erdős encourage mathematical advances?
4. Do you think that Erdős considered mathematics an art?

And for Wiles:

1. How would Andrew Wiles answer our three questions?
2. What is Fermat's last theorem?
3. Fermat's last theorem, while true, is not on its own particularly significant: so why has it been so influential?

After the videos, we discussed the questions and the students had to write up their thoughts on each question in one or two sentences.

In the class after we had finished the videos we read an excerpt from the Socratic dialogue *Meno*. The portion we used was from [6], but the dialogue is widely available including on the *Internet Archive* [9]. We read the scene between Socrates and Meno's slave in which he walks the slave through duplicating a square. We did it as a play in front of the class with myself acting as Socrates and two students acting as Meno and as the slave. This gave us a nice bit of mathematics to discuss (proportions of lengths versus areas) and a way of introducing the Socratic Method and Inquiry-Based Learning. As with the videos, after we had done the reading and had a discussion the students were asked to respond to these questions:

1. What is the Socratic Method?
2. Does Socrates teach the slave anything? If so how does he do so?
3. How would Socrates answer our three questions? Or, at least how would he seem to answer the third question?
4. From the dialogue what did we learn about how area compares to length? When you double or triple length how will it affect area? Do you think this will happen with all shapes?

These helped emphasize the points I wanted them to take away from the reading and draw their focus to the more general questions with which we were concerned.

At the same time we were using class time to watch videos and discuss readings the students had four essays they were assigned to read on their own out of class. These were:

"A Mathematician's Apology" by G.H. Hardy,

"Mathematical Creation" by H. Poincaré,

"The Mathematician" by J. von Neumann, and

"The Locus of Mathematical Reality" by L. White.

All the essays are available in *The World of Mathematics* volume 4 [10], which is readily available in print and is available online at [9]. For these readings I asked them to write up a synopsis of our in-class discussion and to tell me how they felt the author of each essay would answer the three questions we were focused on.

3.2 Essay

With the research done, it was time for the students to answer the three questions a second time. In an attempt to give them as much leeway as possible I asked them to write an essay, comprising 7% of their final grade, that answered the questions:

1. What is mathematics?

2. What does it mean to do mathematics or to think mathematically?
3. Does mathematics exist independently in nature or is it a creation of human thought?

and was graded according to the rubric in Figure 1.

#	Criterion	0	1	2	3
1	Clarity	The writing is unclear and hard to follow.	Frequent awkward phrases, but generally easy to follow.	The writing is mostly clear with only an occasional awkward phrase.	The writing is clear and easy to follow.
2	Consistency	Inconsistent opinions and no real logical support.	Opinions are mostly consistent but poorly supported.	Opinions are consistent and mostly supported.	Opinions are consistent and well supported.
3	Structure/Grammar	Significant grammatical errors or use of informal voice.	Some grammar errors and/or informal voice.	Few grammar errors and/or informal voice.	Consistent use of formal voice and few grammatical errors.
4	External Support	Opinions are given with no external support.	There are few references to outside sources and references are unclear.	References to external sources but they are sparse or unclear.	Opinions are clearly supported by reference to dependable sources.
5	What is mathematics?	This question is not addressed.	This question is not fully addressed.	This question is addressed but not in a clear or consistent manner.	This question is clearly addressed.
6	What does it mean to do mathematics or to think mathematically?	This question is not addressed.	This question is not fully addressed.	This question is addressed but not in a clear or consistent manner.	This question is clearly addressed.
7	Does mathematics exist independently of human thought or not?	This question is not addressed.	This question is not fully addressed.	This question is addressed but not in a clear or consistent manner.	This question is clearly addressed.

Figure 1. Essay rubric

I placed no conditions on length: the rubric did, however, have two categories related to the quality of their writing, two that addressed the consistency and detail with which they presented their opinions, and one for how well they answered each question. I required them to use the rubric to grade their own work and submit this with their essay. The need to encourage good writing habits and complete answers to the questions asked are self-evident. The two categories on the rubric that addressed how they presented their opinions were there for two reasons. First, it is always a good idea to encourage students to offer support for opinions and to be logically consistent (especially in a mathematics class). The second reason that I added the categories is that when I had previously given a variation on this assignment I had gotten feedback on evaluations such as the following:

> Dr. Rocca was very unfair. He gave me a bad grade because he didn't agree with my opinion. You can't grade an opinion, it is just what I think.

In order to avoid this criticism I emphasized to the students that supporting their opinions, explaining how they arrived at their ideas, and being logically consistent, were necessary conditions for a valid opinion. An unsupported inconsistent opinion is no opinion at all.

Once the essays were completed each class focused on the content pertinent to the course. I did not frequently or deliberately bring up the three questions again until the last day of class. On the last day of class, after all the content that would be covered was covered, once more the students took out a piece of paper and answered the three philosophical questions with which we had started the semester. Now with three points of reference it was time to see how student ideas had grown or shrunk over the course of the semester.

4 Perspectives

4.1 Initial Responses

With the exception of a couple of students who were mathematics majors and the one or two who had taken a similar course previously (some students in MAT 113 and MAT 198 had previously taken MAT 110: Great Ideas in Mathematics) the students' initial responses were predictable. The following give the general flavor of the 70 initial responses I collected to the question "what is mathematics?":

> The study of numbers, equations, and finding solutions to numerical problems.
>
> Math is the use of numbers and logic to solve problems or find solutions.
>
> Mathematics is a method of problem solving that relies on logical means as a way of producing a solution.

A vast majority of the responses discussed numbers and equations; occasionally they mentioned problem solving. The students were equating mathematics with arithmetic and algebra. This was about what I expected and part of the motivation for exercise.

4.2 Essay Responses

About ten students withdrew from the course before completing the essay, or finished the course but never properly completed the essay. From the 60 essays that were submitted here are some typical quotes:

> Math, when looked at as a whole of all of its parts, is an art. Not the kind of art that uses paint, clay, or rhymes, but an art that utilizes ideas to show its eternal beauty.
>
> Nature provides us with a pattern and it is up to humans to feel out and formulate the answer in the simplest and most beautiful way.
>
> One of the things that I learned from the play *Meno* is that math isn't just an assumption, its figuring out why you're either right or wrong.
>
> Coming into this class, I thought that math only had to do with numbers, but I now see that it is much more than that. It is an art. It is a gathering of ideas. It is all around us; it is out there to be found.

As you may be able to tell from these quotes, the *Meno* and *A Mathematician's Apology* had the greatest influence on the students. Partly this was because they were the easiest to understand. Also, we read the *Meno* together during class; so everyone present that day had definitely done the reading. But, especially with some of the more artistically minded students, Hardy's description of mathematics as an art where you work in the medium of logic and truth really resonated. At least one writing major also connected with Poincaré's description of needing to go for a walk to help him think, an idea that also arose in "The Proof," since it is clearly a similar phenomenon to writer's block.

4.3 Final Perspectives

Partly due to attrition from the classes and partly because they were collected the last day of classes when the students knew that we were not covering anything new, I only collected 37 final responses. Here is a sample of the responses:

> Math is the purest of sciences that deals with the complexity of numbers and discovery of patterns.
>
> Math is more than just adding and subtracting. It more so focuses on deep thought, logical reasoning and proof.
>
> Math is the logical thought process of explaining why something is true.
>
> Math is the process of using prime and composite numbers in all different kinds of formulas...

Some students slid back to simply associating mathematics with numbers, albeit now dividing them into prime and composite numbers. However, while the responses are not as detailed or flattering as the responses in their essays, most students seemed to retain the idea that mathematics went beyond arithmetic and algebra.

Further, on the first day of class most of the students finished writing their responses in five minutes and answered each question with one or two sentences, if at all. On the last day many students took ten or more minutes to answer and their responses for some of the questions consisted of entire paragraphs. This demonstrates that even if the complexity of their ideas had receded, the amount of thought they were willing to give to answering the questions had increased.

Finally, of all the responses collected on the last day, the nineteen students in the Great Ideas in Mathematics course seemed to retain the most sophisticated ideas about mathematics. This may be significant since they were in the class that was taught using the inquiry-based materials in [7], and so perhaps had the most opportunity to engage in the processes of discovery and exploration.

5 Student Feedback

Based on feedback from student evaluations of the course as a whole and a couple of student responses that I gathered after the semester, student attitudes toward the assignment in its entirety and toward the materials we used for the assignment were mixed.

Student attitudes about the materials we used for their research into the three questions we wished to examine ranged from too hard to too boring to very challenging (but in a good way). In these quotes we can see that they all felt the readings were challenging:

> The first part of class was awful—the readings were too challenging for a math class and the reason we were reading got lost.
>
> The course was difficult in the beginning. The readings [from the *World of Mathematics* [10]] were advanced and hard to follow.
>
> I thought the difficult readings were a bit refreshing from all the dumbed down readings I am normally accustomed to at WCSU.

Clearly, the students had a range of opinions on the value of being assigned challenging essays. Separate from the difficulty of the essays, some students felt that they gained something from looking at the readings directly rather than just discussing ideas abstractly:

> Getting to know mathematicians through their early essays was thought provoking.
>
> The *Meno* reading in class was one of the highlights of the class for me. It really does show the Socratic method in its purest form, and inspired me to use it more often with the students that I tutor.

Based on class discussion and feedback it was generally felt that the essay by Hardy and the *Meno* were, as observed before, the most accessible. It was generally felt that Poincaré was long winded, von Neumann was hard to follow, and the feelings on White were generally neutral. The videos received similar varied feedback:

> I wish there were fewer videos on mathematicians because they were boring and made me not interested in the class. But overall the class was okay.
>
> The movies about Erdős and Wiles were great as well, and served as wonderful tools to introduce the students to the world of higher mathematics.

Some students felt confused about what they needed to do for this essay as exemplified by this response:

> First essay was unclear and ambiguous. The "three questions" were never definitively explained ...

Based on the essays that were turned in and on the time students were able to spend answering the questions on the last day of class it is not clear that this perspective was the prevalent one. There were in fact some students who had a much more positive view on the work we were doing:

> I certainly did enjoy answering the three questions, and they definitely made me think more deeply about the concept of mathematics.

Remaining motivated to helping students broaden their perspectives on what mathematics is and what mathematicians do, how shall we proceed? What changes need to be made in order to make this a more effective exercise? What needs to be kept, tweaked, or dropped altogether?

6 Reflections and Future Plans

In general this exercise in philosophy went well. Most of the students who turned in answers all three times had grown in their understanding of mathematics. Many had lost some of the sophistication in their thinking between the submission of their essay and the end of the semester; this could simply be because the final responses were given off the top of their heads. However, there are some changes I would make.

For each reading, students were asked to answer "How would the author answer our questions?" With such a general prompt the students struggled to focus on how the authors were answering the questions. Therefore in the future I will put together much more specific questions for each of the readings. I will also have the students read only portions of the essay by von Neumann, much of which was too complicated or off our main topic.

I will also make it clearer that in the essay they need to give me their own opinions. This was not a significant problem but two or three students in each class did not understand this and gave me the opinions of the authors we had read. This was redundant since they had already done this in a previous assignment.

In the future I will revisit the questions more during the semester. This semester I didn't spend time reviewing the questions while covering course content; one of my goals had been to see how course content and delivery affected the students' final responses. Future classes completing this exercise will spend more time deliberately tying course content to the questions with which we start the semester. This will also allow me to see how such reinforcement helps students retain the ideas they develop during their initial research.

Finally, I will require students to turn in a rough draft and final draft of their essays. For many students this happened anyway since some students had not completed the assignment correctly and I allowed those who were not happy with their initial grade to redo their work. Outlines and rough drafts will be an integral part of the assignment. This will support writing across the curriculum and push students to think more carefully about their work.

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6

Evangelizing for Mathematics

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1 Introduction: the Audience and the Challenge Therein

Albert Einstein is quoted as saying, “It is a miracle that curiosity survives formal education.” As a formal educator myself, I cringe at the truth of this assertion. It brings to mind images of students languidly sitting in their desks while I drone on about what should be the most exciting concepts and ideas they will ever be exposed to. However, a mishandling of precious material leads to a total disconnect between students and idea. As mathematics educators, we find ourselves all too often in this classroom of disengaged students. I write this chapter in hopes of sharing my approach to teaching some of the most disconnected students that we encounter.

At Ouachita Baptist University, a small, private liberal arts institution of about 1500 students, we offer our Mathematics for the Liberal Arts course as an alternative to College Algebra for meeting the mathematics requirement for many majors. However, these students are some of the most difficult to teach. When approaching them, I consider them initially as mathematically agnostic. They file into the classroom that first day, resentful and resigned, and I know the questions that are going through their heads:

> Why must I take this class? I’ll never use this in ‘real life’.
>
> Will I be able to understand anything this professor says?
>
> What is the minimum I can do in this class in order to survive and never take mathematics again?

You see, I had that same attitude about certain non-mathematical courses that I was required to take. As much as we want an eager audience, it does nothing to support our cause to deny that this is where we must begin. The important question is, how do we instill that seed of curiosity that will bloom into engagement with the material? That is, how do we evangelize mathematics to mathematically agnostic students?

2 Setting a Baseline

Before I share my approach to reaching these students, let me first state my philosophy of mathematics: mathematics is intimately present in our world and beautiful in its own right. I believe mathematics has value for everyone, regardless of discipline or occupation, and the process of discovering mathematics in everyday experiences is not only exciting, but aesthetically pleasing. Trust me when I say that, initially, Liberal Arts Mathematics students do not share these beliefs. Preaching at them is certainly not the answer. I believe the solution lies in two key philosophies: teach the students we have, not the students we wish we had, and make mathematics meaningful in their world. We all hope to have students in our classes who live and breathe mathematics. We might even want students who sit on the edge of their seats waiting for the next problem or the next life-altering mathematical truth. But the reality is far different. To really reach them, we must get to know them and their mathematical viewpoints. (They have them whether they realize it or not.)

One way that I try to reach the students that I have is by collecting what I believe to be critical information the very first day. And among the questions that I pose, I ask them to rate, on a scale of 1 to 5 (with 5 representing "strongly agree"), their agreement with the three statements:

Mathematics is important.
Mathematics is beautiful.
I can do mathematics.

For the spring 2013 semester, I had 31 students in Mathematics for the Liberal Arts, mostly freshmen and sophomores, in two back-to-back morning sections. Their average rating for the importance of mathematics was 4.0. I think, for the most part, even liberal arts majors recognize that mathematics is important. This is something to build on; I want them to regard mathematics not only as important, but also important to them personally. The lowest rating was for the beauty of mathematics, where the average rating was only 2.6. The majority of the students came into the course finding mathematics unattractive. No surprises there. Finally, for the "I can do mathematics" statement, the average was 3.3. Again, no surprises. The students think mathematics is important, but for other people; they don't feel confident they can do it themselves. All this is useful information, but I find it necessary to go even deeper, to the very core of their belief system about mathematics. Before I have introduced the course, really before I have even introduced myself, I ask them a two-part question: "What is mathematics and why won't it go away?" (The latter part of this question is phrased in the negative because I know that is what they want mathematics to do: go away. It also usually warrants some chuckles and alleviates some of the tension.) The students are required to write down whatever comes to mind when I ask the question. A few examples:

A bunch of numbers. Mostly used in the classroom. It's something we are required to learn.

Math is hard, and it won't go away until Jesus comes back to save us from our suffering.

Math is, from my experience and understanding, the painful nerve in the scholastic nervous system.

Mathematics is something that has to do with adding, subtracting, and dividing. The reason it won't go away because we need it to count money, and to buy things we need and other stuff.

Mathematics is a bunch of numbers and letters that make no sense. I wish it would go away.

They turn in their responses before I explain that answering this question will be the theme of the course. I also inform them that the question will reappear on the final exam. This levels the playing field: they have been (hopefully) honest with me about what they are bringing to the table and I tell them upfront that I have every intention of transforming their core mathematical beliefs. I know the challenges that I must overcome to convert them; I am a missionary in hostile territory! They are afraid that I plan to make them change to fit the shape of mathematics. However, I hope to show them how mathematics fits into their world.

3 A Careful Approach

I begin gently and carefully. I realize that many students have been abused by algebra and other previous mathematics courses; so I immediately set them straight, demonstrating that this is definitely neither an algebra course nor like anything they have taken before. The first few days of class are spent with students working in groups to solve problems requiring visualization or modeling but no algebra. A great example of this is the Meanie Genie problem in *The Heart of Mathematics* [1]: given nine stones, seemingly identical in weight and appearance, determine how to find the heavier stone with only two balance weighings. Students begin to get excited when they can work with their hands and visualize a problem and its solution. From here, we begin a foray into numbers. Since most of them equate mathematics with numbers, it is a natural place to begin. One of the highlights of this phase of the course is having them develop and then study the Fibonacci sequence. They are stunned to learn that Fibonacci numbers and the golden ratio are in nature, art, music, architecture, etc. To further bring this home, I give them their first Facebook assignment: they are to find an example of the Fibonacci sequence in the real world and post about it on our Facebook group page. They also make substantive remarks on other students' postings. From their comments, I know that they will never see pineapples or flowers or Da Vinci's *Mona Lisa* the same way again. Many of them, for the first time, find relationships between mathematics and the world they live in.

Along the way, I am addressing my second key philosophy, making mathematics meaningful in their world. One powerful tool to accomplish this is group work, which I employ whenever and wherever possible, emphasizing hands-on learning and discovering. (Our department's choice of *The Heart of Mathematics* as our textbook is very supportive of this approach.) Group work empowers students to take a key mathematical idea and make it personal, by working through a problem or issue and communicating mathematical truth to one another using their own language. One example of this appears when discussing the Fibonacci sequence. I do not give them the sequence right out. I pose the Bunny Problem and have them work in groups to determine how many bunnies there will be in a year. I model a naive approach to solving the problem and then set them loose. When they try to replicate for further iterations, they find the brute force method cumbersome and invariably someone starts looking for a better strategy. I encourage them to come up with a different approach and when someone discovers the pattern, their entire group becomes excited and the enthusiasm spreads to the entire class. Not only is mathematics becoming personal, but they are also finding that they can do mathematics!

To continue to focus on making mathematics meaningful in their world, I liberally use projects more than examinations as a means of assessment. For instance, after reading the fantastic book *Fermat's Enigma* [2] by Simon Singh, groups choose a mathematician and create a YouTube video to express the importance of that person's mathematical contributions and how he or she thinks that their mathematician would answer the what-is-mathematics question. This gives my students the opportunity to be creative and dramatic and, behind the scenes, to connect with a mathematician as a real person with real passions and successes and failures. I am also encouraging them to use a familiar medium, YouTube, to do something mathematics related, bringing mathematics into their world.

After exposing them to some key ideas in geometry and topology, I assign an individual art project, for which they create a piece of art using or displaying some type of mathematics. (Their work must always have an abstract, explaining how their project uses or displays mathematics.) My goal is to introduce them to the aesthetic component of mathematics. There is a lot of enthusiasm about this assignment. Students have come up with beautiful poems, dramatic readings, visual art, musical pieces, and so on. The assignment helps them personalize their understanding of mathematics, relating mathematics to their own talents and abilities. For example, one student loved to knit and chose, for her art project, to knit a Möbius scarf. She will forever understand and appreciate the beauty of this shape she created with her own two hands.

A third project involves community volunteering. In small groups or individually, students do an act of service that involves mathematics in some way for someone outside the university. I am very generous in interpreting what involves mathematics and many students get very excited when they realize they can do mathematics in a way that helps others. A frequent proposal for this assignment is for students to tutor a young person in mathematics. But many students get more creative. For example, I have had multiple students suddenly realize how much mathematics is involved in cooking and baking; they choose to prepare food for the needy. Regardless of their view of their mathematical competence, the project addresses the idea that they personally can and do use mathematics in their world and they can even benefit others with their mathematical skills.

The final project is a problem-solving project where they again work in groups, selecting material not covered in the course and presenting it to the rest of the class. In addition to their presentation, they must also solve four problems requiring understanding of their topic without benefit of my instruction or assistance. My proudest moments occur when I see groups coming up with visual representations and engaging activities to enlighten their fellow students about mathematics new to them. The presentations and solutions demonstrate that they can now learn mathematics on their own and relate it to others as well as themselves. And it means that I have been true to my key philosophies: I took the students that were given to me, with all their mathematical baggage, and facilitated a connection with the world of mathematics that was both meaningful and personal.

4 Response and Assessment

Mathematics for the Liberal Arts is not an easy course. The students are constantly working, required to participate actively every day. This results in a transformation between that first what-is-mathematics day and the end of the course. To culminate the experience for them, 50% of the final exam is an essay revisiting the what-is-mathematics question, supporting their answer with elements of the course that helped shape their understanding. Here are just a few excerpts from and comments on the newly proselytized:

> When I walked into class the first day of the semester, I had no idea what to expect. I have always disliked math, mainly because I had a hard time understanding formulas and concepts. I just wanted to take the class and get math over with for the rest of my life. So when I was told on the first day that math was "beautiful" and "intriguing" I was skeptical to say the least. After the first few lectures, however, I soon found that maybe math itself wasn't the problem, but rather my definition of math.

This student went on to discuss how exploring pineapples and seeing slides of how the Fibonacci sequence occurs in nature challenged him to see mathematics as not only beautiful, but also surprising and interesting.

> I've learned in the process of taking Math for the Liberal Arts, that math is in everything, and that's why it won't ever go away. In countless ways, I owe my existence to things that involve math. I live in the third dimension. That's math. When I'm looking for a specific book in the library, I have to search for it based on its ISBN number. That's math. Sometimes my shoelaces get tangled up in horrific knots, and I have to untangle them. That's math, also. And sitting here, figuring out what I need to make on my final to pass this class is math too.

> Mathematics is a language, sometimes in various symbols, but also sometimes in concepts that have come to define our lives as human beings. Without mathematics, not only would we not have the opportunity to create beautiful music, but we would also not be able to see beauty in nature.

As a music major, this student connected ideas of the course to his personal art and came away from the course with a deeper appreciation of how mathematics fit into his world.

> Mathematics is intricate and it always has us asking questions. There is value in asking the questions though, because math is a tool to help us understand most everything else around us . . .

I love that this student's curiosity was stimulated by the course and she came to understand that mathematics is a language that helps us understand our world.

> Many people dislike math because it is very complicated and vast. Although it seems ungrounded in fact and fickle and confusing, the opposite is true. Mathematics is among the only things we can be absolutely sure of.

From the content of the course and by reading *Fermat's Enigma* [2], this student learned that mathematics is distinguished from science by the aspect of absolute proof. As he went on to say, "There is nothing negotiable about math. If proven, a theorem is and will remain so forever."

Overall, the student essays show me example after example of how the students find elements from the course that connect with them personally and increase their understanding and appreciation of mathematics. Although anecdotal, I feel the essays provide evidence that students' core beliefs about mathematics have been transformed.

At the end of the course, I also try to quantify the progress of my students by again having them rate their agreement, on a scale of 1 to 5, with the three statements assessed at the beginning of the course. So that I can quantify a change, the assessment cannot be anonymous but there is no grade associated with their responses. I feel that I get a reasonably accurate view of their true feelings. Regarding the importance of mathematics, the average rating increased from 4 to 4.8. I think this bears testimony that they come in knowing that mathematics is important in general but get exposed to its vastness and breadth along the way. Where I most need to continue to evolve is with the third statement, "I can do mathematics." Its rating only moved from 3.3 to 3.7. Perhaps in exposing them to some of the rigors of mathematics, I inadvertently make them believe that only doing mathematics at a certain level qualifies as really doing mathematics. I intend to further investigate this result. What I am most proud of and where I see the greatest improvement, however, is with the second statement. At the beginning of the course, the average agreement with the statement that mathematics is beautiful only ranked 2.6, with less than 13% of respondents giving this statement 4 or 5. At the end of the course, it was rated an average of 4.4 and almost 90% of the respondents gave it 4 or 5! I am far from perfect but my approach is definitely connecting with students on a personal level and changing their perception of mathematics. They walk into class that first day, dreading lectures about something they just don't get and instead find mathematics alive and well and within their grasp.

5 Conclusion

My philosophy for teaching Mathematics for the Liberal Arts has evolved over the last few years. However, at its core, it always embraces the reality of our students' initial attitudes about mathematics and endeavors to engage them with mathematical encounters that help them personalize their experience. Unlike many of the equations that we solve on a daily basis, there is no one right answer. Success is measured by the progress that students have made to accept mathematics, not as drudgery, but as something beautiful in its own right and something meaningful in their world.

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III

Developmental Mathematics and College Algebra

Developmental and college algebra courses tend to concentrate on the nuts and bolts of symbolic computation, and seem to have little time for philosophical discussion. However, at times attention to philosophical issues can enliven the course and improve student learning. We have two contributions concerning these courses.

Chapter 7: James Henderson, in “Less is More: Formalism in Developmental Algebra,” finds that his developmental mathematics students have imbibed a simplistic platonism with the mathematics they have learned. Philosophical issues lurk beneath even the most basic mathematics. Just making the transition from counting (listing the numbers in the standard order while moving from one object to another) to treating numbers as objects independent of context involves ontological and epistemological issues: what are these “objects”? How do we gain knowledge of them? Henderson suggests that sometimes explicitly introducing a formalist approach may help students avoid the morass of ontological difficulties concerning the nature of the complex numbers. With concrete examples, he shows students that they have actually done a similar maneuver earlier, in their studies of negative numbers and fractions.

Chapter 8: Lesa Kean’s action research chapter, “Using Beliefs about the Nature of Mathematical Knowledge in Teaching College Algebra,” explores how focusing her mind on introducing students to various beliefs about the nature of mathematical knowledge (that it is interconnected; that some is procedural, some conceptual; that it might be discovered or invented; that it might be justified by logic and proof or by social and cultural means) affected and improved her teaching in college algebra. Most faculty teaching these courses feel pressed to get through all the material that must be covered. It would seem that there’s no time to introduce philosophical concepts, and that students are not sufficiently sophisticated to appreciate them. Toward the end of the activity, Kean shared and discussed with her classes these views on mathematical knowledge, and she discusses how this improved their understanding.

7

Less is More: Formalism in Developmental Algebra

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1 Introduction

When teaching a developmental mathematics class, it may be that adopting one philosophical position or another will sometimes prove helpful, depending on the subject matter. There are four sections of developmental mathematics offered at the University of Pittsburgh-Titusville every term (two each of Beginning Algebra and Intermediate Algebra), each enrolling 25–35 students. By comparison, there is one first-semester calculus class offered each term with an average enrollment of fifteen. Given the relatively large enrollment in developmental courses compared to higher-level offerings, it is especially important to exploit any advantage one can in them.

In particular, what I have in mind is the nature of number. My experience is that most developmental mathematics students have a simple sort of platonism in mind when they consider the character of mathematical objects (if they consider them at all). For them, numbers really exist, though not in the same sense of tables and chairs, and they are out there somewhere, though not located in any physical place. This is all very well when working with whole numbers (and, generally speaking, when dealing with real numbers), but when the conversation turns to, for instance, imaginaries, it can be problematic. The kernel of the problem, I believe, is that developmental mathematics students simply do not know what to make of a number such that if it is squared, a negative number is generated. They can't see it, and this gets in the way of dealing with non-real complex numbers in general. All learning gets put on hold until the mystery of just what kind of thing we are dealing with is straightened out. Incidentally, and for reasons beyond my understanding, this malady seems to afflict non-traditional-aged students to a higher degree than it does traditional-aged ones. To avoid the problem, I advocate a quasi-formalistic approach that centers on the notion that imaginary numbers are simply made up and that a deeper understanding of them is not strictly necessary to manipulate them.

The University of Pittsburgh-Titusville is the only two-year branch campus of the University of Pittsburgh, and it primarily serves as a feeder to other campuses in the Pitt system. Many UPT students are referred to Titusville after applying to the main campus in Pittsburgh, and most require some sort of remediation when they arrive. Because of this, I spend most of my time teaching courses at the level of the calculus sequence or below. This chapter discusses techniques I use in my developmental algebra classes, courses that do not carry college credit, though I suspect they might be beneficial to students in other lower-level courses as well. Most of the students in these sections, then, are not science or engineering majors, but rather non-science majors with limited interest in and confidence about mathematics. It is worth noting that some are nursing students, many non-traditional-aged, in need of sharpening their mathematical skills before moving on to credit-bearing courses.

2 Platonism and Formalism

Before proceeding, it may be useful to say a very few words about platonism and formalism in mathematics. Plato developed his theory of forms in part to account for what he took to be the apparent absolute certainty of mathematics. He took knowledge of mathematics to be real and unchanging, as opposed to mere opinion regarding everyday concerns. The objects of this knowledge, he concluded, must likewise be real, unchanging, absolute, and altogether different from ordinary, earthly objects which, in the words of his pupil Aristotle, are subject to generation and decay. Thus, he posited a non-physical realm of ideas, or forms, that exists outside space and time. These forms, naturally, could not be apprehended through the senses, but rather were grasped by intellectual exercise. Ordinary, garden-variety objects participated in the forms by instantiating them in the world. For instance, a set of twins participate in the number 2 by being a pair.

It is likely that not many of my students would accept Plato's notion that the unchanging, immaterial realm of the forms is the true reality while our impermanent physical world is illusory. Even fewer have considered the possible problems with his theory raised by Plato himself in *Parmenides*, *Republic*, and other dialogues. Still, all reflective humans have puzzled over the problem of universals, the existence of generalizations from the observation of particular instances. Plato's forms—or a very much simplified version of them—are a well-trodden field for those who consider these issues. When we observe individual dogs, we take them to be members of a general set of dogs (each of which exhibits a certain dog-ness), and the generalization seems as real as the dogs themselves. This is equally true of numbers. As long as there are physical objects to which they may refer, most of my beginning algebra students take numbers to actually exist (again, in a non-physical way) and to be, like the truth in "The X-Files," out there.

In stark contrast to platonism is formalism. The best known exponent of formalism is David Hilbert, and justly so; his is the most sophisticated version of formalism on offer. However, I wish to concentrate on an account of formalism given by another German mathematician, Johannes Thomae, initially published in 1880 and revised in 1890 (in *The Elementary Theory of Analytic Functions of a Complex Variable*). The variety of formalism favored by Thomae, known as "game formalism," can be stated succinctly: "For the formalist, arithmetic is a game with signs which are called empty. That means that they have no other content than they are assigned by their behaviour with respect to certain rules of combination (rules of the game)" [1, sec. 95, p. 170]. Thomae takes this view explicitly because it "rids us of all metaphysical difficulties" [1, sec. 89, p. 164].

It is easy to see that he is no platonist. Weir [3] makes it clear that Frege offers serious challenges for game formalism as a comprehensive philosophy of mathematics, but in this setting it is not a completed philosophy of mathematics I seek. Rather, I simply want to clear away some of the ontological brush that keeps my students from mastering techniques of manipulating complex numbers. In this light, Thomae works perfectly well. His formulation instructs us to not worry about the nature of imaginary numbers (or matrices, or whatever sort of mathematical object that is the source of problems). They don't mean anything; they are marks we make on a page or a chalkboard, players in a game we have invented. When we encounter them, the rules of the game allow certain well-defined options. That is all. Viewed from this perspective, there simply are, as Thomae put it, no metaphysical difficulties. This is precisely what I am after in the classroom.

3 Meanwhile, Back at the Lecture

I am choosing to concentrate on the point where students are introduced to imaginary numbers as the dividing line between the usefulness of a platonic mindset versus a formalistic view. This is purely for ease of discussion; the boundary could be placed anywhere concepts unfamiliar or confusing to students arise. Wherever the line is drawn, however, it is generally not the case that these concepts pop up on day one of the new semester. Given that, a lot of groundwork will be laid ahead of time with respect to dealing with the more challenging concepts. In particular, with respect to non-real complex numbers, rules for collecting like terms (when dealing with addition and subtraction of polynomials, for instance), rules for manipulating exponents (when multiplying polynomials in general or dividing monomials in particular), and rules for simplifying expressions involving radicals are singularly efficacious. Prior knowledge of the techniques will be assumed as the discussion develops.

Let us begin with an easy example that often comes under discussion at the beginning of a semester. When the addition of whole numbers is at issue, an attitude of a platonic sort about the nature of whole numbers may be useful. As I indicated, it is my experience that a simple (or maybe simplistic) form of platonism is what most developmental students have in mind when they consider mathematical objects anyway. Here both the mathematical objects, whole numbers, and the operation to which they will be subjected, addition, are relatively familiar to them. In such a case, trying to bring students around to a different way of thinking about these ideas would be a case of fixing something that isn't broken. As a rule, whole numbers are pretty easy to grasp within a primitive platonic framework, and college students—even those needing remediation—are no strangers to addition. Thus, no philosophical work is required of the instructor when what works best for the students is what they are bringing to the conversation already. The same could be said for many topics with which the students are already at least to some degree familiar. Examples include multiplication and division of real numbers, distribution of multiplication over addition and subtraction, and the addition and multiplication properties of equality. In all of these areas and more, there simply is no need to import philosophical material into the conversation when it won't help the students learn the material.

Consider, however, when beginning-level algebra students are first introduced to imaginary numbers. In this case we are investigating numbers such that when they are squared, a negative number is generated. How might a beginner think of such a thing? Whatever it is, it isn't positive since squaring a positive yields a positive. It isn't negative because squaring a negative yields a positive. Finally, it isn't zero; squaring zero yields zero. For the developmental students with whom I have worked, this exhausts the possibilities. Imaginary numbers, then, are totally mysterious in a way that whole numbers are not; they are an entirely different kind of mathematical object from the ones to which the students have been introduced. In terms of Plato's forms, the difficulty arises when we introduce mathematical concepts that, in the platonic sense, cannot be instantiated in the real world in any obvious way. In essence, the problem becomes how to guide students through their metamathematical difficulties in such a way that the learning process is least inhibited, and it is here where mathematical formalism can be of help.

To ease the learning process, I adopt a formalist approach like that of Thomae and ask the students not to think about the character of complex numbers too much. "These is are just marks on a page. You shouldn't get caught up in what you may think they really are because, in the end, it's just not helpful to do so." I try to drive this home with a story I am reminded of about the dangers of being sidetracked by such issues in the course of one's studies. The story itself is due to Nobel laureate Steven Weinberg:

> [W]hile Phillip Candelas . . . and I were waiting for an elevator, our conversation turned to a young theorist who had been quite promising as a graduate student and who had then dropped out of sight. I asked Phil what had interfered with the ex-student's research. Phil shook his head sadly and said, "He tried to understand quantum mechanics." [2, p. 84].

This is precisely the sort of thing I am trying to avoid: I don't want them to spend time trying to understand the numbers themselves, but rather spend time trying to understand what to do with the numbers. To that end, I tell my students working with square roots of negatives is a game and that this game has rules. Rule #1 is "$i^2 = -1$." Rule #2 is "For any positive number a, $\sqrt{-a} = i\sqrt{a}$." The secret is not to understand the nature of i; it is to eliminate higher powers of i and square roots of negatives from the equations and expressions with which they are working. That is all they have to do, and worrying about other issues only gets in the way. Now, eliminating higher powers of i is simply a matter of applying what they already know about exponents and using rule #1. Obviously, a fair amount of chalk is used in clarifying this point in a classroom setting, and a brief example of what I have in mind may be instructive. I will omit no details. First we note that

$$i^1 = i.$$

(This follows from the rules of exponents.) Next, by rule #1,

$$i^2 = -1.$$

Now

$$i^3 = i^2 \times i^1 = -1 \times i = -i.$$

Finally,

$$i^4 = i^2 \times i^2 = -1 \times (-1) = 1.$$

For something a little more challenging,

$$i^{27} = i^{24} \times i^3 = i^{4\times 6} \times (-i) = (i^4)^6 \times (-i) = 1^6 \times (-i) = 1 \times (-i) = -i.$$

There is nothing revolutionary about this presentation, and if one gives a step-by-step account of it, I find most students can follow it—and this is the important part—all without having the slightest idea of what imaginary numbers are.

Now, eliminating square roots of negatives (which leaves square roots of positives, which they already understand) is a simple application of rule #2. This, too, requires some justification, though not as much as the case just examined. Adding and subtracting complex numbers requires using what students already know about adding and subtracting polynomials. It is a matter of collecting the terms that include an i and collecting those that do not. Multiplying complex numbers boils down to what students know about multiplying polynomials and use of rule #1 above. If one chooses to round out arithmetic with complex numbers, and not everyone does, dividing them relies on exploiting what students already know about multiplying fractions and polynomials and eliminating radicals from denominators. This all requires work from everyone involved in the learning process, but learning the techniques is in no way an insurmountable task. Like so many other cases in mathematics, the trick is not to learn how to solve a new kind of problem; it is to learn how to reduce a new problem to an old one. In any event, trying to understand the nature of i simply gets in the way of mastering what the syllabus calls for, and it doesn't make solving problems any easier.

After treating imaginaries (and by extension, non-real complex numbers) this way, it is useful to reassure students that we have not made a radical jump in treating these numbers differently from the real numbers with which they are familiar. I ask them to think back to when they were first learning how to subtract. Early primary school lessons usually take the form of something like "You have five apples, and you give Johnny two apples. How many apples do you have left?" It is hoped that this tried and true method will get the idea of subtraction across to the elementary school student in an understandable fashion. My students all agree that this makes good sense, and then I ask them what six minus nine is. "Negative three," the eager ones will answer, but there is a problem. Given that subtraction is a relation we have just defined in terms of giving away part or all of a set of objects we have, this makes no sense. The issue is made manifest by posing the following brain teaser: "You have six cookies, and you give Mary nine of them. How many do you have left?" Although they haven't realized it up until now, my students come to realize that negative numbers, like imaginaries, may be viewed as just being made up. Why do we make up negatives? It turns out that they are handy bookkeeping devices, as anyone with a credit card (or sometimes a checking account in tough financial times) can attest. Imaginaries are similarly useful in solving problems, just not simple, everyday problems like balancing a checkbook.

Non-integer rationals, too, can be seen in this way. My students are well aware that 15 divided by 3 is 5 because 15 puppies may be evenly distributed among 3 children: each child gets 5 puppies (and 2 unhappy parents). In common parlance, 3 goes into 15 five times. After rehearsing this, I ask, "What is 4 divided by 11?" Now even the eager ones are unsure, and they say "$\frac{4}{11}$?" in a questioning tone of voice. The difficulty, as should be clear, is that 4 puppies may not be evenly doled out to 11 children, at least not in any satisfactory way. Another concern is this: 11 doesn't go into 4. It's just too big (or in the case of "11 divided by 4," 4 doesn't go into 11 without remainder, or evenly). My students find it oddly satisfying that fractions are also just made up because they happen to be useful as bookkeeping devices. That is exactly why we make them up.

The story can go further if one chooses to pursue it. It is well known that the cube root of 27 is 3 because 3 is that number such that if it is cubed, the result is 27. In the same way, the principle square root of 16 is 4 because 4 is that positive number such that if it is squared, the result is 16. "Now," I ask, "what is the square root of 2?" I am usually greeted with blank stares. "It is that number such that if it is squared, the result will be 2," I helpfully inform them. Guarded nods tend to follow. The trouble here is that I can give them no better answer than that. It is not $\frac{7}{5}$; that's just a little too small. $\frac{141}{100}$ is closer, but a brief calculation shows that it is also just a little too small. I don't ordinarily prove that there is no rational number that will work in a Beginning Algebra class. They have suffered enough, and at all events they are at this point prepared to take my word for it. However, a simple construction of a square with a diagonal shows there must be *some* number that does the job. Irrationals, like their negative and non-integer rational brethren, are just made up. Since we have been making up numbers all along the way (though they didn't know it), my students are more at ease with just one more set of made up numbers, in this case imaginaries.

It is beneficial to let them know that they are not the first to face such challenges, and that the German mathematician Leopold Kronecker said, “God made the whole numbers; all else is the work of man.” That is, it's all just made up. But you have to know it for the test anyway. Naturally, none of this precludes the possibility of giving an interpretation later on once the students have a grasp on the properties involved. In trigonometry class I find it helpful to give a geometric interpretation of complex numbers—they can be thought of as points in a plane and even eventually linked to Euler's formula, $e^{ix} = \cos x + i \sin x$.

I'm not trying to give the students a deep grounding in the philosophy of mathematics; these discussions are taking place in a beginning-level algebra class. There are many very difficult open questions in the philosophy of mathematics. An in-depth discussion of them would be sure to confuse developmental mathematics students, and in any case would not be useful as far as progress in lower-level algebra classes is concerned. Thus, a detailed presentation of deep philosophical issues like the nature of number would not only be unhelpful, but positively detrimental. I keep foremost in my mind Niels Bohr's warning that the closer one gets to the truth—in this case philosophical truth—the less clear one is going to be [2, p. 74]. I assign no extra readings or reports as material collateral to the problem solving techniques one usually covers in a beginning algebra class. It's all handled during lecture as a way to help students focus on mathematical problem solving instead of worrying about the nature of number, which, at that point in their mathematical development, isn't helpful anyway.

4 Does It Help?

I don't expound upon the philosophy of mathematics simply because I love the sound of my own voice (though, in fact, I am partial to it). My goal is to ease the learning process for my students, many of whom see mathematics class as a significant emotional experience. In a sense, whatever philosophical guidance my students receive does nothing to make them better mathematicians; their grasp of the material is not enhanced by a knowledge of formalism. This, however, isn't the point. In discussing Thomae, what I want to do is clear away some of the ontological underbrush that might keep them from learning simple problem solving techniques. In this respect, I find that it can be useful.

It is fair to ask if students are ill-served by a minimalist approach to basic algebra. In the long run, will their not having been given the whole story make learning mathematics more difficult? Students may indeed suffer if they never hear more about non-real complex numbers than what I have laid out here. What I have in mind when I present imaginary numbers to my developmental students, however, is a bridge to their next mathematics class. I want to teach them how to manipulate numbers and survive their courses until they are prepared for a full-blown presentation of the material. It is vital that this scaled-back approach designed for developmental courses not be the end of the story and that they hear the rest of the narrative when they are ready for it, usually in a college algebra class.

Oftentimes I get a palpable sense of relief from my students when they learn that they don't have to understand “number” to understand algebra. Since everyone understands games (and not everyone understands algebra), my charges are in more familiar terrain and they feel freer to simply apply the rules to the players we encounter, even if they do this with an attitude of “If I must.” It has been with a sense of relief that I have been told “This just makes it easier.” I think they feel they gain a sense of distance from the material and, frankly, they don't have to know as much. They don't have to know what the numbers are, just what to do with them.

This is a far cry from when I began my teaching career, never spending any time on philosophy. The sense of being overwhelmed with alien concepts simply isn't present now to the extent that it was back in those days. When it occurred to me that issues of the nature of number were a stumbling block to learning what was really central to the course, I began introducing this material to the classroom, and the results have been positive. Over the years I have expanded the discussion to include sets of numbers other than just imaginaries because there was an unease with treating them differently than the rest. This unease disappears when students realize that the vast majority of the numbers they encounter in algebra classes can, at least in principle, be thought of as having the same artificial character. They don't mind that I don't treat whole numbers in this fashion because, I think, they feel there is a naturalness to them that isn't present when dealing with less easily understood sets of numbers.

In sum, a simple exchange on what at first appears to be not central to the topics ordinarily seen as essential can help facilitate the learning process if it isn't too abstract and lifts a burden from the students. In this case, a small digression on philosophy can do just that, if for no other reason than it helps students to know that not all mysteries need to be solved and that others have struggled with the same issues that plague them.

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8

Using Beliefs about the Nature of Mathematical Knowledge in Teaching College Algebra

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How do mathematicians' philosophical views affect their classroom practice? This is an open empirical question [8, p. 275]. However, it does seem clear that they could deliberately infuse their philosophies, or at least an awareness of their beliefs about the Nature Of Mathematical Knowledge (NOMK) into their classroom practices. Many would suspect that such an infusion could affect student perceptions of NOMK [7, p. 174] or even the nature and utility of the mathematical knowledge produced in the student [2, p. 132].

Until recently I have been working under the assumption that my current teaching practices are in harmony with my personal concepts about NOMK, but I had only analyzed it superficially. In the fall of 2013 I decided to teach with a more deliberate awareness of the concepts of NOMK to be presented in my classrooms. In this chapter I will share what I learned from this activity.

1 Focus

Among other responsibilities, I teach mathematics content courses at Butler Community College which has campuses in Andover and El Dorado, Kansas. My focus for this chapter is the three College Algebra classes I taught in the fall of 2013 for this institution. Class size ranges from 11 to 22 students. Two of the three classes I taught that semester were dual-credit classes taught to high-school seniors planning to complete their college mathematics credits early. The classes met twice a week for ninety minutes at a time. Every student in them had successfully completed either trigonometry or precalculus in high school within the last calendar year and would generally be considered successful in high school mathematics. Interestingly, some students perceived themselves to be mathematically challenged. Often I found this meant that they "did get a B that one time." These students thus were academically driven individuals who wanted to understand their subjects and to get good grades.

My third College Algebra class came from the general population of students at Butler. The students in it varied greatly in nearly every respect, including age, number of years since their last mathematics class, and even number of tries typically needed to pass a mathematics class. About half of the students expected College Algebra to be the last mathematics class they would ever take. Another third of the class expected to take only one or two more classes as part of their major requirements. The remaining sixth had plans to go on to some STEM-related career. The class met once a week for a three-hour session.

I went through several steps to gain some insight into what aspects of NOMK were already being addressed in my classes and to decide what I could do to interject a few more. First, I filled out a survey on teachers' concepts of NOMK that I had written as part of my dissertation research [9, pp. 257–258]. Second, I took notes on my use of NOMK references in a class I taught prior to actively attempting to interject NOMK into the classroom. Third, I spent two weeks of intensive attention to the NOMK concepts presented in my classroom, keeping a journal as I did so. While journaling I reflected on what evidence I could find that the students were internalizing the concepts and what I could do to improve their retention. Finally, I evaluated a test I gave them at the end of the journaling unit.

2 The Nature of Mathematical Knowledge

Not everyone means the same thing when they say mathematics [11, p. 90], and I have found that this is also true about NOMK. Different authors, when discussing the philosophy of mathematics or NOMK, have different aspects of mathematical knowledge in mind. Thus I explain here what types of questions I am considering. A few years ago as part of a graduate research project I had the opportunity to read articles by many authors and distill the concepts of NOMK they were describing into eight aspects divided (unequally) over four philosophical questions [9, pp. 12–17].

Question 1: How simply is mathematical knowledge organized?
I list this question first because it is the one that mathematics education researchers and philosophers of mathematics disagree about least. Mathematical knowledge is interconnected. "Mathematics is a web of interrelated concepts and procedures." [4, p. 11]. Although those of us on the professional side of mathematics are aware of this interconnectivity, many students are not. Yet many students as well as professionals can give examples of interconnections within mathematics.

Question 2: What is the source of mathematical knowledge?
Authors answer this question in a variety of ways. Some conceive of mathematical knowledge as discovered. This may be "due either to its existence beyond humanity, waiting to be discovered. . . or to its creation as a logical, closed set of rules and procedures." [12, p. 23]. For individuals who hold this position, mathematical knowledge may be discovered either while attempting to solve problems, or by exploring the rules of a mathematical discipline such as geometry. Either way, discovered patterns can be developed and applied in various settings. For some, mathematics seems *a priori*, or predetermined. "Mathematics is a static and immutable knowledge with objective truth." [1, p. 79]. Other individuals conceive of mathematical knowledge as invented. For them, "mathematics [is] a dynamic and continually expanding field of human creation and invention." [6, p. 328].

The concepts are not mutually exclusive; people may hold more than one of them simultaneously. For example, I studied with a professor who said that some mathematics is discovered, as the Pythagorean theorem was discovered in many cultures independently at many periods of time. At the same time she held that some mathematics is invented, as evidenced by the many axiomatic systems that have been designed simply to be internally consistent, with no intention of any real-world application.

Question 3: What is the justification of mathematical knowledge?
Either it is justified by logic and proof ("a consistent, formalized language without error or contradiction" [12, p. 23]), or by social and cultural means ("It is societal knowledge that must be interpreted in a manner that holds meaning for the individual" (ibid)). Again, it may be too simplistic to expect everyone's responses to this question to fall neatly into only one category. My dissertation research suggested that, in addition to the categories mentioned, some mathematics teachers may consider simple substitution ("Just plug it back in and check!") to justify mathematical knowledge. Still others consider the role of context in justification ("If it's a distance, it can't it be a negative number, right?"). In both cases, the methods were viewed by the teachers not as a subset of logic and proof, but as stand-alone techniques valid on their own.

Question 4: What is the content of mathematical knowledge?
Of the aspects of NOMK I found in my literature review, this was the question more addressed than any of the others. Is it procedural? If so, then perhaps mathematical knowledge is a set of "useful, if [possibly] unrelated rules and

procedures." [5, p. 487]. Is it conceptual? If so, then, perhaps it, "serves more as a way of coming to know than as a set of things to be known." (ibid).

Here again it is important to emphasize that individuals may simultaneously hold both positions to be true. Without this acknowledgment we run the risk of setting up a false dichotomy between aspects of NOMK. It does not seem appropriate to try to pigeonhole a person's concepts as either purely procedural or purely conceptual. For a more detailed discussion, see my dissertation [9].

Since there is no clear consensus about which concepts of NOMK are correct, I am working under the hypothesis that simply the awareness of some of the issues about NOMK may help students be more successful. During the journaling phase of the project I actively interjected concepts of NOMK into the classes. I then observed whether any of them were successfully internalized by my students.

3 Data Collection and Analysis

The Survey

As mentioned before, I wrote a survey intended to capture teachers' concepts about NOMK as part of my dissertation work. In preparation for this chapter I took my own survey. What I found was that I had no particularly strong position on the first three philosophical questions. For example, although I see interconnections in mathematics, I am not convinced the field is fully interconnected. Regarding the source of NOMK, I find each of the discovered position, the invented position, and the *a priori* position plausible in certain instances. I am not particularly concerned whether NOMK is justified best by logic and proof, by social convention, by substitution, or by context. What I did have strong opinions about was conceptual NOMK. I share Boaler's position that doing mathematics is more a living act of conceptualizing than anything else. To paraphrase her, procedures are no more mathematics than a page of printed music notation is music [3, p. 30]. Procedures have their place, just as the grand staff and quarter notes have in music. I just think that we can do a lot more without them than we think, if we just allow ourselves to think conceptually. I believe our students can do this as well.

Preliminary Notes

Prior to my effort to teach with NOMK awareness I wrote detailed notes for the activities in what I would consider a typical class meeting. I took the notes early in the semester in all three of my College Algebra classes. In each class I began by asking students to place any homework problems with which they had struggled on the board. I then asked their classmates to offer their written and oral solutions. During this time I allowed other students to ask questions about the solution either to the solution provider or to myself. Depending on whether or not I felt that the concept has been sufficiently understood, I sometimes asked the class some additional questions for clarification.

On one particular pre-NOMK-awareness day I was teaching students strategies for solving quadratic equations. We started with a list of problems. I reviewed factoring with them and offered some new factoring strategies for those who needed them. Once they were more comfortable with factoring I asked them to scan the page to see which quadratics could be easily factored. Successful students put their solutions on the board for class inspection. Next I worked with the students to develop solutions using square roots, solving equations such as $x^2 = 6$ by taking square roots of both sides. They then searched their list for problems that could be solved using that strategy. Next I presented completing the square as a strategy for reorganizing problems to be solvable using square roots. Again, students looked for problems that could be solved using that method. By this point in the lesson one or more students in each class mentioned having sometimes used the quadratic formula for problems like this in high school. In fact, in each class at least one student was able to state or write the formula fully.

My goal in teaching the lesson this way has always been to provide procedures while highlighting the fact that the procedures alone are often not enough to solve problems of this type. There is strategy involved. In other words, I am actively involving students in exploring patterns, communicating their thoughts, and choosing appropriate procedures—conceptual activities. As the lesson occurred early in the semester, many students were not yet accustomed to having a choice in their procedures. They were more accustomed to having the professor tell them what procedure to use, and when. Even searching through the page to find where to begin was a new task for them, but in the end they seemed to gain some confidence from having accomplished it.

Reactions to this from students vary, but are generally positive. For example, my notes from this semester show that one dual-credit student said, “I always come here [to mathematics class] scared, and leave feeling happy.” Later in the same lesson I had the class go through a process in which we derived the quadratic formula by completing the square from a general quadratic equation. This same student said, “You blew my mind twice, because I got that.”

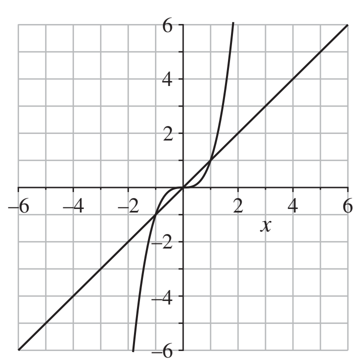
Consistent with the results of my self-survey, my notes from this class showed I was doing well at highlighting the roles of procedure and concept in mathematics but was doing little to make students aware of other philosophical issues. Thus, as I began my two weeks of journaling, I was determined to be more explicit about other NOMK concepts as well.

Journaling Begins: Inverse and Exponential Functions

I began my two-week NOMK-focused journaling activity right after a chapter test when we were beginning a new chapter. The first topic was one-to-one functions, inverse functions, and the composition of a function with its inverse to produce the identity function: $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$. In each class I showed how graphs of inverse functions were related to graphs of the original functions by pointing out the similarities between the definition of function and the definition of one-to-one function. I demonstrated a process for producing a graph of an inverse function by interchanging x-coordinates and y-coordinates and building the inverse graph point by point. I asked the class to determine the domain and range for each function and its inverse based on the appearance of the graph of the function and asked them to hypothesize what the relationship between the domains and ranges were. In other words, I provided them with an opportunity to see lots of interconnections between the various aspects of inverse functions, and between inverse functions and other mathematical knowledge that they already possessed. I was pretty happy with myself for addressing this first question from my NOMK list. I had shown interconnections for this topic by discussing the concepts that drove the procedures. In my notes I discussed whether such a combination of NOMK concepts was necessary:

> I find that when I show interconnections (or have students find interconnections), we need conceptual mathematical knowledge to do so. Maybe this is why so many teachers have difficulty showing the interconnectedness of mathematical knowledge, even if they believe it to be interconnected. If you are trying to teach procedurally, devoid of conceptions, whether to minimize student work or to simplify the types of written work you would need to grade, you cannot show interconnections. Indeed, if you only show one way to do each type of problem, what interconnections are even available to discuss?

One student let me know she couldn’t see what I was saying when I suggested that a function and its inverse reflected across the line $y = x$. To help her I drew the graph for $y = x^3$ on a sheet of paper (see the illustration) and used the bleed-through from the marker to draw $y = \sqrt[3]{x}$ on the back. Then I physically spun it along the line $y = x$ to show the type of reflection involved. At that point I heard many students in the classroom either gasp or say “oh!” The student who brought up the confusion then said, “oh, I see it!”



1. Mathematical knowledge is interconnected.
2. Mathematical knowledge is discovered.
3. Mathematics is *a priori*, pre-determined.
4. Mathematical knowledge is invented.
5. Mathematical knowledge is justified by logic and proof.
6. Mathematical knowledge is justified by social and cultural means.
7. The content of mathematical knowledge is procedural.
8. The content of mathematical knowledge is conceptual.

Table 1. Eight NOMK concepts

I felt rather successful at providing examples of interconnected, procedural, and conceptual mathematical knowledge, but I had still not addressed the other aspects. I decided to create a poster summarizing the eight NOMK concepts (see Table 1) I was trying to demonstrate. I hoped this would help me focus more on all eight aspects, and maybe even entice my students to ask about them. I made several copies and posted them in the classrooms. I did not explain the poster directly at first, wanting to see first if the students' curiosity would draw them to ask about it.

The next topic I covered was exponential functions. I chose to have the students as a class construct $f(x) = 2^x$ and $f(x) = (1/2)^x$ by computing and graphing coordinate pairs. Then I had them generalize about asymptotes and common coordinates over all exponential equations. I split them into groups and had each group graph and produce equations for different bases and transformations. Each student group chose a scribe to put their solutions on the board. I presented them with the more general form of the equation, $f(x) = a \cdot b^{x+h} + k$; then we generalized again. I asked them how to move the graph up and down. They decided a change in the k-value would do that. I then asked how to move graphs right and left. They decided an h-value would do that, but it moved the graph in the opposite direction from what they expected given the sign h appeared to have in the equation. I took these observations as a sign that the students were interconnecting the h- and k-values in this type of function with those in earlier types of functions.

We also generalized formulas for reflecting graphs across the x-axis and finding the functions' domains, ranges, and asymptotes. By having students construct rules about how to manipulate a basic exponential graph, sometimes called "parent" graphs, I gave them the opportunity to discover this mathematical knowledge themselves. They learned by exploration and recognition of the patterns they observed. I felt I had given them a pretty good taste of discovered mathematical knowledge. I felt that I was addressing NOMK more directly than I had before, but I really hadn't gotten much student reaction that indicated the NOMK concepts were taking hold for them.

Journaling Continues: Applications of Exponential Functions

On Wednesday I began my second week of journaling, a day when I was scheduled to teach a lesson that was already three semesters in the making—though it seemed to fit nicely in these weeks of NOMK awareness.

For years I had assumed the application problems for exponential functions were the most difficult ones, thus requiring more direct and procedural instruction. So I would painstakingly review each equation with the students in advance of their use, explaining each of the variables and doing examples of each type of problem on the board. I regularly found students completely exasperated with the homework and often unable to complete it.

Student comments I used to hear about this topic included:

I have no idea what I am doing here.

How did you know to put that value there (in place of that variable)?

Who makes this stuff up anyway?

They must have a lot of time on their hands.

I am never going to be able to do this later.

I got 3.6 (or some value). How did you get (some other value)?

This is what the calculator said, so it must be right.

I just don't see it.

Compound interest formula	Continuous compound interest formula
$A = P(1 + \frac{r}{n})^{nt}$	$A = Pe^{rt}$
A = amount after t years	A = amount after t years
P = principal (original amount)	P = principal (original amount)
r = yearly interest rate (as a decimal)	r = yearly interest rate (as a decimal)
t = number of years	t = number of years
n = number of times compounded per year	$e = 2.7182818284590...$

Malthusian Model of Population Growth	Logistic Epidemiology Model
$P = P_0 e^{kt}$	$P = \frac{M}{1+(\frac{M}{P_0}-1)e^{-kt}}$
P = current population	P = the infected population at time t
P_0 = initial population	P_0 = initial infected population
t = time in years	t = time in years
k = birth rate − death rate	k = constant of contagiousness
	M = theoretical maximum population

Table 2. Equation examples

How can we be expected to do these ourselves when it takes so long to do it in class?

A little over a year ago I decided to try a new approach. I gathered all the equations intended for students' use on a sheet of paper with a brief explanation of the variables involved. (See Table 2 for a sample of equations.) Then I mixed up application problems onto another sheet of paper (see below) and told the students it was their job to decide which equation applied to which problem, derive from the problem which value was to be substituted for each variable, and solve the problems. I arranged the students into groups of two or three and set them loose. I figured the class couldn't possibly go worse than it had in years past.

Application problem examples

For each problem that follows, determine which equation you need. Then solve for the missing variable.

1. In a city with a population of 1,200,000, there are currently 1,000 cases of infection with HIV. If the constant of contagiousness for HIV is 0.4, how many people will be infected in three years?

2. The parents of a newborn child invest \$8,000 in a plan that earns 9% interest, compounded quarterly. If the money is left untouched, how much will the child have in the account in 55 years?

3. The population of the United States is approximately 300 million people. Assuming that the annual birth rate is 19 per 1,000 and the annual death rate is 7 per 1,000, what will the population be in 50 years?

I was not prepared for such a drastic change. The students conferred with one another, but mostly just to clarify what they were seeing already. They got the material done in a fraction of the time. They even seemed to think the class was easy! I was understandably blown away by the difference. So partly to fill the unexpected extra time (but mostly to see if they really understood) I had one volunteer per problem go up to the board and write out their solution. Then we took turns having them explain their work to the rest of the class.

Student comments/questions were now of this type:

Yup, I did that same thing.

I picked that same equation!

(Teacher: How did you know to use that equation?) Because the formula was called (thus-and-such) and the application problem used that same word. Plus, the descriptions of the variables sounded like the same thing as the description of the problem.

I just plugged it all in and it worked!

Is that it? What else are we learning today?

That was way too easy. We must be doing something wrong.

But they hadn't done anything wrong! And the knowledge stuck. They were still able to identify and use appropriate variables and formulas for application problems on later tests, later reviews, and on the final exam. This was the case even when students were not prompted to use a formula, where variables were not specified, or when novel formulas were suggested of which they had no previous knowledge. There had been nothing like that kind of retention in years past.

So what had changed? In the past I had tried to replace the conceptual nature of these problems with increasingly complicated procedures. It seemed to exhaust the students' minds as much as it taxed my patience. Students and teacher alike questioned their ability to continue to function competently in this environment. But once I changed and embraced the conceptual nature of the problems and trusted the procedures to take care of themselves, the students successfully figured out which to apply in each case! From this and other evidence I am convinced that teaching students to trust their common sense and their ability to reason, using their verbal abilities and conceptual powers, is crucial to building mathematical knowledge. And the students both can and will do it when they believe they are expected to do so.

Once again this semester I mixed all my application problems together and had the students determine which equation to use where. Again it went very well with the dual-credit classes. I found many students helping other students. For the most part they only called me to break a tie if two or more students had differing opinions that they couldn't resolve. But in many instances they did a great job of trouble-shooting and defending one solution over another based on order of operations or other conceptually and contextually based arguments.

In the three-hour class, I needed to teach both the exponential equation lesson and the application problem lesson. This class struggled a bit more with the lessons than the other two, as usual. For example, they needed a bit more coaching to see how to manipulate the h-value to move a function left and right. They did eventually see the connection between this work and other equations that can use (h, k) to translate a parent graph in different ways. They also struggled a little more to generalize about domains, ranges, and asymptotes. I had to ask quite a few more scaffolding-type questions to get them there, but in the end they seemed to get it.

For the application problems we used the same strategy I had used with the other classes. And again, it took a little longer, but they seemed to eventually understand. Interestingly, they still had no trouble determining which equation to use for each problem. They also had no trouble deciding what each variable was meant to represent, nor with substituting the values into the equation. I could see all the equations and their appropriate substitutions written on their papers. The problems in the evening class almost exclusively involved using the order of operations appropriately as they entered them into the calculator. We talked quite a bit about doing operations in the right order and the appropriate use of parentheses. It didn't take much coaxing to get them back on the right track. Though the third class struggled more with the lesson, past experience tells me they still got a lot further than they would have if we had worked the problems through, as a group, micro-procedure by micro-procedure.

4 Discussing NOMK with the Classes

The final topic in this section was logarithms. I first presented students with an exponential equation and asked them to find an inverse equation for it. As they had learned to do, they dutifully replaced all the xs in the equation with ys, and all the ys in the equation with xs. Then they worked on trying to isolate y. After several false starts some suggested it couldn't be done. At that moment I presented the logarithm function as a solution to the problem. Once we had practiced changing a few equations from exponential form to logarithmic form and vice versa, I suggested that logarithms had been invented to deal with the desire to produce inverse functions for exponential functions. As such, I formally addressed the concept of mathematical knowledge being invented for perhaps the first time. I followed that quickly with applications of logarithms in several situations. For example, I told them that logarithms were used in pH scales, in the Richter scale for measuring earthquakes, and in the computer coding of digital music production. I saw a few students glancing at the poster I had put up while I was sharing this. So I used that opportunity to briefly discuss the rest of the poster. I premised it with the hypothesis that teachers may teach differently depending on what they think mathematical knowledge is.

Some of the students made observations about the source of mathematical knowledge. For example, one student expressed that mathematics was the way it was in order to differentiate between quantities of items. He showed how one hat was different than two hats. He suggested that if there was a different formula for finding the area of a triangle than the one we know, that it wouldn't actually be a triangle, but something else. Though his argument was a little labored, it seemed that clear that he felt NOMK was the way it was because nature is the way it is, and it couldn't be different. I suggested to the class that his was an example of discovered or *a priori* NOMK, and they seemed to agree.

Another student suggested that she really resonated with the invented concept. She said she often thinks of mathematics as a game. She gave an example of trying to help a friend with some mathematics homework. The friend had asked why a particular procedure was the way it was. My student had responded that it was simply because those were the rules and that was just the way this game was played. The students seemed to agree that she was demonstrating the invented concept.

Then we talked briefly about the justification of mathematical knowledge. Proofs in geometry were suggested as examples of logic and proof. I mentioned that some people think mathematical knowledge is just a social construct. Humans made it up to be this way and it could have been different if we wanted it to be. No one commented about that, but I got several rather enthusiastic head nods from three or four people. We also talked briefly about substitution and context as ways to verify mathematical knowledge. We had talked earlier about an application problem in which a person ate one-third of one-half of a leftover pizza. If the whole pizza had 2000 calories, how many calories had he consumed? We talked about numbers needing to make sense in their context, and about how you couldn't possibly get more calories from a fraction of a pizza than you could from the whole thing. I could tell the students were understanding me when they suggested that, if this were not the case (that is, if the less you eat, the more calories you receive), then we would have discovered the solution to world hunger and it likely would cause universal obesity.

Finally we talked briefly about procedures versus concepts. I asked, what happens when you get out a piece of paper to do a mathematics problem? Is the mathematics the numbers and symbols you write on the page, or is it the thought processes in your head that tell you what to write? Or both?

The second class went similarly, and we even had some time for the students to do their homework. One student came up to me and said, "I don't understand the concepts that support this procedure." I loved it! It suggested that the student had internalized the distinction. She showed me a problem:

$$4^x = \frac{1}{64}.$$

She asked, "How can I know what x is just from that?"

I said, "well, the set-up implies that some exponent on a 4 could give you 1/64."

She said, "So, it [referring to x] has to be negative, to make the reciprocal."

"Yes. But negative what?" I responded.

She said, "2 would make 16 [interpretation: 4 with an exponent of 2 would make 16]. What [exponent on a 4] would make 64?"

"I suppose you can test that," I said.

"Yeah," she responded.

Later she showed me another problem:

$$x^{\log_4 6} = 6.$$

"I need another concept again," she said.

It was getting good! "What if we tried a simpler problem with the same format? This is in exponential form. Could you change it into logarithmic form?"

"Yeah," she said. [She proceeded to do the work: $\log_x 6 = c$.] "Oh, then you could just substitute back for c, right? [she wrote $\log_x 6 = \log_4 6$], so $x = 4$."

"Exactly!" I said. "Yes. I mean, you can see how the two sides [of the equation] are really very 'matchy-matchy,' right?" The student giggled. I continued, "No, that is not the technical term for it. But you maybe can see that..."

"... x must be 4," the student finished the thought.

Many students seemed to have difficulty with this problem. So we ended up doing the same problem on the board with pretty much the same dialogue and more student participants. I did not try to generalize with the students about how to apply our procedure to other problems, but I overheard quite a number of students saying to each other something like "just substitute a variable in place of the weird looking part, then substitute it back in later."

My point is, I did not give the procedure directly, because of my conceptual ideas about the substance of mathematical knowledge. Interestingly, the students did not ask for the procedure. They asked for the concepts. From them, they successfully inferred the procedures.

A little later in class another student was trying to solve a problem and he didn't like the strategies we were using. He said, "I would just make something up. If it were up to me, I would just make stuff up."

I told him that according to the invented aspect of NOMK, he maybe could make some stuff up, but he was not allowed to make up contradictory stuff. So he could make up a few things, but then he would have to make the rest be consistent with it. He seemed to find that interesting.

5 Effects on Student Learning

As part of the lessons on logarithms I teach many properties of logarithms. The department provides a formula sheet for the students to have with them while studying and for the final exam. The sheet contains all the logarithmic properties. However, since introducing logarithms in my NOMK-emphasis weeks, I have noticed that most of my students had no need to refer to the course-approved formula sheet for the logarithmic properties, even though I made it clear that they didn't need to memorize them. This may be because they memorized them anyway or because they understand them conceptually well enough that memorization is not necessary.

In the week that followed my journaling activities I gave each class another test. Their in-class understanding of the application problems carried over to the test nicely. About one-third of the second dual-credit class and one-third of the evening class had difficulty placing the asymptotes in graphs of exponential and logarithmic equations. A good smattering in all three classes struggled with applying logarithmic properties in problems that required more than one property. Even considering this, their test results were generally higher than or equal to those of previous semesters. So even though some of my students continue to struggle with College Algebra, I feel that their exposure to issues of NOMK was beneficial to many of them.

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IV

Calculus and Probability and Statistics

Chapter 9: Margo Kondratieva's chapter, "Capturing Infinity: Formal Techniques, Personal Convictions, and Rigorous Justifications," suggests getting calculus students to start thinking about the subtleties of infinity and the need for rigor. Why, and when, can we apply the same rules to infinitely long expressions that we do to standard expressions? Students are skeptical that $0.1111\ldots = 1/9$. She introduces historical philosophical issues concerning completed infinities and limits to help students understand that there are indeed difficult questions hidden here, and that this is why the mathematical community needed to develop its current rigorous approach to them.

There are many philosophical issues related to calculus, and so it may be surprising that we only have one chapter on calculus in the volume. Perhaps it is because our calculus courses are so full of new concepts, examples, theorems, computations, and applications that we barely have time to breathe. See the Introduction for other chapters that discuss issues in calculus.

Chapter 10: Dan Sloughter, in "Making Philosophical Choices in Statistics," describes the choices all faculty who teach statistics must make, not only between frequentist and Bayesian (subjective) approaches to inductive inference, but between differing frequentist interpretations (that of Fisher versus that of Neyman and Pearson) as well. The approaches have differing interpretations of the meaning of probabilistic statements and disagree about the nature and aim of inductive inference. The role of the instructor is, in part, to clarify the distinctions: what each is measuring, and what kinds of conclusions can be drawn from the results of their analyses.

In many mathematics departments, there is an insufficient number of statisticians to teach all the statistics offerings, and as a result, many non-specialists get involved in teaching them, especially the service courses. This chapter can be helpful background especially for these faculty members. One of the reviewers called this chapter "one of the clearest expositions I have read on this topic."

9

Capturing Infinity: Formal Techniques, Personal Convictions, and Rigorous Justifications

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1 Introduction

Infinity is a provocative concept that has generated a long-lasting debate in both mathematics and philosophy. As Knorr pointed out discussing antiquity, "the interaction of philosophy and mathematics is seldom revealed so clearly as in the study of the infinite." [14, p. 112]. However this is not the case in the contemporary teaching of mathematics: even though students become familiar with examples referring to infinity as early as in grade school, the concept of infinity rarely gets any philosophical treatment. In the teaching of mathematics, infinity is sometimes intuitively introduced as "something beyond any bound." At the same time, some mathematical expressions involving infinity are said to have associated finite values. This approach often leads to confusion among students. Formal methods could be introduced to deal with the expressions in order to associate them with corresponding finite entities. For example, given an infinite periodic decimal fraction one could use appropriate geometric series in order to rewrite the number in a finite form (such as $0.6666\ldots = 2/3$).

The concept of infinity frequently, but not exclusively, appears in the calculus curriculum. Traditionally, calculus has been taught mainly by demonstrating various methods to handle certain types of problems. Even after the calculus reform movement introduced in the 1980s, which suggested more emphasis on conceptual understanding, truly theoretical framework is often left for the third-year analysis courses, where calculus methods become rigorously justified. However, analysis courses are mostly taken by mathematics majors, and the rest of the students are never exposed to the theory and philosophy underlying calculus methods. This curriculum structure is justified by the fact that rigorous theoretical exposition would be difficult for freshmen, especially if it has to be done in large classes that also include students with either poor algebraic skills or little enthusiasm for mathematics, or both.

Nevertheless, having a philosophical-theoretical framework is often critical for students to be able to understand the idea of formal mathematical methods and to interpret obtained results. I argue that a compromise between rigor and accessibility of derivations involving the concept of infinity can be found in some historical approaches.

This approach can be employed for teaching mathematics courses at the first- or second-year university level. It may also be beneficial for future teachers of mathematics, and in fact for all students who want to develop and apply mathematical thinking in their future careers.

Section 2 of this chapter gives examples of infinite objects associated with finite values and discusses some learning issues. Section 3 presents historical and philosophical background related to the concept of infinity. Section 4 argues that historical examples may help teachers to find a balance between accuracy and practicality in mathematical exposition while helping students to deal with their confusions associated with this challenging concept. Section 5 concludes with a discussion of students' reaction to this approach.

2 Mathematical Examples Involving Infinity and Some Learning Issues

Let us first consider several mathematical examples that refer to infinity. The first three demonstrate that a finite numerical value can be represented as a result of an infinite procedure. Here I present typical textbook questions and corresponding formal solutions.

Example 1. Represent $\frac{1}{3}$ as a decimal fraction.

Solution: By applying the division algorithm repeatedly we obtain an infinite periodic decimal $0.3333\ldots$, and we write $\frac{1}{3} = 0.3333\ldots$.

Example 2. Represent $\sqrt{2}$ as a continued fraction.

Solution: Because $1 < \sqrt{2} < 2$ we can write $\sqrt{2} = 1 + \frac{1}{x}$, where $x > 1$. Then, after algebraic simplifications we have $x = \sqrt{2} + 1$. But using the expression $\sqrt{2} = 1 + \frac{1}{x}$, we rewrite $x = 2 + \frac{1}{x}$. Therefore, $\sqrt{2} = 1 + \frac{1}{2+\frac{1}{x}}$. By substituting $2 + \frac{1}{x}$ in place of x each time, this becomes an infinite process and we obtain an infinite continued fraction, which in compact notation can be written as $[1; 2, 2, 2, 2\ldots]$. We conclude that $\sqrt{2} = [1; 2, 2, 2, 2\ldots]$.

Example 3. Represent 5 in terms of an infinite nested root.

Solution: Observe that $5 = \sqrt{25} = \sqrt{5 \cdot 5} = \sqrt{5 \cdot \sqrt{25}} = \sqrt{5\sqrt{5\sqrt{25}}}$. Substituting each time $5\sqrt{25}$ in place of 25 we come to an infinite process and obtain an infinite nested root $\sqrt{5\sqrt{5\sqrt{5\ldots}}}$. We can write $5 = \sqrt{5\sqrt{5\sqrt{5\ldots}}}$.

In all three examples we have equality between a rational or irrational number and an infinite expression resulting from an infinite process. The left side, the number, is an object familiar to students; for instance, it can be defined geometrically in terms of length. In contrast, the expressions on the right side may look counterintuitive, even though the processes to obtain them are formally described and derived using familiar mathematical rules. The only unusual part is that the rules should be applied infinitely many times. Obviously, "there is a tension between the two sides which is the source of power and paradox. There is an overwhelming mathematical desire to bridge the gap between the finite and infinite." [6, p. 153].

Let us now consider related inverse problems: converting an infinite expression into a finite value, given that it exists.

Example 4. Represent the infinite decimal $0.111\ldots$ as a ratio of integers.

Solution: Note that

$$0.111\ldots = \frac{1}{10} + \frac{1}{100} + \cdots = \sum_{n=0}^{\infty} \frac{1}{10^n} - 1 = \frac{1}{1 - 0.1} - 1 = \frac{1}{9}.$$

Here we use the formula $\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$ with $q = 0.1$.

Example 5. Convert the infinite continued fraction $[1; 1, 1, 1, 1\ldots]$ into a real number.

Solution: Note that $[1; 1, 1, 1, 1, \ldots] = 1 + \frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\ldots}}}}$.

Let $1 + \frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\ldots}}}} = x$. Then we can write the equation: $x = 1 + \frac{1}{x}$, or equivalently $x^2 - x - 1 = 0$. Solving for x and observing that $x > 0$, we obtain $x = \frac{1+\sqrt{5}}{2}$. Thus we write $[1; 1, 1, 1, 1, \ldots] = \frac{1+\sqrt{5}}{2}$.

Example 6. Convert the infinite nested root $\sqrt{2\sqrt{3\sqrt{2\sqrt{3\ldots}}}}$ into a real number.

Solution: Let $\sqrt{2\sqrt{3\sqrt{2\sqrt{3\ldots}}}} = x$. Then we can write the equation $\sqrt{2\sqrt{3x}} = x$. Solving for x we obtain $12x = x^4$ and since $x > 0$, we have $x = 12^{1/3}$. Thus we have $\sqrt{2\sqrt{3\sqrt{2\sqrt{3\ldots}}}} = 12^{1/3}$.

Examples 4, 5, and 6 demonstrate mathematical techniques that allow finding a real number associated with an infinite expression. However, many students remain unconvinced about the identity of an infinite object and corresponding numerical value even if such an identity is formally established. This phenomenon has been documented in mathematics education literature for a number of years (see, e.g., [20], [8], [24], [5]), with the majority of examples referring to infinite decimals. Tall [23, p. 221–222] comments that students often "conceive of 0.999... as a sequence of numbers getting closer and closer to 1 and not a fixed value," because "you haven't specified how many places there are," or "it is the nearest possible decimal below 1." Also, it was found that "the majority regarded $0.1 + 0.01 + 0.001 + \cdots = 1/9$ to be false but $1/9 = 0.1 + 0.01 + 0.001 + \ldots$ to be true" because the students were "seeing the expression $0.1 + 0.01 + 0.001 + \ldots$ as a process, not a value." As for the second equation, students sometimes regard the left side (1/9) as a generator of the corresponding infinite expression on the right. A recent study by Ngansop and Durand-Guerrier [18] provides evidence of how some students struggle to accept that a finite number 1 is equal to a non-terminating decimal 0.999.... Students know that "it is said to be true, but remain doubtful concerning the truth of the equality."

The latter article also expresses a concern about the logical validity of students' argumentation "relying on the extension of operations to non-terminating decimals, without having a proof that this can be done." An example of such argumentation is to derive $1 = 0.999\ldots$ from $1/3 = 0.333\ldots$ by multiplying both sides by 3. Another popular derivation [16, p. 119] is based on the proposition that if $x = 0.999\ldots$ then $10x = 9.999\ldots$. By subtraction we get $9x = 9$; so $x = 1$. In [16] this methodology is presented to the reader without any discussion of why all operations are legitimate.

Thus, there are two learning issues that it is necessary to address: (1) whether a student is convinced by formal derivations and (2) whether a derivation itself is fully justified.

In this regard, I propose that many students would benefit from being exposed to related episodes and facts from the history and philosophy of mathematics. I consider some of them briefly in the next section.

3 The Concept of Infinity and Its Historical-Philosophical Treatment

If students are given a historical-philosophical overview regarding the development of the concept of infinity they will realize that infinity was never easy to understand even for greatest minds of past eras, and that difficulties and doubts that students may experience while learning the concept are not unique or specific to them.

In their development of mathematics and science the early Greeks had come to the fundamental question: is matter continuously divisible? In a thought experiment, one can continue to divide a segment into smaller and smaller pieces infinitely many times. However if dividing a material object in this way, will one reach an indivisible piece, an atom? Zeno surprised the ancient world by showing that beliefs of both atomists and their opponents could lead to apparent paradoxes, which demonstrated that the concept of infinity was not properly understood. The paradoxes stimulated ancient thinkers such as Eudoxus, Archimedes, and Euclid to invent more precise concepts and methods of dealing with infinite processes. It should be noted that in his calculation of the area of circular and parabolic regions, Archimedes referred to an infinite triangulation, and nevertheless was able to deal exclusively with finite formulas, by conjecturing the limiting value and proving his conjecture by showing that other values would lead to a contradiction.

Aristotle proposed distinguishing between actual and potential infinities. On the one hand, no bound exists for integer numbers because for every integer there is a preceding and following number. Thus for any finite collection of integers one can always find a larger finite collection. This is an example of a potential infinity. On the other hand, the entire collection of integers is an example of an actual infinity. Even with the introduction of a positional number system by the Babylonians, which allowed a concise representation of very large numbers, the accessibility of actually infinite objects remained questionable. Aristotle argued that mathematical methods deal with potential rather that actual infinity. "Our account does not rob the mathematicians of their science, by disproving the actual existence of the infinite. . . . In point of fact they do not need the infinite and do not use it." [2, Chapters 4–8 of Book III]. Indeed, as Halmos pointed out more than two millennia later, "the genius is—given an infinite question to think of the right finite questions to ask." [1, p. 9].

The concept of actual infinity provoked scientists and philosophers for centuries after Aristotle. In his work *Discorsi e dimostrazioni matematiche intorno a due nuove scienze* [9], Galileo uses the one-to-one correspondence principle to demonstrate that two concentric circumferences of circles have different lengths but contain the same (infinite) number of points. He explains this counterintuitive statement by the fact that "we attempt, with our finite minds, to discuss the infinite, assigning to it properties which we give to the finite and limited." Kant [11] as well appeals to the fact that perception of actual infinity by humans is problematic: ". . . in order to conceive the world, which fills all space, as a whole, the successive synthesis of the parts of an infinite world would have to be looked upon as completed; that is, an infinite time would have to be looked upon as elapsed, during the enumeration of all coexisting things."

Gauss echoes Aristotle by writing in 1831, "I protest first of all against the use of an infinite quantity as a completed one, which is never permissible in mathematics. The infinite is only a *façon de parler*, where one is really speaking of limits. . . ." [25, p. 432].

One can argue that the concept of actual infinity, despite being highly counterintuitive, had been applied in early ages. For example, the method of mathematical induction, which requires one to grasp infinitely many implications, has been used since 1000 A.D. [12, p. 173]. Pascal, when employing this method in his *Treatise on the Arithmetical Triangle*, summarized it as follows: "Although this proposition has an infinity of cases, I shall demonstrate it very briefly by supposing two lemmas. The first, which is self-evident, is that the proposition is valid for the first row. The second is that if the proposition is found in any row then it must necessarily be found in the following row. Whence it is apparent that it necessarily is in all the rows." [21]. In a similar fashion, any statement that includes phrases such as "for all integers" or "for all odd (even) numbers" etc., suggests that a property is valid in some uniform way simultaneously in infinitely many cases, and this implies that one deals with actual infinity. In order to deal with every case at once one "abstracts the situation by the introduction of a variable. The variable is the means of reducing the infinite to the finite. However, . . . the notion of variable is ambiguous." [3, p. 123].

Hegel supported the Aristotelian view on the infinite and pointed out on the "error of holding mental fictions" when using "such abstractions, as an infinite number of parts, to be something true and actual." [17, p. 199, §427]. Hegel explained, with reference to Spinoza, that "the series $0.285714\ldots$ is the infinite merely of imagination or supposition; for it has no actuality, it definitely lacks something; on the other hand $2/7$ is actually not only what the series is in its developed terms, but is, in addition, what the series lacks." [ibid, §566]. Further, Hegel praised the Newtonian method in calculus and considered an infinite series from his dialectical point of view. His logic was as follows. Adding any finite number of terms is a progress towards the infinite sum but it is never the infinite sum itself. In this sense, Hegel stated, infinity is negated by the finite. At the same time, finite is negated by infinity because the counting will never terminate. Since the infinite negates the finite, which itself negates the infinite, he concludes, "the infinite is the negation of the negation." [17, p. 137, §273]. An infinitely travelled circle is a bounded object encapsulating this infinite loop of mutual negation. By recognizing that "the image of true infinity . . . [is] the circle, the line which has reached itself, which is closed and wholly present, without beginning and end," [ibid, §302] Hegel makes a short step toward approval of actual infinity. He proposed that while "finite and infinite are qualitatively opposite" they still could be united in a quantum. In his example, a qualitative relation, the ratio $2/7$, emerges from the potentially unrestricted quantitative process of summation of the series $0.285714\ldots$.

In the nineteenth century Bolzano proposed looking at infinite collections of objects as single entities. He introduced the notion of a set, which allowed speaking of an infinite collection as of an object by itself. At first glance there seems to be little improvement. However it led to a radically different point of view expressed by Cantor: a "potentially infinite concept always points towards a logically prior actually infinite concept whose existence it depends on." [4, p. 404]. This philosophical position significantly influenced the development of modern mathematics based on the set-theoretical approach of Bolzano, Cantor, and their followers. For example, H. Weyl maintained "mathematics is the science of infinite, its goal the symbolic comprehension of the infinite with human, that is, finite means" (as cited in [3, p. 113]). Similarly, according to Peirce, "the prevalent opinion is that the finite numbers are the only ones that we can reason about . . . mathematically" [19, p. 538]: however for those "who do reason, reasoning about infinite numbers is easier than about finite numbers." [ibid, p. 539].

Nevertheless, the great divide in the philosophy of mathematics continued in the twentieth century, in particular in a debate between intuitionists led by Brouwer and formalists led by Hilbert. Intuitionists claimed that in order for a mathematical object to exist it should be possible to construct it in a finite number of steps. In this sense, a number resulting from an infinite process cannot be treated as an object. However, as long as the object can be constructed in another way by a finite construction, its infinite representation doesn't prevent its existence. Thus, in cases when the infinite expansion of an object reflects a deficiency of a particular representation, other means had to be found to handle it. In contrast, the formalists' standpoint was that "symbols of mathematics have an existence independent of whether they refer to actual objects," and one can talk of an object independent of whether or not it exists, "provided that the deductions based on it are logically sound." [22, p. 744]. In particular, for formalists deductions involving infinite-digit numbers were acceptable.

Infinite representations of finite objects, as demonstrated in Section 2, are ambiguous. However, "there can be no modern mathematics ... without a resolution of this paradox." [3, p. 135]. This resolution "was not mathematical so much as conceptual," [ibid., p.137] which led to a contemporary understanding of the real number system as a whole. In particular, it led to the mathematical understanding of continuity, and permitted reasoning such as follows: "because the series of real numbers between 0 and 1 is continuous, there must be a least real number, greater than any number of this endless series $[0.1, 0.11, 0.111, \dots]$," [19, p. 544] which is $0.111\dots$, or $1/9$. Thus, Byers concluded, ambiguity often calls for creative interpretations and "imbues mathematics with depth and power." [3, p. 11].

4 Some Implications for Teaching Mathematics

4.1 Perception of Infinite Objects

We learn several things from the history of philosophical development of the concept of infinity. First, there are objective psychological difficulties with understanding infinite objects. As Woods and Grant explain, "the idea of the infinite seems difficult to grasp, because, at first sight, it is beyond all human experience. ... Mathematics deals with definite magnitudes. Infinity by its very nature cannot be counted or measured. This means there is a real conflict between the two." [26, p. 353]. Imagining or perceiving the limit is difficult because the limiting object may not possess all properties of each individual term of the infinite sequence. For example, $1/n$ is positive for all $n > 0$ but the limit as n goes to infinity is zero; so it does not have this property of the individual terms. This is only one among many examples that may lead to apparent logical paradoxes, confusions, and perceptual doubts. Therefore, mathematical details, definitions, and methods are of particular importance when one deals with infinity and limits. However, such formal definitions and methods would have more meaning for students if they were supported and justified by prior philosophical considerations such as those presented in the previous section. In particular it would be beneficial to our students to understand the distinction between potential and actual infinities as it was discussed by Aristotle and many of his successors. This can be done by including relevant historical facts and philosophical ideas in lessons, along with mathematical examples that deal with infinite objects.

Second, treating objects as potentially infinite allows us to work with the finite entities of which we are taking the limit and thus to apply finite reasoning. Many calculus methods adopt precisely this point of view. The methods also include the step where passing to the limit actually occurs, and this requires understanding the theory of limits, for example its epsilon-delta version. However, as I will demonstrate, the last step can be replaced with another approach.

Third, familiarity with and acceptance of the concept of actual infinity allows one to treat an infinite object as a symbol, which is a subject of deductions. Lakoff and Núñez speak about the *metaphor of infinity* in which "processes that go on indefinitely are conceptualized as having an end and an ultimate result." [15, p. 258]. This is important for setting a theoretical framework for working with objects of an infinite nature and may affect learners' perception of these objects.

There is an epistemological distinction in perceiving a non-terminating expression (e.g., a decimal) as an object versus as a process. The former perception aligns with the metaphor of infinity as described by Lakoff and Núñez. The latter results from the potential infinity standpoint of viewing such an expression as a process not having an end, which essentially prevents the thinker from accepting the fact that the result of the process is a finite number. This observation is true not only in the case of freshmen, who might be new to mathematical culture. A recent example can be seen in the statement made by a faculty member in a letter to the editors of *Educational Studies in Mathematics* [10].

The author claims that "a zero limit cannot be considered as absolute zero" and therefore it would be mathematically correct to say "0.333 . . . tends to $1/3$" rather than "0.333 . . . is equal to $1/3$."

Evidently, the distinction between "tends to" and "equals" lies in perception of the non-terminating decimal expression relying on different philosophical viewpoints. As Dubinsky et al [7] pointed out, "An individual who is limited to a process conception of .999 . . . may see correctly that 1 is not directly produced by the process, but without having encapsulated the process, a conception of the 'value' of the infinite decimal is meaningless. However, if an individual can see the process as a totality, and then perform an action of evaluation on the sequence .9, .99, .999, . . . then it is possible to grasp the fact that the encapsulation of the process is the transcendent object. It is equal to 1 because, once .999 . . . is considered as an object, it is a matter of comparing two static objects, 1 and the object that comes from the encapsulation. It is then reasonable to think of the latter as a number so one can note that the two fixed numbers differ in absolute value by an amount less than any positive number, so this difference can only be zero." [7, pp. 261–262].

Therefore by directing students to see process as a totality, that is, to accept the actual infinity rather than the potential infinity standpoint, may help them to encapsulate the process and to see why indeed 0.999 . . . and 1 are equal. The next section provides mathematical details of a proof based on a philosophical standpoint that 0.999 . . . is an object.

4.2 Rigor and Accessibility

When discussing mathematical derivations, teachers should strive to clarify to what extent a given step is warranted. Indeed, a rigorous theory within which certain operations would be legitimate may not be available for students due to their background. For example, extension of arithmetic operations to non-terminating decimals can be justified via Dedekind's theory of real numbers with cuts. An alternative scenario involves the concept of limit. However, before this extension is established in one way or another, the multiplication by 3 of both sides of equation $1/3 = 0.333\ldots$ is not a valid operation in a strict sense. Nevertheless, at the intuitive level it may support the equality $1 = 0.999\ldots$. In the absence of a rigorous theoretical framework, the argument does not prove the statement. Instead it shows that if multiplication could be justified then the two equations are in agreement. Thus, it demonstrates a consistency between the two statements.

A check of consistency is what actually occurs in all the mathematical examples presented in this chapter, if they are given to students before the concept of the limit of a sequence is rigorously introduced. In Example 1, the process of dividing 1 by 3 proceeds like the ordinary division algorithm, except that we emphasize that it generates a non-terminating decimal fraction. In Examples 2 and 3, an infinite object (continued fraction or nested root) is consistent with the recursive process of substitution of a finite recurrence relation $x = 2 + \frac{1}{x}$ or $25 = 5\sqrt{25}$ respectively. In Example 4, the solution, $1/9$, can be checked by the process of dividing 1 by 9. Similarly, the sum of an infinite geometric series is consistent with formula for finite geometric series, $\sum_{n=0}^{N-1} q^n = \frac{1-q^N}{1-q}$, when combined with our intuition that q^N is negligibly small for a large power N if $|q| < 1$. In Examples 5 and 6 we assign x to be equal to an infinite expression and obtain an algebraic equation for x. The logic of this action is that we do not know if the infinite expression is equal to a number ahead of time, but if it is, the number should satisfy an algebraic relation consistent with the structure of the infinite expression.

A check for consistency is accompanied by a search for possible inconsistency. Any occurrence of an inconsistency or paradox is an indication of inadequacy of the method applied even if it seems intuitively correct. For example, suppose a student argues that the infinite alternating series $\sum_{k=0}^{\infty}(-1)^k$ sums to zero because the sum of the first and the second term is zero, the sum of the third and the fourth term is zero, and so on. However, by regrouping the terms one can show that the same sum is equal to 1. If we agree that a sum can have only one numerical value, this result signals a flaw in the student's approach.

If a rigorous theory is not available to students, a search for consistency that supports their intuition seems to be a natural way of advancing mathematics both at the personal level and historically. For instance, "all of Euler's work on infinite series was justified by the fact that the series developments produced a 'right answer'; however they could not be logically justified until a better formulation of the notions of convergence was created." [22, p. 603]. Nevertheless, as they develop mathematically, students need to gradually become aware of the limitations of methods they apply.

Yet a rigorous treatment of our examples that appeals neither to a formal notion of limit, nor to the arithmetic of infinite decimals, is possible. Let us investigate the expression $0.999\ldots = 1$ in the spirit of Archimedes' proof of his formula for the quadrature of a parabola. (For a modern reconstruction of Archimedes' derivation of the area bounded by a parabola see Katz [12, p. 72].)

Denote by $T(n)$ the finite number with exactly n 9s, that is, $T(1) = 0.9$, $T(2) = 0.99$, etc. For any whole number $n > 0$ we have $T(n) + 10^{-n} = 1$. For example, $T(1) + 1/10 = 1$, $T(2) + 1/100 = 1$, etc. Regarding $0.999\ldots$ as an object rather than a process, denote it by T. In these notations we need to prove that $T = 1$. One would agree that $T(n) < T$ for any $n > 0$, but we do not know anything else about T. In other words, we only suppose that T exists as a subject of deductions and T is a real number.

This proof is by contradiction (double *reductio ad absurdum*). We first assume that $T > 1$, and thus $T - 1 = a > 0$. We can always choose a number m such that $T - T(m) < a$. But this would imply $T - T(m) < T - 1$, and thus $T(m) > 1$. However, the latter inequality contradicts $T(m) + 10^{-m} = 1$, which gives $T(m) < 1$. Since the inequality $T > 1$ leads to a contradiction, it is not possible.

Second, assume that $T < 1$, and thus $1 - T = b > 0$. We can always choose a number r such that $9 \cdot 10^{-r} < b$. On the other hand, $1 - T(r) = 10^{-r} < 9 \cdot 10^{-r} < b = 1 - T$. Thus, $T(r) > T$, which is a contradiction.

Since both assumptions $T > 1$ and $T < 1$ lead to contradictions, we conclude that $T = 1$.

This formal Archimedes-like exposition may still be difficult for some students to grasp, even if each step is relatively simple and logical. Often, it is better to start with outlining the idea of the method. For example, the second step of the proof can be preceded by saying "if you believe that $0.999\ldots$ is less than 1, you must tell how much the deficit is, and when you do, we can show you a large enough chunk of (finitely) many 9s in this decimal which are closer to 1 than the deficit." [16, p. 62].

5 Conclusions

As Tall pointed out, "It is my contention that mathematical thinking uses one of the most powerful and natural constructions of the human mind—the ability to use symbols to switch between concepts and processes." [23, p. 213]. As was discussed in the previous section, the ability to switch between objects and processes is especially valuable when dealing with infinite expressions, and the awareness of various philosophical viewpoints on infinity supports this ability. Further, knowing that other people have similar difficulties perceiving infinite expressions and understanding other people's thoughts about infinity may help students to deal with their own doubts and confusions.

Possible paradoxes that occur from formal manipulations of infinite expressions show students the necessity of having rigorous and consistent theories [13]. Proofs by contradiction (in the spirit of Archimedes, as shown above) demonstrate that one can reason about infinite objects exclusively with finite means. Realization that advanced mathematical methods associated with infinite objects are justified within broader theoretical approaches to the concept of limit makes students appreciate the theories.

Although I use historical-philosophical discussions in connection with the examples given in Section 2 in my undergraduate mathematics courses, I have not yet had a chance to conduct systematic research or collect statistics regarding its effectiveness. However, anecdotal observations and occasional surveys reveal that many students are interested to hear stories about great mathematicians and philosophers, especially if they are focused and given in relation to a mathematical topic they are currently studying. For example, some students say that knowing about the longtime debate on the nature of infinity makes them feel less stressed when they have perceptual or conceptual problems with this notion. The distinction between potential and actual infinities empowers them to think deeper and helps them to decide which standpoint they need to adopt when working with infinite objects. Since such philosophical considerations are not common in other mathematics courses, some students say that they are pleasantly surprised to have the experience.

The subset of students with weak mathematical background and grade-oriented performance who can easily be confused by any deviation from a strictly procedural course delivery may need special attention within this teaching approach, which includes an excursion in philosophy of mathematics. However, often students say that they would rather remember derivations and statements in connection with stories and anecdotes told during the lectures. Thus, including historical-philosophical episodes may improve the retention of course material. In addition, more students become interested in mathematics and motivated to learn it through historical-philosophical examples, which gives me the courage to include them in my lectures.

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10

Making Philosophical Choices in Statistics

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I think most of us who teach mathematics at the undergraduate level tend to believe that we are agnostic as to our philosophical convictions when we are in the classroom. Other than choosing to use classical logic over, for example, intuitionistic logic, it might appear that we have left our philosophical inclinations at the door. However, this is seldom the case. For an analysis course, even at the level of calculus, we have, most likely, chosen a standard approach with limits over a nonstandard approach with infinitesimals. In many courses in analysis and algebra, most of us will have chosen to use the axiom of choice, and its equivalents, freely and without mention. For the most part, the choices are unproblematic; after all, we are simply choosing to present the subject as it is accepted by the majority of mathematicians. However, when we venture into statistics, we are confronted with at least three competing philosophical approaches: the frequentist realist view of R. A. Fisher, the frequentist behaviorist perspective of Jerzy Neyman and Egon Pearson, or the subjective view of a Bayes-Laplace development. No philosophy of statistics has a claim to be the standard approach; indeed, some textbooks will present all three. Moreover, unlike the choice, for example, between standard and nonstandard analysis, the choice we make influences not only our presentation, but the conclusions we reach as well.

1 An Example

Dr. Joy Hazard has disappeared, but the notes of her last experiment remain on her desk. She had been working on an experiment, each repetition of which resulted in either a success or a failure. She suspected the outcomes were equally likely, and there were considerable theoretical reasons why that should be so. Joy's notes reveal that she had performed the experiment six times, the first five times resulting in failures and the last attempt resulting in a success. Three statisticians are called in to evaluate the data.

The first statistician, Dr. F., looks at the data and immediately recognizes a problem: there is no indication as to the design of Dr. Hazard's experiment. If the design were simply to repeat the underlying experiment six times, then, letting X represent the number of successes in six independent repetitions of a Bernoulli experiment, each with probability of success p, Dr. F. would consider the hypothesis $H_0 : p = \frac{1}{2}$. To test this hypothesis, he computes

$$P\left(X \leq 1 \mid p = \frac{1}{2}\right) = \left(\frac{1}{2}\right)^6 + 6\left(\frac{1}{2}\right)^6 = 0.1094,$$

and concludes that, given the truth of H_0, the data are not all that unlikely. Hence there are no grounds for the rejection of H_0.

On the other hand, Dr. F. points out that if Dr. Hazard's design was to repeat the underlying experiment until the first success is observed, and if Y represents the first trial on which a success is observed in independent repetitions of Bernoulli trials with probability of success p, then

$$P\left(Y \geq 6 \mid p = \frac{1}{2}\right) = \left(\frac{1}{2}\right)^5 = 0.03125.$$

Thus, under this scenario, the observed data are very unlikely if H_0 were true. Hence there is some evidence against the true value of p being $\frac{1}{2}$.

Dr. N. sees an additional problem: without a well-specified alternative to the hypothesis H_0, there is no way to determine what Dr. Hazard intended as grounds for rejecting H_0. If the design involved a fixed number of repetitions, and X and p are as in the first case discussed, then one would test H_0 against an alternative $H_1 : p \neq \frac{1}{2}$, at a level of significance no greater than 0.05, by rejecting H_0 if either $X = 0$ or $X = 6$. If the alternative were $H_1 : p < \frac{1}{2}$, then one would reject H_0 if $X = 0$. In either case, the results would not lead to rejecting H_0. If the design involved waiting for the first success, and Y and p are as above, then one would test H_0 against either $H_1 : p \neq \frac{1}{2}$ or $H_1 : p < \frac{1}{2}$ by rejecting H_0 if $Y \geq 6$. Hence the data would lead to the rejection of H_0.

Dr. B. is mystified as to why Dr. F. and Dr. N. need to know the intentions of Dr. Hazard before drawing a conclusion. From other workers in the field, he has inferred that, before any data were gathered, the probability that $p = \frac{1}{2}$ was $\frac{1}{2}$. Given the data, one then simply updates this probability (using Bayes' theorem), obtaining

$$\begin{aligned} P\left(p = \frac{1}{2} \mid \text{Data}\right) &= \frac{\left(\frac{1}{2}\right)^6 \times \frac{1}{2}}{\left(\frac{1}{2}\right)^6 \times \frac{1}{2} + \frac{1}{2}\int_0^1 s(1-s)^5 ds} \\ &= \frac{1}{1 + 2^6 \int_0^1 s(1-s)^5 ds} = 0.3962, \end{aligned}$$

regardless of the intentions of the missing researcher. Hence Dr. B. concludes that, although the evidence decreases the chances that $p = \frac{1}{2}$, the probability of the truth of this hypothesis is, nevertheless, still substantial.

2 Three Approaches to Hypothesis Testing

The above example is illustrative of three distinct philosophical approaches to hypothesis testing in statistics. The first, due to the statistician, evolutionary biologist, and geneticist R. A. Fisher (1890–1972), combines a frequentist view of probability with a realist attitude toward scientific knowledge. For Fisher, the aim of an experiment is to uncover the most likely state of nature that underlies the phenomenon under investigation. For example, a biologist may design an experiment with the goal of discovering whether or not there is evidence to support the hypothesis that, in an infinite population of a specified type of mouse, a certain type of mating will produce a white mouse with probability $\frac{1}{2}$. The second, due to the statisticians Jerzy Neyman (1894–1981) and Egon Pearson (1895–1980), also begins with a frequentist interpretation of probability, but with a nominalist slant that leads to a behavioristic development of statistics. In this approach, the relative frequencies of probability refer only to actual or potentially actual sequences, never to hypothetical populations that could never be generated. The grounds of scientific reasoning derive from the world of controlled repeated sampling, such as is found in industrial quality control procedures. The third, usually referred to as the Bayesian view, but with roots due more to Pierre-Simon Laplace (1749–1827) than to Thomas Bayes (1701–1761), rejects a frequentist view of statistics in favor of a subjective approach. That is, probabilities are measures of our knowledge, or ignorance, about the workings of the world. The job of statistics is to make the calculation of probabilities coherent, updating them as more data become available. As ultimately these probabilities rely on the initial, *a priori*, beliefs of the investigator, this approach results in fundamentally different interpretations of many of the basic concepts of statistics.

2.1 Fisher's Approach

A typical situation for Fisher's approach to hypothesis testing might run as follows: given a random sample $X_1, X_2, \ldots, X_n$, suppose we wish to test the hypothesis H_0 that the sample is from a distribution with density f. Moreover, suppose we identify a statistic $T : \mathbb{R}^n \to [0, \infty)$ with the property that $T(X_1, X_2, \ldots, X_n)$ tends to be small when $X_1, X_2, \ldots, X_n$ is, in fact, from f. Given observed data $x_1, x_2, \ldots, x_n$, we call the probability

$$P(T(X_1, X_2, \ldots, X_n) \geq T(x_1, x_2, \ldots, x_n) \mid f),$$

by which we mean the probability that the statistic T will exceed the observed value $t = T(x_1, x_2, \ldots, x_n)$ when the sample is from the distribution f, the p-*value* of the test. If the p-value is exceptionally small, Fisher says we are faced with a simple disjunction: either we have observed an exceptionally rare event, or H_0 is, in fact, false [5, p. 42]. Hence a statistical inference is a probabilistic version of an argument by contradiction.

So, on Fisher's view, small p-values provide evidence against H_0. We may see this is as a variation on Émile Borel's *single law of chance*: "Phenomena with very small probabilities do not occur." [1, p. 1]. However, we must not interpret a p-value as an evidentiary measure. That is, we cannot necessarily compare p-values between different tests. In particular, we must not interpret large p-values as establishing the truth of the null hypothesis. As Fisher insists, "[I]t is a fallacy . . . to conclude from a test of significance that the null hypothesis is thereby established; at most it may be said to be confirmed or strengthened." [3, p. 73]. At best, a large p-value indicates that the data are not inconsistent with the hypothesized distribution, but this does not imply that the hypothesized distribution is the true state of nature. Indeed, typically the data will be just as consistent, or even more so, with a range of possible distributions. For Fisher, our strongest claims to experimental knowledge arise from the observation of small p-values. From this view, we may infer the falsity of a hypothesis H_0 only when we can design a test that reliably yields small p-values: "we may say that a phenomenon is experimentally demonstrable when we know how to conduct an experiment which will rarely fail to give us a statistically significant result" [4, p. 14].

2.2 Neyman and Pearson

A typical situation for the Neyman and Pearson approach might be: suppose $X_1, X_2, \ldots, X_n$ is a random sample from a distribution with density f_θ, where $\theta \in \Theta \subset \mathbb{R}$, and we wish to test the hypothesis $H_0 : \theta \in \Theta_0$ against the hypothesis $H_1 : \theta \in \Theta_1$, where $\Theta = \Theta_0 \cup \Theta_1$. Before observing values of the random sample, we choose $C_0 \subset \mathbb{R}^n$ and $C_1 = \mathbb{R}^n \setminus C_0$ and decide that we will reject H_0 (equivalently, accept H_1) if we observe

$$(x_1, x_2, \ldots, x_n) \in C_0$$

and accept H_0 (equivalently, reject H_1) if we observe

$$(x_1, x_2, \ldots, x_n) \in C_1.$$

Let

$$\alpha = P((X_1, X_2, \ldots, X_n) \in C_0 \mid \theta \in \Theta_0)$$

and

$$\beta = P((X_1, X_2, \ldots, X_n) \in C_1 \mid \theta \in \Theta_1).$$

We call α the *significance level* of the test and $1 - \beta$ the *power* of the test. Typically, C_0 is determined by a choice of a test statistic and the desired value for α, in which case the only control on β is through the choice of the sample size.

For our initial example, we assume that $X_1, X_2, \ldots, X_6$ is a random sample from a Bernoulli distribution with some probability of success p. That is, for $i = 1, 2, \ldots, 6$, $X_i = 1$ if the ith trial resulted in a success and $X_i = 0$ otherwise. If we let $\Theta_0 = \{\frac{1}{2}\}$ and $\Theta_1 = \left\{p \ : \ 0 < p < 1, p \neq \frac{1}{2}\right\}$, then we wish to test $H_0 : p \in \Theta_0$ against $H_1 : p \in \Theta_1$. To obtain a significance level less than 0.05, we may let

$$C_0 = \left\{(x_1, x_2, \ldots, x_6) \ : \ \sum_{i=1}^{6} x_i \leq 0 \text{ or } \sum_{i=1}^{6} x_i \geq 6\right\}$$

and

$$C_1 = \left\{(x_1, x_2, \ldots, x_6) \ : \ 0 < \sum_{i=1}^{6} x_i < 6\right\}.$$

The resulting significance level is

$$\alpha = P((X_1, X_2, \ldots, X_6) \in C_0 \mid \theta \in \Theta_0) = \frac{1}{2^6} + \frac{1}{2^6} = \frac{1}{32}.$$

This computation is possible only because H_0 completely specifies a probability distribution, namely, a Bernoulli distribution with probability of success $\frac{1}{2}$. On the other hand, H_1 does not specify one distribution, but rather a family of Bernoulli distributions with various probabilities of success. Hence we could compute β, the power of the test, only if we had some prior probability distribution on Θ_1, which would take us out of the realm of frequentist statistics and into the world of Bayes and Laplace. As a substitute, we may instead consider the *power function* of the test,

$$\pi(s) = P((X_1, X_2, \ldots, X_6) \in C_0 \mid p = s),$$

which provides the probability of rejecting the null hypothesis as a function of the probability of success p. See Figure 1.

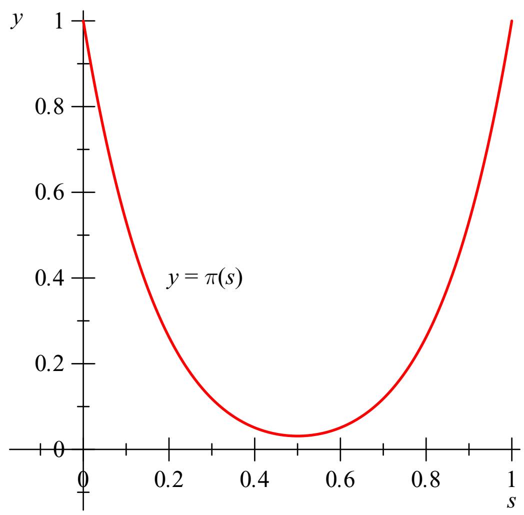

Figure 1. Graph of the power function π.

As seen in the example, α is computable only when Θ_0 is a single point. If Θ_0 is not a singleton, one may take

$$\alpha = \sup_{\theta \in \Theta_0} P((X_1, X_2, \ldots, X_n) \in C_0 \mid \theta \in \Theta_0).$$

Similarly, $1 - \beta$ is computable only when Θ_1 is a single point. It is typically more useful to consider the power function

$$\pi(\theta) = P((X_1, X_2, \ldots, X_n) \in C_0 \mid \theta).$$

The logic of the Neyman-Pearson scheme is as follows: α and β measure the probabilities of certain types of errors that one can commit when making an inference. In particular, α measures the probability of the error of rejecting H_0 when H_0 is in fact true (called a *Type I error*) and β measures the probability of the error of accepting H_0 when H_0 is in fact false (called a *Type II error*). Of course, much hinges on how one interprets probability in the given circumstances. For Neyman and Pearson, probability refers to relative frequencies for sequences of events that are actually or, at least, potentially realizable. Hence if a given experiment were to be repeated indefinitely, one would expect to commit a Type I error about $100 \cdot \alpha\%$ of the time and a Type II error about $100 \cdot \beta\%$ of the time.

In a standard setup, a Type I error is considered to be more severe than a Type II error. For example, consider the quality control problem for a manufacturer shipping batches of widgets. Here H_0 would be the hypothesis that the batch is of acceptable quality and H_1 the hypothesis that that the batch does not meet standards. The company would like to minimize the chances that a shipment is rejected when it is in fact acceptable, a Type I error. Although the company would also like to keep the probability of shipping a defective batch, a Type II error, as small as possible, such an error may not be considered as serious as the first. For this reason, the value of α is usually determined in advance and the region C_0 chosen to ensure a Type I error no greater than α.

2.3 Bayes-Laplace development

A typical application of the Bayes-Laplace methods is as follows: suppose $X_1, X_2, \ldots, X_n$ is a random sample from a distribution with density f_θ, where $\theta \in \Theta \subset \mathbb{R}$, and we wish to test the hypothesis $H_0 : \theta = \theta_0$. Moreover, suppose our initial beliefs about the values of θ are described by a probability measure m on Θ. We will suppose $m(\{\theta_0\}) \neq 0$, that is, that we have some positive belief that θ_0 is the exact value of θ. Then, given data $x_1, x_2, \ldots, x_n$, we use Bayes' theorem to compute

$$P(\theta = \theta_0 \mid \text{Data}) = \frac{f_{\theta_0}(x_1)\cdots f_{\theta_0}(x_n)m(\{\theta_0\})}{f_{\theta_0}(x_1)\cdots f_{\theta_0}(x_n)m(\{\theta_0\}) + \int_{\Theta'} f_\theta(x_1)\cdots f_\theta(x_n)dm(\theta)},$$

where $\Theta' = \Theta - \{\theta_0\}$. We now have an updated measure of belief in the hypothesized value of θ.

Before starting an analysis of experimental results, a Bayesian statistician must first have determined a probability model for the data, along with an initial set of beliefs (represented by the measure m) concerning the possible values of the underlying parameters. The probabilities given by m cannot be thought of as relative frequencies, unless one wishes to consider some type of sampling from multiple possible universes. Rather, m quantifies the investigator's state of knowledge. Once data are available, Bayes' theorem provides the mechanism for updating the prior probabilities to reflect the knowledge gained from the experiment.

There is no additional logical structure introduced here. That is, the Bayes-Laplace approach reduces inductive inference to deductive logic, statistics to probability. Such a reduction is very attractive to mathematicians, but, as noted below, requires adopting a philosophical view of probability, and of the world, that they might not find to be as attractive.

3 The Philosophical Differences

The three schools of statistical inference result from fundamental philosophical differences: not only differing interpretations of the meaning of probabilistic statements, but also on the nature and aim of inductive inference itself. Fisher was interested in uncovering the true state of nature, seeing probabilities as relative frequencies that would emerge if certain experiments, under certain conditions, were to be repeated indefinitely. For example, he says,

> If, in a Mendelian experiment, we say that the probability is one half that a mouse born of a certain mating shall be white, we must conceive of our mouse as one of an infinite population of mice which might have been produced by that mating. The population must be infinite for in sampling from a finite population the fact of one mouse being white would affect the probability of others being white, and this is not the hypothesis that we wish to consider; moreover, the probability might not always be a rational number. Being infinite the population is clearly hypothetical, for not only must the actual number produced by any parents be finite, but we might wish to consider the possibility that the probability should depend on the age of the parents, or their nutritional conditions. We can, however, imagine an unlimited number of mice produced upon the conditions of our experiment, that is, by similar parents, of the same age, in the same environment. The proportion of white mice in this imaginary population appears to be the actual meaning to be assigned to our statements of probability [2, p. 700].

This is reminiscent of the position held by C. S. Peirce in his later years. For example, in explaining what is meant by the statement that the probability of obtaining a three or a six on the roll of a die is $\frac{1}{3}$, Peirce said,

> I mean, of course, to state that the die has a certain habit or disposition of behaviour in its present state of wear. It is a *would be* and does not consist in actualities of single events in any multitude finite or infinite. Nevertheless a habit does consist in what *would* happen under certain circumstances if it should remain unchanged throughout an endless series of actual occurrences. I must therefore define that habit of the die in question which we express by saying that there is a probability of $1/3$ (or odds of 1 to 2) that if it be thrown it will turn up a number divisible by 3 by saying how it *would* behave if, while remaining with its shape, etc. just as they are now, it *were to be* thrown an endless succession of times [15, par. 8.225].

Neyman and Pearson also saw probabilities as relative frequencies, but only for sequences of events that are actually or potentially realized. In contrast, Fisher and Peirce recognize that the "unlimited number of mice produced upon the conditions of our experiment," or the endless succession of throws of a die "while remaining with its shape, etc. just as they are now," can be neither actually nor potentially realized. Neyman considers the statement "the probability of getting six points on the die when the die is thrown" to be ambiguous: it might refer to a completed sequence of throws, in which case the probability is simply the ratio of the number of observed sixes to the total number of throws; to a sequence of throws to take place in the future, in which case the probability is unknown until the throws are actually carried out; or to a hypothetical sequence of throws [9, p. 5]. For the latter case, the probability of rolling a six still refers to the ratio of the number of sixes rolled to the number of rolls. That is, even for a hypothetical set F_1 of n tosses of a die, to say that the probability of getting a six on one throw of a die is $\frac{1}{6}$ means "that among the n throws in F_1 there are exactly $\frac{n}{6}$ with six on the top face of the die." [9, p. 6]. In contrast to Fisher's view (see the discussion earlier in this chapter) of probability, the probability of rolling a six must be a rational number. Hence, for Neyman, the probability of an event refers to the frequency with which the event has either occurred in some sequence of repetitions, or to the frequency with which the event will occur in a sequence of repetitions.

For the Bayesians, probability is a subjective measure of belief in the state of nature. As indicated, inductive inference is then nothing more than an application of probability to the updating of the beliefs when confronted with additional data. The philosophical differences between frequentists and Bayesians have been much studied (see, for example, [6] and [7]). For this reason, and because Bayesian statistical methods are not treated in much depth, if at all, at the undergraduate level, I will not go into much detail on subjective approaches to probability and how they are used in statistics. Suffice it to say that, if one adopts the Bayes-Laplace framework, one must either accept an interpretation of probability as a measure of the state of one's knowledge of the world, and not as a measure of the frequencies with which events occur in the world, or opt for a worldview that sees our world as one of many possible worlds. But even in the latter case, a frequentist view of probability is not tenable. As C. S. Peirce wrote,

> The relative probability of this or that arrangement of Nature is something which we should have a right to talk about if universes were as plenty as blackberries, if we could put a quantity of them in a bag, shake them well up, draw out a sample, and examine them to see what proportion of them had one arrangement and what proportion another. But, even in that case, a higher universe would contain us, in regard to whose arrangements the conception of probability could have no applicability [14, p. 714].

The narrow view of frequencies formulated by Neyman and Pearson, involving only sequences that are actually or potentially instantiated, fits well in the framework of quality control sampling, the prototype of Neyman-Pearson statistical inference. If θ is some measurable characteristic of an object manufactured on an assembly line, one may set up a fixed testing procedure that will result in incorrect inferences with known probabilities. That is, Type I errors, inferences that the process is not within the control bounds when it in fact is, and Type II errors, inferences that the process is within the control bounds when it in fact is not, will occur with predetermined frequencies. In such a situation the license for, and the consequences of, the inferences are clear: a known percentage of all the inferences made will be in error. But the situation is not so clear when the same ideas are applied to scientific inferences, inferences about the state of nature. In this realm repetitions, even if possible, are not intended. The goal is not to minimize risk through adjustments of testing parameters to control the frequency of Type I and Type II errors, but to make a statement of fact about the world. Neyman and Pearson are aware of this weakness in their conception of frequencies. Consequently, Neyman prefers to speak of "inductive behavior" instead of "inductive inference" (see [10, p. 291] and [8, p. 1]) and Pearson speaks of statistical testing procedures as providing guides to our decisions and suggests that we treat an individual inference as but one among all the inferences made by a larger community [11, p. 142].

Fisher was highly critical of this approach to statistical inference. In particular, he believed that the repeated sampling paradigm, although well suited for industrial quality assurance procedures, did not apply to scientific inquiries. Here is but one example of the type of rhetoric employed by Fisher:

> I shall hope to bring out some of the logical differences more distinctly, but there is also, I fancy, in the background an ideological difference. Russians are made familiar with the ideal that research in pure science can and should be geared to technological performance, in the comprehensive organized effort of a five-year plan for the nation. How far, within such a system, personal and individual inferences from observed facts

> are permissible we do not know, but it may be safer, and even, in such a political atmosphere, more agreeable, to regard one's scientific work simply as a contributary element in a great machine, and to conceal rather than to advertise the selfish and perhaps heretical aim of understanding for oneself the scientific situation. In the U.S. also the great importance of organized technology has I think made it easy to confuse the process appropriate for drawing correct conclusions, with those aimed rather at, let us say, speeding production, or saving money. There is therefore something to be gained by at least being able to think of our scientific problems in a language distinct from that of technological efficiency [3, p. 70].

If this language seems extreme, I should note that Egon Pearson's father, Karl (1857–1936), once compared Fisher with Don Quixote tilting at the windmill, claiming that "he must either destroy himself, or the whole theory of probable errors." [12, p. 191]. This was in reaction to Fisher's correction to the elder Pearson's goodness-of-fit test in the case where some parameters are estimated from the data.

Fisher did not see p-values, or levels of significance, as fixed measures of the strength of the evidence provided by the data. As he says, "[I]n fact no scientific worker has a fixed level of significance at which from year to year, and in all circumstances, he rejects hypotheses; he rather gives his mind to each particular case in the light of his evidence and his ideas." [5, p. 45]. For Fisher, the decision on what constitutes a small probability in a given situation is not a matter of the perceived risks involved, for scientific research is

> an attempt to improve public knowledge undertaken as an act of faith to the effect that, as more becomes known, or more surely known, the intelligent pursuit of a great variety of aims, by a great variety of men, and groups of men, will be facilitated. We make no attempt to evaluate these consequences, and do not assume that they are capable of evaluation in any sort of currency. [3, p. 77].

4 Implications for Teaching Statistics

Few statistics textbooks differentiate between Fisher's approach to statistical inference and that of Neyman and Pearson, perhaps, at least in part, because a frequentist interpretation of probability underlies both. As a consequence, many books present a mishmash of p-values and errors of Type I and Type II. I believe it is important to clarify the distinctions.

How one approaches these philosophical difficulties in a statistics course will vary, depending, in part, on the type of institution and the objectives of the course. In my case, I teach at a small college with a bias towards the liberal arts. We have only two statistics courses, a lower level course with no particular prerequisites (which I have taught only a few times) and an upper level course with prerequisites of vector calculus and a probability course (which I teach frequently).

In both courses, I stress p-values and their proper interpretation. That is, a p-value measures nothing more than the probability with which a deviation as great, or greater, than that observed would occur if the null hypothesis were true. The connection between p-values and significance levels is tenuous. As discussed above, Fisher and Neyman do not even attach the same meanings to statements of probability. Moreover, a p-value is not, in all cases the smallest significance level with which one would reject the null hypothesis under the Neyman-Pearson paradigm. For example, in a much discussed disagreement on how to evaluate contingency tables, Fisher would compute a p-value based on fixing the marginal totals and Pearson would respond that that is not the probability associated with repeated random sampling (see, for example, [16] and [11]).

One objection to stressing p-values is that they are often misinterpreted as some type of evidentiary measure, such as the posterior probability of the truth of the null hypothesis. However, if we avoided every part of mathematics that is subject to misinterpretation (starting with the basic notions of limits), we would not have much left. Indeed, I would argue that a large part of what we do, as teachers of mathematics, is to clarify such concepts.

Although my philosophical tendencies incline toward Fisher's point of view, the machinery of the Neyman-Pearson approach, such as Type I and Type II errors, fixed significance levels, and power curves, fits well in a discussion of quality control. For the most part, I prefer to omit discussions of the machinery of the Neyman-Pearson paradigm in a lower level course, but they work nicely in an upper level course, where one might, for example, compare testing procedures by plotting their power curves. However, transferring the Neyman-Pearson logic to the sphere of scientific testing is philosophically delicate. This is in part true because p-values and Neyman-Pearson significance levels, as indicated above, do not always refer to the same probability. Related to this difference is the problem of how to handle the information gained from a single scientific experiment that (unlike the case with quality control sampling) will not be repeated on a regular basis.

Pearson, in a discussion of 2 × 2 contingency tables, attempts an explanation as to how his logic of statistical inference can make the move from the realm of repeated sampling to that of scientific inference:

> It seems clear that in certain problems probability theory is of value because of its close relation to frequency of occurrence . . . In other and, no doubt, more numerous cases there is no repetition of the same type of trial or experiment, but all the same we can and many of us do use the same test rules to guide our decision, following the analysis of an isolated set of numerical data. Why do we do this? What are the springs of decision? Is it because the formulation of the case in terms of hypothetical repetition helps to that clarity of view needed for sound judgement? Or is it because we are content that the application of a rule, now in this investigation, now in that, should result in a long-run frequency of errors in judgement which we control at a low figure? [11, p. 142].

Although he protests that "I should not care to dogmatize," it is clear from these statements and what follows that Pearson considers a probability statement in a singular situation to have two possible meanings: a way of providing a paradigm to guide our thinking, or one event in a sequence of events in which the individual, or, perhaps, the community, will make statistical decisions. The latter view is very reminiscent of the position of C. S. Peirce in his early writings (see, for example, [13]).

Deborah Mayo [7] has developed the notion of a severe test as a way to bridge the gap between the Neyman-Pearson paradigm and the needs of scientific inference. However, such a treatment is, for me, too far afield to bring into my courses. Moreover, without a realist development of probability, such as that in the later writings of Peirce or as is implicit in Fisher's writings, I see Mayo's logic reducing to a version of Pearson's first option, and, hence, reducing scientific inference to a type of reasoning by analogy.

I have found that students are very much interested in hearing about the philosophical disagreements between the founders of modern statistics. There are few places in the undergraduate mathematics curriculum where one can see such clear delineation between philosophical schools. Students tend to see the content of their courses in mathematics as a given, something that could not be other than it is. Talking in class about some heated disagreement between prominent statisticians of the past will usually lead to the liveliest classroom discussion of the term. When students hear the rhetoric employed as Fisher and Karl Pearson argued about the proper form of the chi-square test, or as Fisher and Neyman went back and forth on the logic of hypothesis testing, they become more engaged with the topics themselves. At the same time, I view such discussions as lagniappe, an addition to the course which is not part of the formal course material. In particular, I would not want to turn such discussions into more material for an exam.

The neat, and often disingenuous, treatments of statistical methodology in textbooks gloss over fundamental philosophical differences between the various approaches to analyzing statistical data. As one consequence, students often see statistical procedures as dry algorithms to be followed slavishly. Although this may agree with the inductive behavior approach promoted by Neyman and Pearson, I do not think this is how we should approach the problem of learning from data. Data seldom lead to precise, definite conclusions. Rather, data serve to increase or decrease support for a hypothesis. How we measure that support, and whether or not we think it tells us something real about the world about us, is ultimately a question for philosophy. If you teach statistics, you are making philosophical choices, whether you admit it or not.

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V

Logic, Foundations, and Transition Courses for Mathematics Majors

Chapter 11: Jeff Buechner, in "How to Use Ideas in the Philosophy of Mathematics to Teach Proof Skills," discusses using concepts from the philosophy of mathematics in a course in logic taken by a range of majors. Since the course is not strictly for mathematics majors, he considers arguments in English, not just mathematical proofs, and how to evaluate the soundness of any type of argument. When he introduces the logical operators, he wants students to learn that rules of inference depend on what kind of world they are being applied to and what kinds of objects there are in that world. So he introduces intuitionistic logic to bring out philosophical questions about the nature of mathematical objects and mathematical reasoning. (Buechner is a philosopher, teaching in a philosophy department, and thus he has a much broader philosophical interest when teaching logic than most readers of this volume. However, we feel his perspective is worth considering seriously.)

The chapter assumes some background in logic. So while it gives significant food for thought for someone teaching a logic course (who presumably is either a logician or has taken several courses in logic), it is not necessarily as adaptable to a brief trek through logic as part of an introduction to proof course. In addition, while the author describes aspects of intuitionism, before one can use his approach a faculty member would need to do some additional reading in intuitionism (and the author supplies references). On the other hand, many who have a significant interest in philosophy of mathematics will have already read about intuitionism, and a bit of these ideas can be introduced into an introduction to proof course if the instructor has this background.

Chapter 12: Gizem Karaali introduces "An 'Unreasonable' Component to a Reasonable Course: Readings for a Transitional Class." The component considers the original observation by Eugene Wigner that mathematics seems to be unreasonably effective in physics, and a range of other articles that have taken up the theme. She uses the component in introductory linear algebra, which functions as the introduction to the mathematics major at Pomona College, even though the topic is not directly related to the material of the course. So the component could be used with almost any course. However, it is perhaps particularly appropriate for a course where students have a range of majors, but may, in the end, decide to major in mathematics: hence, at about the sophomore level. But it would work well in a history of mathematics course as well. Why mathematics is useful, and why it is useful even in unexpected places, leads to a discussion of many implicit assumptions about mathematics: whether it is invented or discovered, whether it is describing the universe or is a game played by rules we choose.

Karaali provides an annotated list of articles she uses in her course, which should be useful for anyone trying to incorporate her ideas into a course.

11

How to Use Ideas in the Philosophy of Mathematics to Teach Proof Skills

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1 Introduction

This chapter will argue that a basic distinction in the philosophy of mathematics—between a realist view and an idealist (constructivist) view of mathematical objects—manifests itself in how logical connectives and quantifiers are understood (via classical versus intuitionistic logic), that how the latter are understood determines distinct kinds of mathematical proofs, that failure to understand these notions is a cause of poor proof skills in mathematics and that acquiring proof skills in mathematics can be greatly facilitated by seeing a comparison between distinct kinds of mathematical proofs and their connection with the meanings of the logical connectives and quantifiers (meanings partially determined by views in the philosophy of mathematics).

In "The Role of Logic in Teaching Proof" [2], Susanna Epp argues that a principal reason for the poor performance of mathematics students at the university level in writing and understanding proofs is that they fail to understand elementary concepts of logic that play an essential role in writing proofs. For instance, they are easily confused by different, but equivalent, ways of writing a conditional sentence, such as "If A then B" and its equivalent, "A only if B." What shocked the mathematics community in Epp's work was that failures in proof ability begin at such a basic level. It is not a matter of not understanding certain mathematical ideas, such as the idea of a function or of a relation (though this too can feature in failures of proof ability). In my teaching of logic to university students I have profited from the insights and findings Epp has provided. In particular, in teaching logic to mathematics majors who will at some point need to write proofs in their mathematics courses, I have devised various exercises and thought experiments that, in my view, considerably improve students' comprehension of fundamental concepts of logic.

Epp assumes that students can acquire a satisfactory understanding of basic concepts in logic without being exposed to alternative logics (that is, logics other than classical propositional and first-order logic). However, I think that alternative logics can be pedagogically useful in teaching students elementary logic concepts, such as the semantics of logical connectives (truth functions) and quantifiers. For instance, classical negation is quite different from intuitionistic negation or negation-as-failure in non-monotonic logics. In considering alternative (that is, non-classical) conceptions of negation, one better understands the role and function of classical negation. But how are alternative logics to be motivated? That is where the philosophy of mathematics can play an integral role—in showing students why there are alternative logics and what the point of an alternative logic happens to be. A student who understands the reason why an alternative logic is in place will not feel that it is an idle exercise in locating and describing various logics within logical space.

2 Where and What I Teach

I teach at Rutgers University-Newark, which is both a research university and a state university. The student body is one of the most ethnically diverse in the United States. Many students major in business. The mathematics department is fairly small, and there is no computer science department. Students wishing to take advanced courses in computer science usually do so at the neighboring New Jersey Institute of Technology, with which we have a reciprocal course registration policy. Although I am a philosopher teaching logic in a philosophy department, much of what I say can be adapted to logic and introduction to proof courses taught in mathematics departments.

The course in logic that I teach is worthwhile for almost the entire student body, since I emphasize the connection between logic as a formal language and reasoning in a natural language such as English. Students learn a basic method for critically assessing arguments of any kind—a method that is essential for almost any course they take at Rutgers. My view is that if presented properly, all will find it interesting and important. Additionally, one value of a university education is to become acquainted with ideas one would probably not encounter outside the university.

The emphasis on reasoning is relevant to psychology majors, the emphasis on philosophy is relevant to philosophy majors, and the emphasis on using the philosophy of mathematics to sharpen proof skills is relevant to mathematics majors. There are about forty students in each introductory logic class I teach. Typically, only a few are philosophy or mathematics majors. Perhaps ten will be psychology majors (the most popular undergraduate major after business and economics). There will also be a handful of students who wish to pursue a career in teaching (in grades K-12). Some will teach mathematics; of those, some (but not all) are mathematics majors. Some are also mathematics majors who do not intend to teach. Over the years I have maintained contact with many students because I have an interest in knowing whether the skills they acquired in my logic course have been useful in their careers. All of my students who have gone on to pursue a graduate degree in mathematics have told me that my course in logic helped them with mathematics proof skills in graduate school.

3 The Philosophy of Mathematics and Proof Skills

As I mentioned in the introduction, although I'm teaching students classical first-order logic, I use an alternative logic to help them gain a deeper understanding of aspects of our standard logic. The alternative logic that I employ is intuitionistic logic, because it presents an alternative answer to an important and basic philosophical question about the metaphysics of mathematical objects. It also provides an important idea to the student about the nature of reasoning—viz., that the nature of the reasoning process (and not simply the content of one's reasons) will differ depending upon what is in the world. It would be bad pedagogy to start with this view, however, since students would not really understand what it says until they have a better idea of the nature of some other reasoning process, such as the reasoning in classical propositional logic. Becoming acquainted with the reasoning involved in simple propositional (or sentential) logic is a necessary condition for introducing the requisite ideas from the philosophy of mathematics. Although the mathematical community has overwhelmingly rejected intuitionism, in large part because to adopt it would involve rejecting most of modern mathematics and require a dramatic revision of most of mathematics starting with calculus, intuitionistic logic is an excellent tool for introducing these philosophical issues.

The consideration from the philosophy of mathematics is whether mathematical objects exist in the world (whether an abstract world or the actual world) independently of the existence of human beings, or whether they are objects that are constructed by human beings, and thus whose existence is dependent upon the existence of human beings. On the view that mathematical objects exist independently of human beings, mathematical objects could exist in a world even if there were no human beings in it. On the view that they are constructed by people, mathematical objects can exist in a world only if human beings do. The virtue of employing this distinction is that it is graphic and easy for a student to grasp.

To be told that the nature of one's reasoning processes depends on which view is correct must be somewhat startling for a student. Until substance is put into the idea by use of both theory and examples, it will be difficult for almost any student to see just what it comes to. In what follows, I will describe in detail the explicit pedagogical steps taken to prepare the students for the introduction of ideas from the philosophy of mathematics.

It is essential for an instructor teaching logic to make it clear from the first day of class what the notion of argument validity is and what needs to be done to determine whether an argument is or is not valid. My method is to start with arguments in English, and discuss how one determines whether the reasons for the claim being made in an argument are good: whether there is sufficient support, or sufficient evidence, for the claim. The method is that of counterexamples: one looks for situations in which the information in the arguments' premises is present, but in which the conclusion is false. Each counterexample gives one a reason to think that the argument might be invalid, since it reveals a deficiency in the information expressed in the premises of the argument. That is, it shows that there are alternative hypotheses compatible with the premises, in which the conclusion of the argument does not hold. Information rules out possibilities; thus, where possibilities are not ruled out by the available information, there is a deficiency (of some sort) in the information. It is useful at this point to introduce the distinction between deductive arguments and inductive arguments. Proofs in mathematics take the form of deductive arguments; they are valid, which means that there are no counterexamples. If a proposed proof has a counterexample, then it is not a deductive argument, and thus not a genuine proof. (At this point, something might be said about probabilistic proofs of mathematical theorems, especially theorems from elementary number theory, and about computer-generated proofs, such as the proof of the four-color theorem.) A few words should be said about what a deductive proof offers over inductive "proofs." In particular, in deductive proofs with true premises, the conclusion is certain. That is, mathematical proofs starting from self-evident axioms (or theorems that have themselves been proved on the basis of self-evident axioms) yield conclusions (theorems) that are certain. This is the highest standard of knowledge available. Knowledge arrived at in any other way does not meet this standard.

4 Introducing Classical Logic

It is better to start with the most general—how to critically evaluate an argument of any kind—and then proceed to more specific kinds of arguments, such as mathematical (deductive) arguments and inductive arguments (such as statistical syllogisms), than to start with specific arguments. The reason is that the ideas from the general setting will carry over to specific arguments. Also, by seeing arguments of various kinds, some of which are not mathematical, a student will appreciate and understand what it is that a mathematical proof provides—that is, the epistemic value of a mathematical proof.

This is a sobering experience for a typical university student, since most will never have been taught how to do a proper critical assessment of an argument. (In my twenty years at Rutgers University I have found that most students never learn how to do it while at the university.) It also shows them that one must be sensitive to word meanings, since it is the information that is expressed in the words in the premises of an argument that is used to determine the goodness of its support.

Once students have such an experience, they are ready to be introduced to a radical idea—that logic can determine the validity of arguments without always needing to examine the words out of which the argument is built. One can eliminate almost all information from an argument, except the information that there are sentences in certain positions within the argument, and the words and phrases "not", "and", "or", "if . . . then . . . ", and "if and only if." Students are incredulous that validity can be determined with just this apparatus. In their experience determining validity in arguments in English, they know that it is the information expressed by the words in the argument that is used to determine the support of the premises. If one loses most of the information, by eliminating most of the words in the argument, how could one possibly determine its validity?

Their amazement dissipates considerably when they learn the definitions of truth functions—the logical connectives—and how certain forms of arguments preserve truth when structured in certain ways. They begin to see that validity is partly a matter of truth-functional structure, and does not depend on word meanings and information beyond that structure. Their next encounter is with rules of inference in propositional logic, and how they are the ways in which all human beings are able to achieve valid reasoning (unless they are intuitionists). They are, at this point, introduced to valid argument forms, such as *modus ponens*, *modus tollens*, hypothetical syllogism, and disjunctive syllogism.

At this point in their education in logic, many might think that the rules of inference in propositional logic are universal—that they apply anywhere in the universe, and to any universe, and do not depend upon what objects exist in that universe. After all, logic abstracts away from word meanings and information. So how could it depend on what objects there are in a world, since what objects there are is a descriptive matter, and thus a matter of what information is available concerning that world?

It is here that introduction of the distinction in the philosophy of mathematics between realism and constructivism (or intuitionism) is most efficacious. Mathematical realism is the view that mathematical objects exist independently of the human mind, while mathematical intuitionism is the view that mathematical objects are constructed by the human mind. How can the distinction in what there is in a world make a difference concerning the nature of logical reasoning in it?

If mathematical objects exist independently of the human mind, then for any mathematical object, it is a determinate matter concerning whether a statement about it is true or false. Just as it is determinate whether a statement about the sun is either true or false, so, too, it is the case for all mathematical objects. On the other hand, if mathematical objects are constructed by a human mind, then they are subject to the limitations of that mind. If a human mind is finite in its capacities, then it cannot complete an infinite process (even though it can both conceive of and reason about infinite processes). Thus, it cannot construct an infinite collection of mathematical objects if it has to construct each item in it. And so statements about infinite collections of objects might not be determinately true or false, since no human mind can construct one.

One dramatic difference between realist mathematics and intuitionistic mathematics has to do with what theorems of classical mathematics can be proved in each. A standard item from the calculus repertoire—the intermediate value theorem—cannot be proved in intuitionistic mathematics. (See the later part of section 6 for a detailed discussion of this issue.) This might suggest to some students that there is something irremediably wrong with intuitionistic mathematics. While acknowledging that mathematicians have largely rejected intuitionism, the instructor should show that it is a coherent viewpoint by explaining that the intermediate value theorem cannot be proved in intuitionism because of the constraints intuitionists place on the definitions of the logical connectives and the quantifiers, which follow from the underlying philosophy of mathematics accepted by intuitionists. Intuitionistic mathematicians characterize the logical connectives in terms of proof relations, while realist mathematicians characterize them in terms of truth values. This difference is what accounts for the failure in intuitionistic mathematics to prove certain classical theorems that can be proved in realist mathematics. (I do not encourage my logic students to adopt intuitionistic logic. In over 25 years of teaching logic, none of my students has, to my knowledge, adopted it. Based on my classroom experience, it is not realistic to worry that students will adopt intuitionistic logic.)

Characterizing the difference between intuitionistic and classical proofs in mathematics can open the door to a penetrating discussion of what proofs are, what they are supposed to show, and the nature of mathematical objects. (For example, how is it that on some views of mathematical objects one kind of proof—a classical proof—is adequate, while on another view, it is not adequate?) There is not something irremediably wrong with intuitionistic mathematics, but rather different philosophical views can underlie different ways of doing mathematics. A student will see the importance of philosophy and of philosophical argument for arguing for those philosophical views about mathematics.

Students are first introduced to the classical truth functions, defined in terms of truth values, that are the basis of a realist philosophy of mathematics. Most are easy for students to understand because the natural (untutored) way to think about a truth value is that it connects a proposition with the world. A proposition is either true or false, depending upon whether it correctly (truly) describes the way (some feature of) the world is. (Of course, there is an impending problem in using the term "correctly", which points to a circularity in the definition. This would also provide an opportunity to enter into a discussion of definitions (both implicit and explicit), predicative versus impredicative definitions of mathematical objects, and various kinds of constructions in mathematics. It is worthwhile for all students in the class to think about what it is that a definition of a mathematical concept does, and how the definition achieves that function.)

Take, for instance, the definition of negation. On the classical (truth-functional) definition, negation simply reverses a truth value. If P is true, then not-P is false, while if P is false, not-P is true. The basic realist intuition is that what happens in the world determines whether a proposition is true or false. If a state of affairs happens in the real world, the proposition expressing that state of affairs is true, while if it does not happen, the proposition is false. Thus, the definitions of most of the classical truth functions in terms of truth values capture this basic realist intuition. On the other hand, the definition of the material conditional (which is the semantic counterpart of the syntactic "implies": $A \Rightarrow B$) poses a problem. The problem has to do with the truth value that the material conditional has when its antecedent (A) is false. A student will think that when the antecedent is false, nothing can be said about the truth or falsity of the material conditional, since the condition or state of affairs expressed by the antecedent is not in place. Take the conditional: "If you go into room R, you will see someone handing out $100 bills." Suppose you (or anyone else) never go into room R. How, in those circumstances, would you determine whether the conditional is true or false?

It is imperative that an instructor reveal to the student why it is that the material conditional is assigned the truth value "true" when its antecedent is false. Recall the definition of validity (in terms of truth values): an argument is valid provided the conclusion is true whenever the premises are true. Valid argument forms capture this condition. But what about the case in which a premise is necessarily false? In such cases, it is impossible to have a situation in which the premises are true and the conclusion false. Thus, any argument whose premises are necessarily false must be valid. At this point, students need to be told that any argument can be rewritten as a material conditional (by conjoining all of the premises and making it the antecedent, and making the conclusion the consequent). Thus, any argument can be evaluated for validity by evaluating the corresponding material conditional. If the material conditional were assigned the truth value "false" when its antecedent is false, it could not be used to determine argument validity, since it would give the wrong answer for any argument whose premises are necessarily false.

After introducing the classical truth functions and rules of inference, it is useful to ask students to observe that propositional logic does not employ quantifiers (that is, quantity concepts) and does not describe logical structure any finer than an atomic sentence, that natural languages do employ such structure, and that valid arguments employing it are ubiquitous and fairly easy to find. (An example should be provided, employing a simple syllogistic form with universal quantifiers. Here is a very simple one. Consider the following argument: All mathematicians prove theorems. Kurt is a mathematician. (Conclusion) Kurt proves theorems. In each sentence there are no truth functions—one cannot find "not", "and", "or", "if . . . then . . . " or "if and only if" in any sentence in the argument. Hence each sentence is atomic in propositional logic: its translation into symbols would be a single letter. Therefore, if the argument is translated into propositional logic, it becomes simply A, B, therefore C, which is invalid in propositional logic.) The concept of predicate logic can then be introduced. These remarks will be important at this point in the course, both to indicate to the student that predicate logic naturally follows propositional logic and to provide a basis for the discussion of the intuitionistic truth functions.

5 Introducing Intuitionistic Logic

Once the propositional truth functions have been introduced and discussed, it is time to introduce the intuitionistic logical constants, such as intuitionistic negation. (See [3] for a good introduction to intuitionism.) Their introduction should be motivated by some of the intuitionists' criticisms of realism in the philosophy of mathematics. Introducing both intuitionism's logical constants and criticisms of realism in tandem is better than introducing each separately, since learning about one will reinforce learning about the other and the student will see more clearly the nature of the difference between classical logic and intuitionistic logic. It is helpful for students to think about the nature of physical objects in the physical world, for there are undoubtedly objects in their experience that do not satisfy the common definition of a physical object. Mathematical objects are such objects. Still, what will a student think of the idea of denying that mathematical objects are objects that exist in the real world? Even if mathematical objects do not satisfy the properties which define a physical object, students will find something unnatural in thinking of mathematical objects as akin to gods, spirits, and the soul. However, there are various ways of making the denial, each of which leads to a different philosophy of mathematics. To avoid complication, the intuitionistic way of denying the realist view should be used, since the contrast with a realist philosophy of mathematics is more pedagogically salient than other views. If the human mind creates mathematical objects, then mathematical propositions are true if they accord with a construction process, or false if it can be proven that they do not accord with a construction process. In that case, how would an intuitionistic definition of a logical connective in terms of truth values be different from a realist definition? To avoid the apparent subjectivity of intuitionism, the bold move made by intuitionists is to define the logical connectives in terms of proof processes. (Of course, whether the element of subjectivity is eliminated depends upon what intuitionists mean by a mathematical proof.)

For example, consider *intuitionistic negation*. A proof of $\neg P$ is a proof of $P \Rightarrow$ "$0 = 1$." The definition says that a proof of the intuitionistic negation of P is a proof that P implies a contradiction of the form "$0 = 1$." But notice that the definition in terms of proofs introduces intuitionistic implication, which must be defined without the use of intuitionistic negation. Here is the definition of *intuitionistic implication*: a proof of $P \Rightarrow Q$ is an effective operation (i.e., an algorithm) that transforms every proof of P into a proof of Q. Thus for any proof of P, one can find a proof of Q. It appears that the definition of intuitionistic implication uses a universal quantifier. If so, we would need to define *intuitionistic universal quantification*. Here is the definition: a proof of $(\forall x)F(x)$ is an operation that, for every numeral n, produces a proof of $F(n)$.

However, in this definition of intuitionistic universal quantification, we appear to be using intuitionistic universal quantification! So the definition of intuitionistic universal quantification appears to be viciously circular. It is worthwhile to expose the student to this claim, since it might easily be made by students. However, it is false: there is no vicious circularity in defining intuitionistic universal quantification, nor is there a need to explicitly use it in defining intuitionistic implication. What accounts for the appearance that there is such a need, though? It is this: we use intuitionistic universal quantification in the metalanguage, the language in which the definitions of the intuitionistic truth functions are explained. Intuitionistic universal quantification is needed to state, in the metalanguage, that the proof conditions for intuitionistic implication hold for all proofs of P. But we do not use intuitionistic universal quantification in the definition of intuitionistic universal quantification. Similarly, we do not use universal quantification in the truth table for the truth function "and", even though, in talking about the truth table for "and" (in the metalanguage) we do use universal quantification. That is, we say (in the metalanguage) that, for all well-formed formulas A that are true and for all well-formed formulas B that are true, the compound well-formed formulas "A and B" are true.

An example of a proof using intuitionistic implication comes from Arend Heyting [4]. (What is in parentheses in the statement of the theorem is the closest standard meaning of terms; there are technical definitions, of course, in intuitionism for these terms.) Theorem: given two real number generators (that is, two real numbers given as sequences), a and b, that lie apart from each other (that is, that are demonstrably not equal), if $a = a'$ (that is, if the sequences a and a' can be shown constructively to converge to the same real number) then a' also lies apart from b. Proof: by the definition of apartness, there are n and k for which $|a_{n+p} - b_{n+p}| > \frac{1}{k}$ for every p. Since the sequences a and a' can be shown constructively to converge to the same real number, an m can be found such that $|a_{m+p} - a'_{m+p}| < \frac{1}{2k}$ for every p. Letting $h = \max(m, n)$, we have $|a'_{h+p} - b_{h+p}| > \frac{1}{2k}$ for every p. We effectively transformed a proof that two real number generators a and b lie apart from one another into a proof that a', which is equal to a, also lies apart from b.

An example of intuitionistic negation also comes from Heyting [4], a proof that $\sqrt{2}$ is not rational. Let the proposition P be that $\sqrt{2}$ is rational. So one should be able to construct (i.e., find) integers a and b such that $a^2 = 2b^2$. Suppose further that a and b are relatively prime. We observe that a must be even. Since it is, 4 divides a^2. So 4 must also divide $2b^2$. So b must also be even, and 2 will divide both a and b. This yields a contradiction. Hence $\neg P$ is provable. This example illustrates how the definition of intuitionistic negation is applied in practice, and also illustrates that standard mathematics can be developed from the point of view of intuitionism.

6 Comparing Realism and Intuitionism

This discussion affords an opportunity to talk about the advances in logic that resulted from formalizing in the object language (of a theory strong enough to express elementary arithmetic) concepts that were employed, informally, in the metalanguage. One might talk about the work of Kurt Gödel, in particular, how he was able to express metalanguage concepts, such as that of provability, in the object language. This will open up a valuable discussion about the distinction between informal and formal concepts, informal and formal proofs, and the idea, in general, of a proof. One important question to consider is whether there are features of designing a formal system that are invariant under a change in one's philosophy of mathematics. For instance, is the role of definitions in a formal system invariant under a change in one's philosophy of mathematics?

It would be useful, at this juncture, to tell students about the founder of intuitionism, L.E.J. Brouwer, and Brouwer's views about intuitionism. In particular, that Brouwer thought all intuitionistic mathematics took place in the human mind, though not all of it in the medium of language. (So defining the intuitionistic logical connectives in language—the description of proof conditions—incurs problems that might be avoided by eliminating the use of language. But then, if language is eliminated, definition of the intuitionistic logical connectives becomes incommunicable.) It would also be appropriate at this juncture to note that the mathematical community rejected the intuitionistic approach because it requires the abandonment of a huge proportion of modern mathematics (and bizarre modifications of mathematics since Newton).

It is worthwhile, given the work of Suzanne Epp cited earlier, to have some students, especially those interested in mathematics, work on conditionals in English using intuitionistic semantics and realist semantics. The exercise would consist of showing the conditions under which certain English conditionals (about fairly mundane states of affairs) are true or false with respect to realism and with respect to intuitionism. For example, consider the conditional "If Peter is given the opportunity to help John, who will be in need at exactly 3 p.m. this Friday, he will help him at exactly 3 p.m. this Friday." Under realism, one would observe Peter when given the opportunity to help John, who will be in need at exactly 3 p.m. this Friday. If Peter helps him, then the conditional is true. Under intuitionism, the situation is more complex: there would need to be some algorithm that transforms every proof of "Peter is given an opportunity to help John, who will be in need at exactly 3 p.m. this Friday" into a proof of "Peter will help him." What exactly would it be to provide a proof that Peter is given an opportunity to help John, who will be in need at exactly 3 p.m. this Friday? If I verify that Peter is given an opportunity to help John, who will be in need at exactly 3 p.m. this Friday, have I proved it? A student at this point might wonder what it means to prove something. For instance, is verifying a proposition always a proof of it? In mathematics, that is not always the case. There are theorems that can be verified to be true without the verification providing a proof that they are true. But it should be made clear to students that a verification of the truth of a theorem can only take place after there is a proof of the theorem, not before. For instance, verifying that the sum of the first n integers is $n(n + 1)/2$ by choosing a value for n of 10 and making the calculation does not show that the theorem is true for all values of n. However, once there is a proof of the theorem (for all values of n), then choosing an arbitrary value of n and making the calculation illustrates the truth of the theorem. This is also an opportunity to discuss why it is that choosing an arbitrary value in proving some theorem can be warranted in some circumstances, but not in others.

However, many perceptual propositions in natural language are taken to be true when the method of verification is to observe that they are true. In these cases, has the perceiver thereby proved that the proposition is true? One might argue that is the case. Here one might introduce ideas about skepticism with respect to perception from epistemology. In doing this, one can show the student that an issue about the reality of mathematical objects joins, at some point, with epistemology and skepticism. For instance, one can talk about the epistemic modality of one's knowledge: what is the modality of knowing a specific proposition? Is the epistemic modality that of certainty? Is the modality that of having good empirical grounds? And what kind of epistemic modality does a proof provide and how is it different from the epistemic modality that perception provides? It is important not just to define what a proof is, but also to talk about what the pedigree of knowledge is that it delivers. How is the knowledge that a proof of a theorem in mathematics delivers different from the knowledge delivered by a scientific theory, such as the theory of evolution or quantum mechanics? In philosophy, one can formulate various skeptical claims about knowledge. For instance, one can easily formulate skepticism about our knowledge of the empirical world. But can one formulate skepticism about mathematical knowledge? These questions are interesting, and they reinforce points made about the nature of proofs in mathematics.

The difference between classical (or realist) negation and intuitionistic negation is stark. In the former, the difference between P and $\neg P$ is that one is false, while the other is true. That is, one of the two propositions correctly describes the world, while the other does not correctly describe the world. But in intuitionism, the difference between P and $\neg P$ is that in the first case there is a proof of P, while in the second case there is some operation that transforms any proof of P into a proof of a contradiction. That also leaves room for neither to be true: there might not be a (known) proof of either. For example, consider the twin primes conjecture (call it T). It might never be proved that there are infinitely many twin primes, nor proved that any given pair of primes is the last one. In classical mathematics, $T \vee \neg T$ would be true, while in intuitionistic mathematics, $T \vee \neg T$ would have no truth value, or be considered indeterminate. (Gödel proved that, in all first-order classical theories strong enough to express arithmetic, there are true propositions of arithmetic that do not have proofs, nor do their negations have proofs. The results hold in both classical and intuitionistic theories, though there are interesting differences).

To show that an argument is valid in classical propositional logic, one can either build a truth table for the material conditional paraphrase of the argument, or else use rules of inference to prove the argument is valid. What means are available for proving an argument in intuitionistic logic to be valid? Not all valid inference rules in classical logic are valid in intuitionistic logic. For instance, inferring $(\exists x)\neg F(x)$ from $\neg(\forall x)F(x)$ is not intuitionistically valid, even though it is classically valid. This is because, given a proof that $(\forall x)F(x) \Rightarrow$ "$0 = 1$," it does not follow on intuitionistic grounds that there is a proof of $\neg F(n)$ for some specific n, since one does not necessarily have a way of finding a specific n for which $\neg F(n)$. For another example: from the premises $A \Rightarrow B$ and $\neg A \Rightarrow B$, one cannot validly infer B, even though in classical logic B can be validly inferred from them.

So far, the material introducing ideas in the philosophy of mathematics can be understood and appreciated by students who are not mathematics majors. But to show the mathematics major the intrinsic connection between the philosophy of mathematics and mathematics, examples from mathematics must be employed. We will introduce the concept of limit and the intermediate value theorem from first-semester calculus. The student of mathematics is now in a position to see how the definition of limit differs for the realist and for the intuitionistic mathematician. The definition of limit is

For each $\epsilon > 0$ there is a $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - c| < \delta$.

What does "whenever" mean? It has to be rephrased in terms of the material conditional: if $0 < |x - c| < \delta$ then $|f(x) - L| < \epsilon$.

This can be rewritten as $(\forall\epsilon > 0)(\exists\delta > 0)(0 < |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon)$.

The realist and the intuitionistic mathematician will understand this formula of first-order logic differently. In particular, to establish that such a limit exists, the intuitionist has to show that for every numerical value of ϵ there is a numerical value of δ such that $0 < |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon$ can be proven for those values of ϵ and δ. On the other hand, the realist mathematician can understand the definition in terms of truth conditions, which can be read off the first-order formula directly. There is no need for a realist mathematician to rethink the first-order formula in terms of proof conditions. A realist can settle for proving that for any ϵ there is evidence that such a δ exists, without necessarily finding its value. Nonetheless, many people teaching calculus will have their students find a value of δ given a value of ϵ.

It is a valuable exercise for students to think through the definition of limit in both ways. In doing so, they will pay careful attention to the quantifiers and the conditional statement in the matrix of the first-order formula. Indeed, they will pay much more careful attention to these items than when they examine the formula solely from a realist perspective—which is to say, when they examine the formula under standard classroom conditions for a course in calculus. In making the comparison between the realist and the intuitionistic understanding of the definition of limit, students will understand better the realist understanding because they are able to contrast it with an intuitionistic understanding. Comparison cases in a learning context are invaluable, since by finding similarities and differences between two items (objects, processes, ideas, etc.), one notices more than one would otherwise notice in examining a single object (process, idea, etc.) on its own without making a comparison.

The student is now ready to see how realism and intuitionism differ with respect to theoremhood in mathematics. The stock example to illustrate this is the intermediate value theorem of elementary calculus. It is provable in classical mathematics, for which the philosophy is mathematical realism, but not provable in intuitionistic mathematics. The set of objects over which the theorem is classically true is the set of real numbers. If f is any continuous function, and $a < b$, for any value v such that $f(a) < v < f(b)$, there is a real number c such that $a < c < b$ and $f(c) = v$. An example of its application is where the function f is such that $f(a)$ is negative and $f(b)$ is positive. Then there is some number, c, for which $f(c) = 0$. That is, the graph of the continuous function intersects the x-axis.

After providing a classical proof of the intermediate value theorem, an instructor can now show that there is no intuitionistic proof of the theorem. This exercise will be useful in helping the student understand proofs by seeing the difference between a classical proof (committed to mathematical realism) and an intuitionistic proof (committed to mathematical intuitionism). The comparison will improve a student's understanding of the classical proof. One way to provide the proof involves first showing that there are real numbers p and q for which the trichotomy law fails intuitionistically. That is, none of $p < q$, $p > q$, $p = q$ can be intuitionistically proved. Suppose that $d = q - p$. Then none of $d < 0$, $d > 0$, $d = 0$ can be intuitionistically proved. It then follows that many functions for which p, q, d are values of it will fail to satisfy the conclusion of the intermediate value theorem. Some functions will satisfy the conclusion, such as the identity function $f(x) = x$. Other functions will not satisfy the conclusion. Roughly, what is needed is a function such as

$$f(x) = \begin{cases} d + x - 1 & \text{if } x < 1 \\ d & \text{if } 1 \le x \le 2 \\ d + x - 2 & \text{if } x > 2. \end{cases}$$

If $d < 0$, the function crosses the x-axis to the right of 2; if $d = 0$, it is on the x-axis from 1 to 2, and if $d > 0$, it crosses to the left of 1. Since we don't know where d is, we cannot use the standard bisection method to find the "intermediate value." We can't even close in on it within one unit! We need to modify the function a bit to make this intuitionistically meaningful, however. For details, see [3, pp. 138–139].

Another theorem on which realists and intuitionists differ is Markov's principle: for any binary sequence (i.e., a sequence all of whose entries are either 0 or 1), if not all the terms in it can, without contradiction, be 0, then a term exists that equals 1. For the intuitionist, given a proof that all the terms cannot be 0, there must be an algorithm that allows one to construct an index n such that some term a_n in the binary sequence equals 1. Searching through all the terms a_i guarantees that such an a_n will be found. However, there is no prior bound on the search, and so it is not intuitionistically admissible. It is trivial for a realist to assert that there is some $a_n = 1$ if not all the terms in the binary sequence are 0. (See [1] for discussion.)

7 Teaching Methods

I employ a combination of lectures, discussions, puzzle and thought problems and challenges (for bonus points), and, on occasion, student presentations. In lectures, it is important that the material come alive for students. They need to be able to see the concepts and supporting framework. But this happens only if they are sufficiently prepared. They must have had exposure to the material beforehand so that they can think about the problems and concepts, even if what they think is incomplete or incorrect. If they go into a lecture knowing virtually nothing about the topic, the lecture will not do much good unless it provokes some prior interest that just happens to connect with the subject matter in some way. But this is haphazard; it would be foolish to count on it to be in place. To make exposure less haphazard, material can be assigned as reading that is required if students wish to optimize their chance to win bonus points.

Discussions are intended to fulfill several goals. First, and primarily, they allow students to talk with me one-to-one and to express their ideas, concerns, worries, problems, aspirations, and much more, without worrying about intruding on class time and without worrying about reactions from other students (other than those in their group). I break the class up into groups of five, and pose a problem for each to think about. Then I migrate from group to group. I might ask each group to write a page on something that we have been working on. For instance, I once asked the discussion groups to write the classical truth function "and" in terms of the Sheffer stroke (not both P and Q) and to think about what advantages, if any, are afforded by using the Sheffer stroke instead of the standard classical truth functions.

Puzzle and thought problems are introduced throughout the course. Some are relatively easy, while others are extremely difficult and challenging. An example of an easy problem is to define "inclusive or" in terms of "exclusive or" and other truth functions. (Most textbooks cover the converse problem: define "exclusive or" in terms of "inclusive or" and other truth functions). An example of a deeply challenging problem: describe a conditional sentence (of the form "If A then B") whose antecedent is true, whose consequent is false, and for which our natural intuition is that it is true. The student will no doubt have to tell a background story about the conditional to convince a listener that it is true. On the other hand, since such a conditional would show *modus ponens* has a counterexample, students quickly realize that there must be some problem in taking it to be true. (It should be easy for a student to see that a proposed conditional is a counterexample: it has a true LHS, a false RHS, and yet intuitively evaluates to "true.")

On occasion, I have students present material to the class. For instance, after showing students the interactions between the universal quantifier and the truth functions "and" and "or" (such as that the universal quantifier distributes over "and"), I select a small group to describe the interactions between the existential quantifier and the truth functions "and" and "or."

When I present material on the use of the philosophy of mathematics to better understand the classical truth functions by introducing the intuitionistic logical connectives, I select students who are majoring in either mathematics or education to talk about some of the intuitionistic definitions of logical connectives. I also give them inference rules that fail intuitionistically and ask them to talk about why they fail. I also have mathematics majors talk about the intuitionistic understanding of the intermediate value theorem and the intuitionistic definition of a limit. These occasions provide bonus points for the students who participate in them; all students are eligible to participate.

8 Difficulties and Changes

One difficulty is that, on occasion, a student will ask about the relevance of introducing material on the intuitionistic logical connectives and quantifiers. That at least one student has asked the question is a good reason to infer that other students have a similar attitude, though they do not raise the question explicitly. In response to this difficulty, I have rethought how to talk about introducing the intuitionistic logical connectives and quantifiers. One change is to make sure that the students have a firm grasp of classical logical connectives and quantifiers before introducing the alternative conception. One test that they have a good understanding is to ask questions and give short exams. A second change to motivate introducing the alternative conception is to discuss the idea of a proof of an arbitrary proposition, whether it is mathematical or not, and the philosophical idea of how a mind can construct a world. We play a short game: imagine a world in which there is nothing and you have to think into existence the objects in it. What would you need to do this? How would you do it? The game is fun because the rules are not precise and encourage creative thinking to find precise rules and to think about how things can go wrong because of their imprecision.

There is also an issue intrinsically tying logic to the philosophy of mathematics. It is the issue of whether there is one true logic, or whether there are alternative logics. If the latter, then there are alternative rules for writing proofs. There might be sentences that can be proved in one logic, but not in another logic. This raises questions about which logics are legitimate, and which theorems are genuine, as well as deep metaphysical issues about the nature of logic and the nature of reasoning.

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[3] George, Alexander and Velleman, Daniel J., *The Philosophies of Mathematics*, Wiley-Blackwell, Malden, 2002.

[4] Heyting, Arend, *Intuitionism An Introduction*, 3rd ed., North-Holland Publishing Company, Amsterdam, 1971.

12

An "Unreasonable" Component to a Reasonable Course: Readings for a Transitional Class

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1 Introduction and Overview

Mathematics instructors often face the inevitable student question: "Why should I care?" How we respond to it can make or break a course. Students who are not sufficiently convinced that the course material is relevant to something outside of the classroom may have difficulty focusing on or caring about it. Behavioral consequences might arise as well as performance issues. In this chapter I describe a reading component that I incorporate into a traditional linear algebra course in order to address this issue. The readings focus on "the unreasonable effectiveness of mathematics", a phrase made famous by Eugene Wigner's article [27].

In developing the component, my first goal was to keep my students connected to the vital fact that mathematics is and has historically been essential to our understanding of the universe. I wanted to make sure that even as they tackled the abstract notions of vector spaces and linear transformations, they would not lose sight of the relevance of mathematics to the world around them and to the disciplines that they could be interested in studying.

My second goal was to introduce a liberal arts (more specifically, a philosophy) component to a rather straightforward mathematics course. Teaching at a liberal arts college, I am always amazed at the cross-disciplinary nature of the courses that some of my colleagues teach. Many instructors on my campus manage to tie their courses to the world outside their classrooms and engage their students in an intellectual pursuit that goes beyond what is in their textbooks. I wanted to do just that for my own classes.

I believe that the concept of a reading component accompanying a mathematics course fits well within the general liberal arts college classroom. Students are encouraged to think beyond the numbers and the algorithms, and are invited to view mathematics as a human endeavor that is intrinsically tied to the way we see and understand the (physical, biological, economic, etc.) world around us. Thus, the readings provide a natural way to engage students' interest in the place of mathematics in our lives. Furthermore as the readings introduce a philosophically interesting question, students are called upon to deal with an intellectual challenge of a kind they have not yet confronted in a mathematics course. Most, if not all, happily rise to the occasion.

Here is an overview of what follows: Section 2 describes the course where I use the reading component, and provides some facts and figures about the institutional context of the course and the students who take it. In Section 3, I comment on the philosophical questions motivating the component and the ones that arise as students tackle the reading assignments. Section 4 lists the articles used for the reading component. I describe how I incorporate it into my courses in Section 5 and present a brief summary of my observations on its effects and outcomes in Section 6. Section 7 is a selection of student reactions. The last section summarizes and concludes, with a view toward possible adaptation at other institutions and for other courses; I also report on some of the difficulties I've encountered and modifications I've made in response. At the end of the chapter I give a brief description of each of the articles I use.

2 Setting and Background

I teach at Pomona College, a small and selective liberal arts college serving approximately 1560 students. Our student to faculty ratio is quite low (7 to 1), and many classes tend to be small, though the mathematics classes (as well as the number of declared mathematics majors) have been growing in size for the last few years. Our students are predominantly traditional students coming straight out of high school, and the vast majority lives on campus. Our students often go on to professional schools after graduation, many go into graduate school, and some eventually find themselves teaching.

The reading component described here is from my version of Pomona's Math 60, our first course in linear algebra. Students taking the course have diverse interests, and even though the course is also our gateway to the mathematics major, many students will choose to declare other majors. Most of the sciences and some social sciences either require or strongly recommend the course for their majors, and so the clientele for the course may be diverse in their prospective major plans. The typical student in the course is a freshman or a sophomore, is interested in mathematics (at least what s/he perceives to be mathematics), and has already taken at least a two-course series of single variable calculus. However s/he is not necessarily mathematically sophisticated or mature, and might or might not be considering a mathematics major.

I had been contemplating for some time the fact that Math 60 is our introduction to the mathematics major. I wondered if I could somehow make the course more about mathematics in general. Students were learning to read and write (very elementary) proofs (in the context of linear algebra) and this allowed them to see how mathematics is done in real life by real mathematicians, but I wanted to encourage them to reflect also on what mathematics is in broader terms. Therefore I decided to add a reading component to the course. It is centered around a general theme, and is not directly related to the content of the course but mainly to the composition of the course audience. That is to say, I use the component for this course precisely because it is a course taken by a wide range of students with differing interests who have yet to develop their own mature understanding of mathematics and its place in the world. One could therefore conceivably integrate it into a range of lower level or transitional level courses such as college algebra, calculus, and differential equations. Introducing distinctly philosophical arguments about mathematics at an early or transitional stage of the students' mathematical lives enriches their mathematical experience while deepening their intellectual commitment to the discipline.

3 Philosophical Questions

The first question addressed by the reading list and the accompanying assignments is the original one Eugene Wigner raises: how does mathematics developed to solve certain problems turn out to be effective in solving problems in a totally unrelated context? Accompanying this is the problem of unforeseen applications of mathematics developed purely for reasons internal to mathematics. These two are important questions that intrigue many people today as can be seen from the ubiquity of the phrase “unreasonable effectiveness of mathematics.” A simple Google search done on August 21, 2013, for this phrase yielded about 660,000 results. The interest is real and is contemporary: two paper sessions at the 2014 Joint Mathematics Meetings held in Baltimore revolved about this theme.

Underlying the problem of the unreasonable effectiveness of mathematics is another, perhaps a more fundamental question: why is mathematics useful at all? If we humans are simply theorizing and creating these abstract mathematical structures from our intellect, as some philosophers suggest, why should any of them correspond to anything external to ourselves? And even if we might be inspired by some aspects of our surrounding world, and the real world is what triggers us to start the mathematical process, as others propose, why do our resulting theories still apply to the world, predicting events and phenomena so effectively?

Careful readers of Wigner's original piece will know that the last question was not the one he raised. Wigner took for granted that if we model a real world problem mathematically, it is natural to expect the model to help us solve the problem and its relatives and predict phenomena related to them. For him and many others, this is reasonable effectiveness; for if we model a real world phenomenon with mathematics, why shouldn't it give useful predictions about it?

One might notice that any approach to the question of effectiveness has some implicit assumptions. First there is the issue of whether mathematics is created or, alternatively, discovered by humans. Clearly people disagreeing on this question will disagree on many others about the nature of mathematics. If the universe and its laws are written in mathematics as Galileo suggests, then the mathematician's task is simply decoding this source code. In that case, mathematics is merely discovered by the human mathematicians, and the effectiveness of mathematics is not an issue at all. Yes, mathematics is useful, because it has to be. It is simply the language of the universe and there is nothing surprising about its usefulness, as it is ingrained in the very fabric of the universe. (Of course the question of just what it might mean to say that mathematics is a language is in itself a difficult one. See below for more on this.)

However if we think that mathematics is a formal game played according to rules some humans have devised – in other words if we espouse some version of a formalist philosophy of mathematics – what happens then? Why should mathematics be applicable? To be more explicit, if mathematics is mainly a mental game with ad hoc rules, why should it apply to our world? How can our brains abandoning the real world and simply creating abstract mansions in the air frequently come up with patterns and constructs that turn out to be useful? It seems that the question of the unreasonable effectiveness of mathematics is a tough one for the formalist.

The humanist who accepts that the phenomenon of mathematics is purely human, a social construct, also has to respond to this crucial question. How can a purely human endeavor correspond so well to the external world that exists and persists outside the human realm?[1] If our human psychology and our social norms are major players in our mathematics, then why do the results we reach still seem "objective" and "correct"? Why do the answers our human mathematics gives to problems of the non-human world work?

The platonist is perhaps the one most at ease. According to the platonist, mathematicians are working with a blueprint of the universe as they abstract things. In other words, their work focuses on the ideal case for the universe, and it naturally applies to it. The mathematical objects resemble the real world objects we see and touch every day. Thus our abstractions of them also resemble the real world and hence can eventually say something about the real world objects in return. On the other hand, some philosophers may argue that the problem is most extreme for platonists, since it is not automatically clear why (and how) some abstract realm of Forms (or wherever these abstract mathematical objects should reside) should have anything to do with our physical world. A modern platonist perspective on the problem of applied mathematics can be found in [2]; see especially Chapter 4.

Another question the reading list raises (as witnessed almost invariably in my students' discussions around the time we get to Hamming's article [10]) is the question of the true nature of mathematics. What is mathematics really? Many wax poetic (or otherwise assert knowingly) about it being the language of our universe.[2] But just what does it mean for mathematics to be the language of the universe?

Accompanying these questions is one about the epistemology of mathematics. How do we come to discover or get to know mathematics? Is it already embedded in our world? Are we born with some mathematical structures and mathematical thinking paths embedded in our brains? If it is really about our universe all along, how can many mathematicians live seemingly immune or completely blind to the realities of the world, and at the same time make sensible statements about it?

All these questions and their relatives come up in the discussion forums over the course of the semester. Students puzzle over them and flex their little-p philosopher muscles.[3] They are gentle with one another but still are unafraid

[1] Of course one may argue about the independence of the human realm from the external world, or even debate whether there are two such distinguishable realms. These may be interesting questions for our students. Today intriguing and complex partial answers to such questions are offered, among others, by the notion of *embodied cognition*. A mathematically relevant source accessible to the general reader is [15].

[2] See for instance [11, p. 157]: "[M]y title is also a riff on Wigner's well-known aphorism about the unreasonable effectiveness of mathematics in physics. We can understand the effectiveness of mathematics: it is the language of physics and any language is effective in expressing the ideas of its subject."

[3] The idea that there is *Philosophy* and then there is *philosophy* is a main theme in the Philosophy for Children (P4C) movement. Philosophy with a big P is reserved for big names, wise and mostly dead people, and, of course, academics. On the other hand, philosophy with a little p can be a part

to question one another's assertions. Along the way the class sees clearly that mathematics is a philosophically rich discipline that does not allow for simplistic answers.

A Sidenote

The organizers of the JMM 2014 MAA Contributed Paper Session titled "Is Mathematics the Language of Science?" pointed out in their call for submissions[4] that:

> Discussions of Wigner's article often reflect the assumption that mathematics has relevance only as a means of exploring the physical world: as Wigner puts it, "in discovering the laws of inanimate nature." Many mathematicians would find this an unacceptable restriction on the definition of their pursuits and activities.

I too belong to the group of mathematicians who would be concerned with the assumption that the sole reason for the relevance of mathematics is its applicability. Still I believe that the focus on applications is a good idea for this level of students. The applicability of mathematics is a deeply interesting philosophical problem on its own (see for instance [23]), but more relevantly, it is often the main external justification for most students at this level for learning mathematics. Furthermore, several of the readings bring up other reasons to pursue mathematical activities. Many students appreciate finding other reasons to do mathematics, and most are surprised by the mere existence of a philosophical dimension to the whole mathematical enterprise. All in all, I believe the reading component encourages in young minds a more nuanced understanding of mathematics and its place in our world. There is more to read and think about of course; this is just a start.

4 The Readings

Here is the basic skeleton of the reading list I use in my classes; see the end of this chapter for a brief description of each:[5]

An excerpt (pages 5–10), What is mathematics, chapter 2 from *The Mathematician's Brain* by David Ruelle [21].

Eugene Wigner, The unreasonable effectiveness of mathematics in the natural sciences, *Communications in Pure and Applied Mathematics*, 13 (1) (1960), 1–14.

R. W. Hamming, The unreasonable effectiveness of mathematics, *American Mathematical Monthly*, 87 (2) (1980), 81–90.

Sundar Sarukkai, Revisiting the "unreasonable effectiveness" of mathematics, *Current Science*, 88 (3) (2005), 415–425.

Alex Kasman, *Reality Conditions*, Mathematical Association of America, Washington DC, 2005.

Arthur M. Lesk, The unreasonable effectiveness of mathematics in molecular biology, *The Mathematical Intelligencer*, 22 (2) (2000), 28–37.

Arthur M. Lesk, Letter to the editor: Compared to what? *The Mathematical Intelligencer*, 23 (3) (2001), 4.

Lev R. Ginzburg, Christopher X. J. Jensen, and Jeffrey V. Yule, Aiming the "unreasonable effectiveness of mathematics" at ecological theory, *Ecological Modelling*, 207 (2007), 356–362.

K. Vela Velupillai, The unreasonable ineffectiveness of mathematics in economics, *The Cambridge Journal of Economics*, 29 (2005), 849–872.

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of anybody's life as the underlying components are in all of us from the beginning: a wonder about the world and curiosity that seeks knowledge through inquiry. See [12], [18] for an elementary introduction to P4C.

[4] jointmathematicsmeetings.org/meetings/national/jmm2014/2160_maacall, accessed August 22, 2013.

[5] Readers might wonder about copyright issues. Indeed this is a complex question. However, I am able to provide in our learning management system direct links to most of the original articles, provided through our library. This is the ideal way to provide access to students and it takes care of most of the readings. It is occasionally the case that a source will not be covered by an institutional subscription; in these cases I upload to the learning management site the digital copy that I have obtained through other (legal) means and signify that the students' use is covered by fair use standards. Fair use is a complex notion and no case is clear-cut; each individual has to make the judgment for their own context. Readers may find more information on fair use here: copyright.gov/fair-use/more-info.html (accessed June 22, 2015). In case of doubt or concern, institutional librarians will typically be happy to provide guidance.

In one of the sections of the course we finished the term with

James Gleick, But aren't truth and beauty supposed to be enough? from the August 12, 1986 issue of the *New York Times*.

In another we wrapped the term up with

Vicki Kirby, Enumerating language: "the unreasonable effectiveness of mathematics", *Configurations* 11 (3) (2003), 417–439.

Here are a selection of other possible readings (listed in alphabetical order by first author) for instructors to consider for their own version of this component:

Thomas R. Anderson, Progress in marine ecosystem modelling and the "unreasonable effectiveness of mathematics", *Journal of Marine Systems*, 81 (1-2) (April 2010), 4–11.

Mark Colyvan, The miracle of applied mathematics, *Synthese*, 127 (3) (2001), 265–278.

Steven French, The reasonable effectiveness of mathematics: partial structures and the application of group theory to physics, *Synthese*, 125 (1–2) (2000), 103–120.

Catherine A. Gorini, The natural role of mathematics in the sciences: how Maharishi's Vedic science answers the question of the unreasonable effectiveness of mathematics in the sciences, *Humanistic Mathematics Network Journal*, 25, 12–19.

Ivor Grattan-Guinness, Solving Wigner's mystery: the reasonable (though perhaps limited) effectiveness of mathematics in the natural sciences, *Mathematical Intelligencer*, 30 (3) (2008), 7–17.

Alon Halevy, Peter Norvig, and Fernando Pereira, The unreasonable effectiveness of data, *IEEE Intelligent Systems*, 24 (March/April 2009), 8–12.

Roland Omnès, Wigner's "unreasonable effectiveness of mathematics", revisited, *Foundations of Physics*, 41 (11) (2011), 1729–1739.

Max Tegmark, The mathematical universe, arxiv: arxiv.org/abs/0704.0646v1.

Jean Paul van Bendegem and Bart van Kerkhove, The unreasonable richness of mathematics, *Journal of Cognition and Culture*, 4 (3–4) (2004), 525–549.

This is merely a subset of possible readings that could complement the semester-long project. I would be very much interested in reader suggestions to add to the list.

5 How I Used the Readings in My Course

The first time I used the readings I simply assigned them as reading outside class, and included a simple (and perhaps too vague) essay prompt in the take-home final exam:

> In several readings for this course, you have tackled the idea of the "unreasonable effectiveness of mathematics." Describe the evolution of this phrase in the readings, focusing on at least three of them. Your write-up should be an essay of at least one page. Make sure to add your own examples and counterexamples to the discussion.

Not being trained as a writing instructor, I did not really know what would make a good essay prompt. Indeed, the above was not a great one, but it led to some interesting essays that I really enjoyed reading. Nonetheless I was concerned that the reading component was not fully and systematically integrated into the course; it sort of dangled on its own at the end of the semester. I thought a bit about how I could incorporate the project into the course more seamlessly. I also wanted to make sure the students engaged with the readings throughout the semester in a sustained manner. If the only time they needed to be accountable for the readings was the take-home final, students could conceivably read a handful of them at the very end and still manage to respond to the essay prompt in a reasonably thoughtful manner. However since my goal was to encourage students to think about the main philosophical questions raised in the readings while they were actually learning linear algebra, I really did not want to enable this as a feasible option.

In order to address these two concerns, I decided to require that students comment on a discussion forum associated to SAKAI, the course management system we use for our courses here at Pomona. More specifically, after each reading assignment (which came to approximately one reading per week or two), I would start a new discussion thread on the course management system discussion forum, and ask that students write thoughtful comments on the reading. My prompts for these threads were even more vague than the essay prompt quoted above. Here is an example:

> This is the discussion forum for the first reading assignment: “What is Mathematics?”, an excerpt from David Ruelle’s 2007 book titled *The Mathematician’s Brain*.
>
> The reading was assigned on January 25, 2013, Friday, and the discussion comments are due by February 1, 2013, Friday. Simply click on *Post New Thread* to start a new line of discussion related to the reading. Or if you have something to add or respond to motivated by an earlier post, you can simply *reply*.

Then after the deadline for each reading assignment, I read student responses and assigned grades. Students were graded only according to whether they made a substantive comment on the reading or responded thoughtfully to what their classmates had written before them. In other words I did not try to evaluate their contributions; as long as it seemed as if they made an effort to join and contribute to the conversation, they received full credit for the assignment.

Here is how I first introduced the reading component in my printed notes for our first lecture:

> There is also a reading component to the course. This series of assigned readings will be included in the SAKAI site for the course. You are expected to write a brief (250 word minimum) response message about each reading on the relevant SAKAI thread. It might be more interesting and helpful to respond also to the students who wrote before you. You can, if you feel the urge, write more than one message per thread of course.

The reading component forum comments totaled up to 10% of the whole course grade. This was enough to make it an incentive for most students to contribute regularly, but it was also not an exorbitant amount; some students only occasionally contributed to the discussion forum, most often when they thought they had something original to say. For readers who might be wondering, I should add that I did not have any complaints about the amount of work the component involved or the percentage of the overall course grade.

The final exam contained an essay prompt that resembled the one quoted above. However this time I also wanted to encourage students to tweak it if they felt moved to do so. In other words, they could come up with and then write on a different prompt if they found that would allow them to touch upon the main theme of the unreasonable effectiveness of mathematics. I did not have many takers for this flexible option.

6 Effects and Outcomes

Here is how one student summarized the reading component in his final (Spring 2013):

> The notion of the “unreasonable effectiveness of mathematics” may seem somewhat silly at first glance. How can something be so effective as to seem unreasonable? Surely a field as applicable to so many parts of life as mathematics should be considered successful when new and diverse applications to other fields are discovered. However, throughout the course of the class readings on the subject, several authors do demonstrate just how uncanny the applications of mathematics are to fields both obvious and foreign. Through an examination of the course readings, it’s clear to see that the “unreasonable effectiveness of mathematics” can grow in a person’s understanding from little more than an abstract concept bordering on the absurd, to a truly fascinating and slightly uncanny phenomenon spanning multiple and varied disciplines.

By forcing the students to think critically about the philosophical issues around the applicability of mathematics in our world, the reading component leads students to a more comprehensive view of mathematics. It helps them develop for themselves a more sophisticated, or at least a more nuanced understanding of what mathematics is.

I also surmise that the reading component may lead students to ponder why they are taking the course and what they are getting out of it. I push them to tackle the question explicitly by another essay question in the take-home final. Here is a part of the relevant essay prompt:

> Why did you take this course? Where does linear algebra stand in your future career? Explain with examples. Emphasize what you learned and how you expect to use it.

This kind of personal introspection I believe allows for a more reflective, more conscious, and thus a more effective learning experience. Accompanied by the diverse selection of applications of linear algebra we study in class, the philosophical discussions have some backing and students can make their own arguments about what mathematics is and where it situates itself within their own world.

Furthermore, almost all students find at least one essay that focuses on the role of mathematics in a field they care deeply about (physics, biology, and economics are the top three majors among those who are not considering a mathematics major). Thus each student seems to be intrigued by at least one of the essays and takes a personal interest in the discussion from that point onward. One of the Spring 2013 students listed the following as one of his three general recommendations to future students for succeeding in this course:

> Do the course readings. Even though reading in a math class might seem like a waste of time and a pointless exercise, [t]he readings serve as [a] bridge between the abstract concepts of linear algebra and its applications to the world at large, and you never know which article you might find unexpectedly fascinating.

7 Student Reactions

Here are some student comments from the anonymous year-end evaluations; there were many others on the same general tenor:

> [Readings] broadened my view of what math is truly about.

> [The] reading component was interesting and thought provoking.

> I think the investigation into what math is and why it is so effective is important. I know these questions may never be answered, but thinking about them and studying them gave me a new view of math as a whole and a much more philosophical outlook on the subject.

> Liked all of the philosophy – very interesting for math class.

> [Due to the readings] I definitely have a greater appreciation for the complexity and beauty of math.

Here are some excerpts from students' final exam essays:

> The reading component gave me a good insight on the reasonably [sic] effectiveness of mathematics, as I like to see it.

> [I]t seems worth noting that we are living in the world of mathematics without really understanding why it is effective. Just look at MacBook I'm using or Iphone [sic] I used to text and call.

> Before taking this course, I have always thought that math was very natural. . . . What has really changed my perspective is that sometimes when we run into roadblocks, such as taking the square root of negative one, we simply "imagine" a solution and move on. What is amazing to me is that this imaginary number ever turns up in the real world in which we exist.

> It is quite amazing how much I have learnt from readings and discussions on the forum. We started the year with a discussion about what mathematics [was] at its core. I argued that mathematics itself was not a single

> discipline, but rather a language that all other disciplines could be expressed in. While I do not think that was an incorrect claim to make, I think that our subsequent readings have shed more insight into the nature of mathematics and that my perspective has shifted.
>
> As the phrase was unraveled throughout the year through various readings, I began to focus on what it was that made mathematics so effective in areas like the natural sciences and ineffective in others. I really believe that it stems from it being able to unreasonably manipulate the complexities of the real world into simplified models.

Overall the students engaged carefully with the readings and I think they took much from them. Furthermore, the forum discussions and the final essays were a delight to read. Inside jokes, ongoing dialogues, and shared examples and experiences in the online conversations added to the creation of a sense of community for the class.

Moreover, I have found that the readings leave a permanent impression on many students. Students from earlier versions of the course occasionally write to me to share with me essays and other written material that remind them of our reading component and the conversations that ensued.

8 Odds and Ends: Adaptations, Adjustments, and What Else?

The reading component was my first attempt at incorporating non-mathematical reading assignments in a mathematics course. I have since devised different reading assignments for other courses I teach regularly. However, this one remains my favorite. The questions that the reading component introduces are almost timeless and yet very timely for students who are just encountering new mathematical horizons, beyond calculus and computations to abstractions, proofs, and mathematical structures. A transitional course is a perfect time for them to ask questions about the nature of mathematics and its relationship to the rest of the world. And it seems that most who earnestly engage with the component gain something from it.

There is always room for improvement of course. A Spring 2013 student in the end-of-term evaluation pointed out that the forum discussions were still somewhat disconnected from the rest of the course. ("Might have been nice to integrate reading component more with what we were doing in class.") I agree. I think that I as the instructor need to do a bit more to bring the philosophical discussion back to the classroom, and relate it to our common experience doing mathematics. In the current format, students engage with the material in a sustained manner, as they tackle the readings almost weekly. But the conversations remain too detached from what is going on in class. It might be a good idea to bring some of the philosophical ideas from the reading component back home to where the mathematics is done, and to specifically connect them to and underline their significance for what we are doing in the course. Half hour discussions after each assignment is due might be a good way to do this.

In previous versions, I have used rather general prompts through the semester to start forum discussion threads on the readings. It is conceivable that this may occasionally lead to repetitious commenting and circular discussions. Therefore, instructors may consider varying directions and getting more specific with prompts as students get more deeply engaged with the topic and are presumably able to handle more sophisticated discussions.

I noted earlier that I do not participate in the discussion forums to respond to student comments. Some instructors may choose to be more involved with the ongoing conversations, in order to get students to think more deeply, or to commend individuals on a thoughtful comment.

A natural question may arise in the minds of instructors about course content. The introductory course in linear algebra can have a jam-packed syllabus. In my course I often cover the same amount of material as I did before I introduced the reading component. (Of course, if I indeed incorporate in-class discussions in the future, as mentioned above, this may change.) I also have not felt the need to reduce the mathematical homework I assign weekly. The philosophical nature of the component and the limited amount of time it takes (approximately an hour a week) seems to ensure that most students do not view it as unwelcome extra work. Moreover inspired by what they view as a novel approach to mathematics, most of them perform as well as, if not better than, usual.[6]

The reading component has a general theme, and so it is independent of the specific content of the course I use it in. It could easily be integrated into a range of lower level or transitional level courses such as college algebra, calculus,

[6] I have no pre-post data to prove this assertion, but only my sense of the range of student performance in my courses to verify.

and differential equations. The associated semester-long project could be adapted to smaller or larger classes with no changes, as long as there are at least five students involved. In classes with fewer than five students, the web interface (online discussion forum) may be unnecessary; in that case students might be encouraged to work offline. For larger classes, the main issue becomes the depth and breadth of the conversations; the instructor may wish to split the class into smaller collaborative groups (of 25–30 each). Grading might be time-consuming for the instructor as the number of participants goes up, but I grade solely in terms of effort and participation, which is quite straightforward.

The component only depends on a collaborative online platform with a discussion forum feature. Educational technologies keep evolving and platforms seem to morph frequently into new ones, but I trust that a discussion forum feature will always be an option. Thus the instructor who wishes to adopt this as an online component can easily do so. On the other hand, real-life face-to-face discussions are valuable experiences. Having led such discussions in other contexts, I would suggest that mathematics students are often keen on tackling philosophical questions together with their classmates in person. If an instructor chooses to bring them into her classroom, I believe they would be productive.

Details on the Readings (Listed Here Alphabetically by Author's Name)

Thomas R. Anderson, "Progress in marine ecosystem modelling and the 'unreasonable effectiveness of mathematics'" [1]. This is an article about the power and limits of mathematical modeling in marine systems science. In argument and approach it is much similar to the ecological modeling article [5], though the focus here is more on complexity of models that might obscure the truth, whereas the earlier one focused more on the priority and importance of the ecological context.

Mark Colyvan, "The miracle of applied mathematics" [3]. This is a modern philosophical account of the original problem, which argues that the issues are far from resolved today despite some claiming otherwise. In particular, the author argues that the problems are not limited to a formalist perspective, but persist for more modern approaches.

Steven French, "The reasonable effectiveness of mathematics: partial structures and the application of group theory to physics" [4]. This article takes a specific case of applied mathematics and analyzes it to argue that the effectiveness of mathematics is not unreasonable.

Lev R. Ginzburg, Christopher X. J. Jensen, and Jeffrey V. Yule, "Aiming the 'unreasonable effectiveness of mathematics' at ecological theory" [5]. The authors provide examples of theorization and mathematization in ecology, and "argue on pragmatic grounds against mathematical literalism as an appropriate constraint to mathematical constructions: such literalism would allow mathematics to constrain biology when the biology ought to be constraining mathematics."

James Gleick, "But aren't truth and beauty supposed to be enough?" [6]. This is a short essay on the true nature of what mathematicians do, revolving around four Fields medalists. There is emphasis on the purity of the mathematical enterprise and its focus on elegance as opposed to applicability. All the medalists in the article reject the computer as a mathematical tool.

Catherine A. Gorini, "The natural role of mathematics in the sciences: how Maharishi's Vedic science answers the question of the unreasonable effectiveness of mathematics in the sciences" [7]. This article argues that "because mathematicians are studying the same principles of order and intelligence that are studied by science, but in a subjective and abstract way, mathematics is the natural language for scientists." The argument is based upon ideas from Vedic science and Vedic mathematics, and finds the power of mathematics in its dependence on human consciousness.

Ivor Grattan-Guinness, "Solving Wigner's mystery: the reasonable (though perhaps limited) effectiveness of mathematics in the natural sciences" [8]. The author argues that Wigner and others who focus on the unreasonable effectiveness of mathematics are misguided: "Instead of 'effective but unreasonable', read 'largely reasonable, but how effective?'." By reviewing the history of applied mathematics, the effectiveness of mathematics is easy to explain, but if we want to be honest, mathematics is not in fact as effective as we often claim.

Alon Halevy, Peter Norvig, and Fernando Pereira, "The unreasonable effectiveness of data" [9]. This article offers a brief discussion of the failure of mathematization of the sciences other than physics, and claims that instead of looking for elegance in simplicity, we should embrace complexity and the most powerful mathematical tool we have for dealing with it: data science.

R. W. Hamming, "The unreasonable effectiveness of mathematics" [10]. Hamming suggests that "we must begin somewhere and sometime to explain the phenomenon that the world seems to be organized in a logical pattern that parallels much of mathematics, that mathematics is the language of science and engineering." And he tries to formulate his answer, which he admits is incomplete and inconclusive. He provides new examples of the unreasonable effectiveness of mathematics and offers separate approaches, three among which are that we see what we are looking for, that we select the kind of mathematics to use, and that science (and math) indeed answers comparatively few questions. Hamming's mathematics is a human construct that surprisingly explains the universe around us.

Alex Kasman, "Unreasonable effectiveness," in [13]. This is a delightful short story about a young mathematician who goes on a search for a mysterious character who lives on a faraway island and just happens to collect all published mathematical knowledge in the world. The story proposes a surprising explanation to the unreasonable effectiveness phenomenon: that indeed the laws of our universe (or at least the part of it we can be aware of) adapt to our mathematics.

Vicki Kirby, "Enumerating language: 'the unreasonable effectiveness of mathematics'" [14]. This is a humanist's take on the assertion that mathematics is a language. As someone engaged with language through the perspectives of cultural and critical studies, the author mainly analyzes an essay by the mathematical philosopher Brian Rotman on mathematics [20].

Arthur M. Lesk, "The unreasonable effectiveness of mathematics in molecular biology" [16]. Lesk asserts that Wigner's original surprise is unjustified, and that mathematics being effective in physics is no surprise. Then he moves on to discussing the place of mathematics in molecular biology and gives examples of what mathematics can do in the field.

Arthur M. Lesk, Letter to the editor: "Compared to what?" [17]. Triggered by the contents of a textbox included alongside his article [16], Lesk distinguishes between parts of biology where mathematics can play an effective role and parts where it might be bound to remain on the sidelines. His main argument is that "historical accident plays much too important a role" in biology for the latter to be completely condensed into and described by mathematical language.

Roland Omnès, "Wigner's 'unreasonable effectiveness of mathematics', revisited" [19]. This article brings us up to date in terms of mathematics and its role in today's physics, and defends the validity and significance of metaphysical inquiry in science.

David Ruelle, "What is mathematics?" which is Chapter 2 from [21]. This is an introduction to an inquiry about the true nature of mathematics and sets the stage for the rest of the semester. The author describes the fundamental place of proof in mathematics and concludes with "mathematics, as it is currently practiced by mathematicians, is a discussion (in natural language, plus formulas and jargon) about a formalized text which remains unwritten."

Sundar Sarukkai, "Revisiting the 'unreasonable effectiveness' of mathematics" [22]. The essay begins with a rephrasing of Wigner's original question as "how is an activity of humans, driven as it is by rules we create and with human-centred ideas such as beauty, so well matched with the natural world?" The author argues that the debate around the issue "reflects as much a confusion about the mysteriousness of what mathematics is as much as its applicability." A broad overview of several philosophical approaches to mathematics follows, which introduces platonism, formalism, and intuitionism. The author concludes that "mathematics is a product of human imagination, is grounded in our experience with the world and functions like a language." In this framework, he argues, Wigner's question is much easier to respond to.

Max Tegmark, "The mathematical universe" [24]. Written by a theoretical cosmologist, this article is an inquiry into the true nature of our universe and the place of mathematics in it. More specifically, the central argument of the paper is that our universe is a mathematical structure, conceived of and perceived by human intellect, as captured by the "Mathematical Universe Hypothesis (MUH): Our external physical reality is a mathematical structure." This is compared and contrasted with the "External Reality Hypothesis (ERH): There exists an external physical reality completely independent of us humans."

Jean Paul van Bendegem and Bart van Kerkhove, "The unreasonable richness of mathematics" [25]. This is an article that emphasizes the importance of the study of the practice of mathematics. Only by trying to understand how real mathematicians do mathematics can we really understand what mathematics is.

K. Vela Velupillai, "The unreasonable ineffectiveness of mathematics in economics" [26]. The essay begins with a recap of the first few readings above, with a focus on the two Lesk pieces. Then the author argues, through historical and philosophical justifications and concrete examples, that "mathematical economics is unreasonably ineffective. Unreasonable, because the mathematical assumptions are economically unwarranted; ineffective because the mathematical formalisations imply non-constructive and uncomputable structures."

Eugene Wigner, "The unreasonable effectiveness of mathematics in the natural sciences" [27]. This is the article that introduces the now-famous phrase "unreasonable effectiveness of mathematics." Here Wigner discusses the notions of laws of nature and mathematics as a language to code our theories. Wigner is concerned with two related issues: that mathematical concepts show up and turn out to be useful in unexpected places and that there seems to be no reason for a functional theory to be the uniquely appropriate one. Wigner gives many examples and speaks clearly about the dearth of our understanding on these issues. His final verdict is that "[t]he miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve."

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VI

Geometry

Geometry has a rich history in the philosophy of mathematics, from informal geometry to Euclid's axiomatic approach. Philosophical issues range from questions about the role of proof and of diagrams, Kant's assumption that Euclidean geometry not only is the geometry of our physical space but that we inherently intuit it, to questions about the parallel postulate, the development of alternative geometries, and the discovery that Euclid's postulates were incomplete for describing his geometry. Thus there is a wealth of opportunity to consider philosophical issues in a geometry course. We have three chapters that concentrate on geometry amd several others that include it. Yet each chapter is quite different from the others.

Chapter 13: Brian Katz, using an IBL (Inquiry-Based Learning) approach to teaching Modern Geometry, works on "Developing Student Epistemologies." His aim is to bring students closer to adopting an "expert" epistemological viewpoint in mathematics (on the nature of mathematical truth, the role of proof, examples, and so on), basing his approach to epistemological growth on William Perry's work. In describing his course he also gives a good introduction to the IBL approach. Students develop their own WikiTextbook as they prove theorems and find counter-examples to conjectures. To get students to rethink their assumptions about the mathematical paradigm, they also have some readings about the foundational period, logical positivism, and Gödel's theorems. He uses students' concept maps at the beginning and the end of the course to assess students' change in their understanding of the mathematical process.

The pedagogical approach discussed in this chapter is designed to push students to develop critical thinking skills, create a personal mathematical epistemology, and engage with mathematical concepts in a more independent manner than may be typical. At the same time, they are focusing on central philosophical questions: the nature of truth, its foundations, and the methods for its development. While this is all discussed in the context of geometry, in the final section the author discusses how at least part of the philosophical aspect can be adapted to other courses.

Chapter 14: Paul Dawkins, in "Helping Students Develop Conscious Understanding of Axiomatizing," takes a formalist approach (although he is not a formalist), using a geometry course to teach students how to produce mathematically acceptable proofs and to understand the mathematical process of proving. He focuses on the use of axiomatics as an insightful example of modern proof-oriented mathematical practice, which is particularly valuable for preservice teachers who will themselves be teaching proof. What is the nature of axiomatizing and how can it be done in a fruitful and efficient way in the classroom? The course starts with a few axioms and a wide range of example "planes" that satisfy the axioms. He gradually introduces new axioms or guides students to formulate possible axioms and students investigate, in small groups, which of the example planes that they are considering satisfy the new axioms as well. The class also discusses issues such as whether to count something as a new axiom or look for a proof of it. At other times, students work on what would be the best way to frame a new axiom or theorem. Philosophical concepts focus on the role of axioms, how they differ from theorems, the role of examples, and implications of assuming or not assuming particular axioms.

Chapter 15: Unlike most of the chapters in this volume, Nathaniel Miller's "The Philosophical and Pedagogical Implications of a Computerized Diagrammatic System for Euclidean Geometry" is less about how to use philosophical issues in mathematics to teach mathematics, than about a partial resolution to a philosophical issue that we believe everyone who teaches mathematics, and particularly geometry, should be aware of. This is the issue of whether diagrams can be part of a rigorous justification of a mathematical statement. We are all aware of cases where use of a diagram is misleading—but diagrams are not unique in this. There are many of purely algebraic "proofs" that $1 = 2$, for example. What Miller has been working on, starting with his Ph.D. thesis, is showing that it is possible to develop a purely diagrammatic system in which all the proofs in Euclid's *Elements* can, in principle, be carried out. This is not the same as saying that a diagram is a proof: for the most part, we use diagrams more as motivation, or an intuitive aid, when we do proofs. But his system captures the rigor that underlies the geometric arguments in the form of how one moves from one diagram (array) to another. Further, with his computer proof system, CDEG, he has taken this a step farther: by making it sufficiently formal that it can be written as a computer program, he is showing that there is, in fact, nothing missing, nothing informally assumed.

We called this a partial answer to the question because it does not show that all diagrammatic reasoning can be made rigorous. Marcus Giaquinto, in *Visual Thinking in Mathematics* (see the Introduction for full bibliographic information), argues that it is unlikely that visual reasoning in analysis can ever be made similarly rigorous. However, faculty teaching geometry should be aware of Miller's contribution, and may find it interesting to look into the computer system. Our chapter about it is a bare introduction, to give readers a feel for the system, but full references to more detailed information are provided. The chapter also gives a good explanation of the philosophical issues involved, and some suggestions on how discussion of such issues can be brought into a geometry course. That being said, however, this chapter (and one other chapter, the final one) is here less because it is of immediate practical use than because the editors feel it is important for faculty teaching mathematics to be aware of its implications.

13

Developing Student Epistemologies

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1 Framing the Question

At the end of the undergraduate curriculum in mathematics, our majors should have learned to think mathematically. This broad skill certainly encompasses many components; looking at our curriculum, it is clear how students are expected to develop some of them, such as rigor, precision, and abstraction. There is, however, a subtle and important facet that does not make an explicit appearance in many descriptions of our major: the habit of thinking about truth like a mathematician. In other words, students should develop an epistemology that approximates that of a mathematician to hold, hone, and use as graduates. I believe that this habit is one of the most critical sub-skills and one of the most jarring when missing. For our purposes, we will use a non-standard definition of an epistemology as a perspective on the nature of truth, its foundations, and the methods for its development. In this chapter, I will describe my efforts to understand the epistemologies our majors develop, my approach to designing a course to move the students closer to an expert perspective, and some evidence of the changes I saw in my students' thinking as a result.

2 The State of Things

First, I must ask if the structure of our curriculum already causes students to develop an appropriate epistemology. Personally, I cannot think of a moment from my own undergraduate or graduate career in which students and experts discussed the nature of mathematical truth explicitly, but I did eventually develop an expert perspective. This approach worked for me, but my sense is that it is not enough for many of my students.

I teach at Augustana College, a small liberal arts college in the midwest. For the past several years, I have taught a course called Modern Geometry that serves as an upper-division elective for our majors and is required for the pre-service teachers in our department. These students have generally studied both abstract algebra and real analysis before taking Modern Geometry, and for many of our students, geometry is the last proof-based course they will take as an undergraduate. The course enrollment of approximately ten students is biased toward our pre-service teachers, but otherwise the course is perfectly positioned to give me access to a group of majors who are nearing the end of the curriculum.

Now that I have selected a group of students to study, I must gain access to their beliefs about truth in mathematics and determine what kinds of phenomena will count as evidence of holding a perspective that approximates that of an expert. I see at least two main approaches to this question. In the proof validation approach, I would try to understand the ways that my students determine if specific mathematical claims are true. This is a fascinating approach being taken by researchers of undergraduate mathematics education. For example, Alcock and Weber use Toulmin's framework [15] to understand student [2] and expert [16] approaches to proof validation. They find that many advanced students will adopt an expert-like approach to proof validation, at least when prompted by questions like those in a clinical interview.

The other main approach to understanding student epistemologies asks about the ways that students view truth in mathematics on a larger scale. I see this second approach as an abstraction of the first. The students have just spent two years learning to construct and validate rigorous proofs, which leads to a natural question: what is the nature of the products of the process we call proof? In the language of Sfard [14], students may have interiorized a basic conception of proof, may have condensed many of the repeated processes of proof, and may have reified proof, changing it from a process they employ to an object they could study. This abstract approach asks about students' thoughts concerning proof as an object.

The proof validation approach to the question of student epistemologies has the benefit that it is clear what kind of task would incite (or fail to incite) the desired behaviors. However, my interest in the epistemological issue stems from the vague sense that the students are missing the (mathematical) forest for the (proof) trees. The abstract approach attempts to measure the students' vision of the forest, and it aligns more closely with my observation that a disciplinary epistemology is not explicit in the way we discuss our curriculum.

Having selected a scale for my query, I need a method to bring student thinking about the connections between the big conceptions in mathematical arguments to the surface, and I was inspired by scholars who use concept maps [11]. For our purposes, a concept map will be a collection of terms that are arranged and displayed graphically and are connected with directed arrows, which are labeled to explain the connection. I had heard that students often struggled with the task of concept mapping, so I scaffolded the task heavily. On the first and last days of class, I gave the students the minimal set of terms that needed to be included in their map as a list: Mathematical Truth, Logic, Proof, Rigor, Lemmas, Axioms, Intuitions, Definitions, Deduction, Examples, Construction, Consistency, Decidability, Undefined Terms, Theorems and Corollaries, and Contradiction. I asked the students to put them and any other terms on index cards so that they could think freely about their organizational structure. Past experience indicates that students do not immediately have an image of a completed concept map, so I provided a high-quality example (about making concept maps [11]) to guide their process. I explicitly use the example map to ask students to avoid vague language such as "is" and "does" in favor of "articulates" and "decomposes" and other more concrete and specific verbs. The mapping task took the students approximately thirty minutes; there were few questions about concept mapping and no questions about the meanings of the terms included.

The maps produced by the students are tremendously diverse, which immediately indicates that our curriculum is not causing them to develop an epistemology that matches one particular expert perspective (or the task is not eliciting it). The reader will likely be aware that there is debate among mathematicians as to the nature of mathematical truth. On one end of the spectrum, the platonists see mathematical truth as a part of the world beyond the human mind, and humans observe this truth. On the other end, truth is entirely dependent on the formal human-made system in which it resides, and it is nothing more than the product of a syntactic game. The student concept maps seem to align most closely with a platonist perspective because they make an implicit assumption that mathematical assertions are already true or false. The student conceptions diverge from the expert platonist perspective in the implicitly assumed source of truth. The students are describing mathematical truth from the perspective of a novice learner, and their assumption of pre-existing truth seems to rest on the authority of experts like their teachers more than on a belief about truth in the universe beyond the human mind.

It is not surprising that the student concept maps communicate a student perspective. What did surprise me was the level of difficulty I had interpreting the student maps. Aside from a general sense that they were describing the ways that they come to learn and accept mathematical truth, the overwhelming majority of the connections in these maps felt odd or vague. I could find a plausible way to interpret many of them from an expert perspective, but such an effort was clearly going beyond the student's original consciously communicated thinking. In addition, there were a handful of recurring connections in the maps that contradict all expert perspectives known to me. Consider this sample student pre-term map in Figure 1.

The pseudo-platonist perspective can be seen in the way that, for the student, *Mathematical Truth* introduces *Definitions*. For an example of an odd or vague connection, see how the student has *Examples* provide *Logic*. I suspect that the student is pointing to the important role of examples in understanding logic or to the familiar learning trajectory

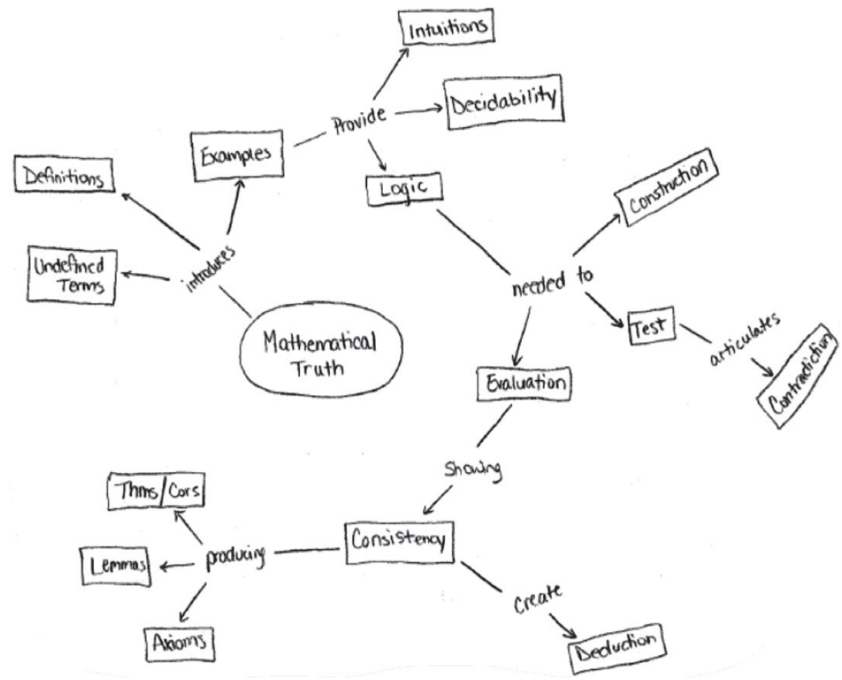

Figure 1. Student pre-term map

from specific examples to general logical patterns. With these interpretations, I am viewing the connections from the map as a piece of the mathematical process, but that piece is not central to the nature of mathematical truth as I see it. The map also contains a connection that directly contradicts expert perspectives: the student places *Axioms* as a sister to *Thms/Cors* and *Lemmas*. It is possible that this student thinks of *Consistency* as producing each of these terms in fundamentally different ways, but the shared arrow and the symmetry of that piece of the map indicate that this student believes that each of these terms is a similar kind of statement in mathematics. This structure indicates to me that this student thinks that each of these is a conclusion, possibly in a slightly different context. In the maps built by some other students, the structures communicate a similar belief about *Definitions*. In general, the structures that contradict expert perspectives center on the connections to and from *Definitions* and *Axioms*.

This perspective is quite discordant to the expert, but it is highly reasonable, even pragmatic and efficient, from the learner's perspective: definitions, axioms, theorems, corollaries, and lemmas are all examples of previously developed statements that can be used freely as true when building new proofs. Moreover, it is not clear how to tell if a statement is an axiom or a theorem without context. For example, in real analysis sometimes the axiom of completeness is assumed as part of the set of rules for working with $\mathbb{R}$. It is called an axiom but framed as part of a definition. However, in other contexts the same property is a hard-fought theorem about the way that the reals were constructed from the rationals. While collapsing definitions, axioms, and theorems into one category may be pragmatic, no expert would do so globally. I believe that this perspective explains why students often say "because the definition is true" when they mean "because the definition is satisfied" and may explain why correcting this language never seems to make a lasting change to their usage.

3 Designing a Course

The student concept maps demonstrated to me that the students could think about the question of mathematical truth but that their perspectives were not clear and coherent frameworks that approximated expert frameworks. Their maps were vague from my perspective, contained a few structural errors, and focused on the act of learning existing mathematics rather than on assessing the foundations on which that mathematics rests or on demonstrating the truth of some potentially new mathematics. I should point out that I am in no way criticizing the students' previous courses with this assessment. I would have been very surprised to find that the students had developed an expert-like epistemology at this point in the curriculum. This learning objective is not a top priority in any of their previous courses, though aspects of it are certainly foreshadowed or embedded. I think the students need these previous experiences even to be ready to think about the bigger epistemological issues actively and productively. Sfard's theoretical framework [14], applied to proof, specifically conjectures that students would only be prepared to consider mathematical truth as they were mastering and internalizing proof construction habits.

With this assessment in mind, I set out to design a course that would bring the epistemological concerns of mathematics to the surface so that the students could engage them explicitly. Fortunately, Modern Geometry is also an excellent course in which to explore the nature of mathematical truth. The geometry content is familiar to the students; so we can focus on the nature of our knowledge instead of on acquiring it. Historically, there has been explicit concern about the status of Euclid's fifth postulate. Moreover, because of Hilbert's work, geometry is often the example considered when professionals discuss mathematical truth. Humans have access to various models of geometries through computer software. Moreover, geometry purports to be discussing the (unique) structure of the universe, but my students have experience with living on a plane and a sphere, with projected maps, and even with perspective drawing. Most importantly, geometry is the most axiomatic content domain in the department, making it possible to attend to the structure of our results. Finally, in our department the students also bring experiences with proof in diverse content areas and with diverse faculty perspectives, which supports the reflection we will need.

3.1 Challenges and Responses

Taken together, these observations led me to take a purposeful approach to helping my Modern Geometry students develop appropriate mathematical epistemologies. This approach is a combination of two facets. One is a scholarly facet that engages developmental psychology research about college students. The other is a pragmatic facet that acknowledges the cognitive demands of this abstract goal and actively supports the students in managing the demands.

At the center of the scholarly facet is William Perry's research [12], which established a scheme for thinking about the cognitive and epistemological growth of college students by describing positions from which they perceive truth and knowledge. Perry's scheme has been criticized and revised, most notably to reflect the role of gender and voice in a student's development, such as in *Women's Ways of Knowing* [5]. However, there are four core epistemic stances from his framework that appear in some form in this family of theories of college student development. Students in a position of *dualism* view knowledge as black and white and believe that authorities know the difference. When in a position of *multiplicity*, students notice that much of knowledge is context-dependent and come to believe that any perspective is a valid source of knowledge. In a position of *relativism*, students begin to demand that other perspectives be justified; as a result, knowledge becomes procedural, the result of argument and evidence. Those who take a position of *relativism-with-commitment* acknowledge that their arguments must be grounded on accepted assumptions, and they become concerned with establishing appropriate and personal precepts.

It seems to me that mathematics attracts many students who believe that truth is more certain in our discipline. In other words, they see mathematics as an opportunity to maintain a dualistic perspective. A geometry course in college presents an unusual opportunity in this developmental scheme. On the one hand, the students are familiar with the core content from high school, and many of my students imagine teaching geometry themselves in the near future. As a result, they bring a strongly dualistic perspective with them to the beginning of the course: "the facts are known and I must prepare to become an authority." On the other hand, they have just accepted proof as the core methodology of mathematics. So they hold a procedural understanding of what it means to do mathematics.

Fortunately, Perry's research also illuminates at least three kinds of experiences that impel students to move to later positions: encountering questions without known answers or about which authorities disagree, engaging a pluralism of ideas among peers, and rigorously justifying claims and questioning assumptions. The first challenge is to build these experiences into the course. As a graduate student, I encountered the themes and experiences from Perry's research when I simultaneously took and taught inquiry-based courses, in which the role of authority is dramatically re-envisioned. In my experience, inquiry-based learning (IBL) courses naturally include the experiences that Perry's research indicates will move students toward the more nuanced epistemic positions. As a result, I decided to teach the geometry course in an IBL format.

To understand the connection between the desired experiences and this course design decision, you may need a more detailed description of IBL [1]. In an IBL course, the focus of the class meetings is on students exploring and explaining mathematics with their peers. The fundamental difference between an IBL course and a lecture-based course is the amount of time that students are asked to interact directly with the mathematics, as opposed to interacting with mathematics that is filtered by an expert [3]. In my courses, in place of a reference textbook and lectures, students are given a carefully designed sequence of questions that they explore and extend while developing the course material. In preparation for class each day, the students work through a portion of the text by exploring questions and proving theorems. They come to class prepared to discuss their findings and arguments with their peers. I usually ask them to work in small groups for several minutes to test out ideas on their peers and correct any errors they discover. Then the majority of class time is spent with students presenting their findings to the full group, assessing and improving the arguments, collectively accepting results, and extending their conclusions with new explorations.

The scholarly facet of the course design led me to an IBL course structure. Now we turn our attention to the pragmatic issues, some of which are the result of the decision to use an IBL structure and others of the level of abstraction needed to reflect on the nature of truth in mathematics. The IBL structure generally requires that students not engage with reference texts; without a text, some students feel that they have no solid ground on which to build. In an IBL classroom, students who are less comfortable talking in class can have fewer opportunities to demonstrate understanding and to experience success. In an IBL course, because students are always working on new theorems, there could be a lack of reflection and synthesis on the work that the group accomplished, and it could be difficult to get students to incorporate feedback to improve their previous work. The goal is to address these challenges while simultaneously building student thinking about the bigger epistemological issues.

First: what aspects of the task of reflecting on the nature of mathematical truth require scaffolding? I am trying to have my students think about their paradigmatic assumptions. They are difficult to see from inside a worldview, and hence students will need to encounter multiple worldviews about truth in mathematics. The faculty in the department could potentially give a diverse set of perspectives. However, precisely because the students do not have experience talking about their mathematical epistemologies, they are not able to perceive the diverse worldviews by observing and questioning an expert. The challenge is to find clearly articulated perspectives that the students can analyze and compare. Thus, I chose to include in the course Rebecca Goldstein's intellectual biography *Incompleteness: The Proof and Paradox of Kurt Gödel* [8]. The book does an excellent job of contrasting the prevailing views on truth, especially the Vienna Circle's logical positivism, Hilbert's formalism, Wittgenstein's mysticism, and Gödel's platonism.

Unfortunately, *Incompleteness* is a tough read for the students at this point in the curriculum because they cannot call to mind and examine a detailed example of a formal or axiomatic system. Geometry is tailor-made to play this role, but simply studying geometry will not be enough. The challenge is to turn geometry into an object that can be viewed as a whole and studied. Pulling the remaining challenges together, the students need an external object that encodes geometry, they need to reflect on the accomplishments from each day of an IBL class, they may need something in place of a reference textbook, and some students may need a way to do high quality work other than speaking in class. My idea was to have the students build a reference textbook together in the format of a wiki; we called this project the WikiTextbook [10]. I now have all the components that I need to describe my course design.

3.2 The Course

Inspired by the findings in Ken Bain's *What the Best College Teachers Do* [4], I framed Modern Geometry around a central question: what does it mean for a (mathematical) statement to be true? For about 70% of the term, the course looked like an IBL course. The students prepared theorems for class and then we spent our time together determining collectively if their arguments werere sufficient. This portion of the course used Chapters 1–7 of David Clark's IBL textbook *Euclidean Geometry* [6], which guides our exploration of the categories of congruence and similarity and applications such as a rigorous definition of π. Clark's book takes an axiomatic approach to plane geometry with a system of axioms that is somewhere between the axioms used by Euclid and those of Hilbert. Most of the axioms that guarantee the existence of intersections are missing, but the system is otherwise amenable to pushing the axiomatic rigor quite far. The missing axioms are actually a useful attribute, setting the students up to make claims that they cannot justify, which allows us to talk about having to add more assumptions and trying to minimize the number of times we must take such a drastic step. Unlike Euclid and Hilbert, the axioms are not stated at the start of the development. Instead, they are selected when we encounter a question the group deems unanswerable with our given tools. This determination usually requires either an exhaustive analysis indicating that none of our known statements could produce the needed information or that all constructions we are considering fail in a model of geometry, such as the one built into the software package NonEuclid, indicating the presence of a non-trivial but unarticulated assumption. While these methods do not prove that the question is unanswerable, the psychological impact of these moments certainly supports the students' later understanding of incompleteteness. I also like the unusual order in which the axioms appear; Clark waits for many chapters to measure angles. This choice requires the students to prove that all right angles are congruent. They want this result to be true, and this ordering adds a great sense of catharsis to the moment when we finally add the claim to our list of results. Between class meetings, the students prepared new arguments and contributed to the WikiTextbook some of the content that they had built together in class.

In the remaining 30% of the term, between classes the students polished the WikiTextbook and reflected on its structure by adding chapter summaries and a foreword. They also produced a visual representation of the hyperlinking structure of the WikiTextbook to serve as a concrete representation of the structure of an axiomatic system. In class, we explored hyperbolic geometry through one of its models. The students and I then leveraged this detailed, externalized system of truth by comparing it to other possible geometries. The students built an appendix to their textbook that classifies results as Euclidean, neutral, or hyperbolic. Finally, we used the detailed example and the experience of contrasting it with others to reflect on the nature of mathematical truth while reading *Incompleteness*.

Along the way, I realized that I wanted to use the language of Perry's scheme when talking with my students. So we read a summary of his scheme and connected it to the structure of our course. I also realized that the students would get a lot more out of the axiomatic development of geometry if they were aware of the epistemological themes at the beginning of the term, even though I claimed above that they needed the experience with geometry to understand the epistemological discussion. On the recommendation of a student and of a colleague, I read the graphic novel *Logicomix: An Epic Search for Truth* [7], which offers an explanation of the life and work of Bertrand Russell in connection with Whitehead, Frege, Cantor, Wittgenstein, and even Gödel. In the second incarnation of this course, I asked the students to read the graphic novel in preparation for the course, and we engaged it at both the beginning and end of the term.

When I talk about this course, I am sometimes asked about my workload and about teaching a diverse cohort of students. Courses taught with inquiry-based learning methods often require a substantial effort from the instructor up front to establish the new norms and practices in the classroom; in my experience, once established, the patterns make the course easier to sustain. Some faculty are concerned that IBL courses only serve the strong students, but Laursen's data [3] show that courses like them have similar or more positive learning and affective outcomes for students on average when compared to non-IBL courses and actually have a strong positive impact on underrepresented students' experience and previously-low-performing students' future grades. One possible mechanism for this effect is the flexibility to differentiate instruction in an IBL classroom; in my experience, I feel more able to support the weaker students while challenging the stronger students simultaneously. For example, every student is required to contribute original content and revisions to the wiki as part of the daily assignments; in a normal week, each student publishes three or more polished proofs, some proof summaries, and some revisions to others' published content. I give students feedback on the level of challenge of the work they contribute for the assignments so that the students are working near the top of their ability levels. The students have drafted, discussed, and revised the proofs I read carefully, so this work tends to be both quick and pleasant.

The design elements described above are intended to provide the three kinds of experiences that move students forward in Perry's scheme, starting with encountering the unknown and disagreement. The course provides daily opportunities for the students to make conjectures; some of them are new questions about geometry that I cannot answer. So we explore them together (which is my favorite part of any week). Furthermore, we wait as long as possible to give a parallel postulate, so that the students encounter a few undecidable statements that will become Euclidean theorems. At a more visceral level, some of the undefined terms come under fire; what exactly is a "straight line," and have the authorities been using the term consistently? Geometry also provides colorful examples of large-scale disagreement among authorities in the form of multiple two-dimensional geometries and their models. In particular, the students read about the attempts to prove the parallel postulate and the eventual creation of incompatible versions of axiomatic geometry. Building the appendix allows the students to directly contrast two systems of truth. But the rabbit hole goes deeper: are we discovering or are we inventing theorems and proofs? The students begin to engage in this larger historical debate.

The other key experience identified in Perry's research is justification, which is at the core of all the course activities. The axiomatic approach puts special emphasis on questioning the results quoted at each step. The WikiTextbook makes the quoting of results explicit through hyperlinking, and the students use this structure to produce a visual representation of the logical dependencies among the results in the wiki. The students are also continually revising the arguments in the WikiTextbook, adding summaries of chapters, and building a foreword to the document that articulates its objectives and epistemological stance. More than any other course in the department, Modern Geometry provides opportunities for the students to question their assumptions. The parallel postulate, which seems self-evident, is under siege and unable to repel the attacks. The incompleteness theorems have consequences for Hilbert's dream of formalizing geometry and all mathematics as well as for the relationship between mathematics and the human mind. So the readings impel the students to think about the nature of mathematical truth. The students engage this larger question by proving that six versions of the parallel postulate are equivalent in neutral geometry. In other words, they prove that six different axiomatic systems for Euclidean geometry contain the same set of truths.

4 Outcomes

I am very excited about the course design described above, but I must ask if it accomplishes the goals for which it was designed. I have been consistently astounded by the quality of the final WikiTextbook and the changes in the students' thinking about mathematical truth. For evidence of this, I asked the students to complete the concept mapping task again. In general, the post-term maps contain mostly clear and accurate connections with only a few vague structures. Figure 2 gives the (impressive) post-term map from the student whose pre-term map was in Figure 1.

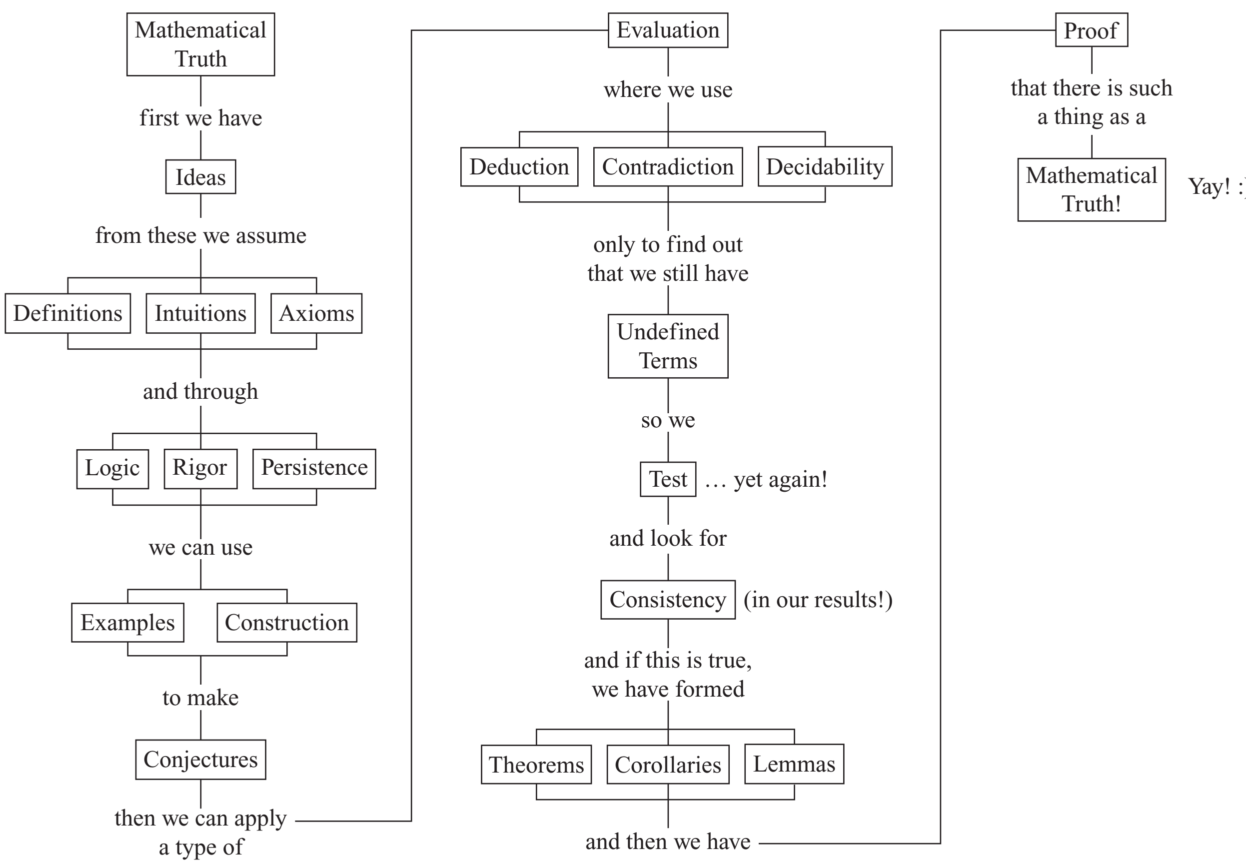

Figure 2. Student post-term map (reformatted for readability)

There was still diversity in the student post-term concept maps. Much of it comes from the lens through which the students interpreted the mapping task. Some students framed their maps around the mechanism by which an individual mind comes to learn about the truth of certain mathematical conceptions: I call this a *psychological frame*. Some students framed their maps around the actions by which the body of mathematical truth is developed and extended: I call this a *procedural frame*. Finally, some students framed their maps around the logical connections that allow truth to flow through an axiomatic system: I call this an *epistemological frame*. The post-term map in Figure 2 is an example of a consistently procedural frame. This map looks linear, though many of the students commented that their later maps were more cyclical, which I interpret as commenting on the recursive nature of mathematical inquiry.

Despite the diversity in frames, many of the post-term maps show an important commonality related to the roles of Definitions and Axioms discussed above. In all three frames, students separate these concepts from Theorems and Corollaries and Lemmas. They use language such as "assume" and "choose" to label arrows leading to Definitions and Axioms, and they use language such as "based off of" and "called upon" to label arrows from them to the rest of their maps.

In other words, many of the maps frame Definitions and Axioms as human choices as in the example above. There are at least two ways to integrate them into an epistemic position. The majority of the students came to believe in something like Kant's "synthetic *a priori* truths"; for them it is possible for axioms to be true in an absolute sense, and all theorems derived from them are true as a consequence. The rest of the students came to believe that axioms are arbitrary choices and that mathematical truth is relative to this context. In either case, these positions are reminiscent of Perry's position relativism-with-commitment. It is perhaps too easy to read the intention of commitment into a concept map. So I will support the claim that some of the students are thinking about the connections between their assumptions and the truth of their conclusions by quoting from the writing in the foreword to the WikiTextbook, written and edited collectively by the students.

> In order for mathematical statements and theorems to be proven true, a starting point needs to exist to allow ideas to build off of one another. Axioms are the starting point in an axiomatic system, but in order to get a grasp on what your starting points could be, we need some basic rules of how things work. Constructions are the ideas that establish those rules. . . . The main difference of a definition from an axiom is the fact that it does not require a proof, but rather gives a name to a general grouping of characteristics that was once discovered in mathematics. . . . That is really all mathematical truth is, the deductions we reach from objects we already know (or assume) to be true. As long as we reach those deductions in such a way that they are iron-clad and cannot be argued, we have found a new truth in our system.

The students write about "building," "rules of how things work," and "deductions"—language that indicates a belief that knowledge is the result of these student-vetted processes. The students also write about a "starting point" of ideas that are known or assumed to be true—language that indicates the belief that arguments must be grounded on chosen commitments. Impressively, this writing indicates an awareness of the active role of commitment with the parenthetical comment in the penultimate sentence. While this student writing from the foreword shows significant progress in Perry's scheme, there are important differences between the perspective it articulates and that of a mathematician. The students make it clear that a definition does not need a proof, but they simultaneously suggest that axioms do need proofs. Similarly, the choice of the word "objects" jumps out: have they selected a confusing term to reference mathematical statements, or are they referring to objects such as triangles?

There is definitely still room for improvement in the course. The concepts of Decidability, Consistency, and Undefined Terms, which are discussed in Incompleteness, do not appear to have been clear for the students. Moreover, none of the students explicitly integrated the incompleteness theorems into their concept maps of Mathematical Truth. I also think I can do far more to help the students make use of their reading of *Logicomix* before and during the term.

5 The Big Picture

I think that the direct approach detailed above for teaching students about the foundations of mathematics requires that they be familiar with an axiomatic system and hence cannot be copied directly into many other courses. However, working on this project has made it clear that I do try to develop student epistemologies in other courses. An introduction to proof course focuses on transforming students' beliefs about mathematical justification [9], and I often explicitly engage the belief that examples cannot prove and that a list of true statements can fail to be a proof [2]. Abstract algebra allows me to discuss the human role in building definitions and choosing whether or not to make them minimal. Real analysis can be particularly fruitful because of the appearance of the axiom of completeness and the abstract ways that proofs demonstrate existence. I think that IBL courses in general help students negotiate the norms and practices of the mathematical community as they develop mathematical autonomy [13, 17], and I discuss Perry's scheme with my other students and advisees.

I have framed the course above as trying to move the students forward in Perry's scheme. This work was recently acknowledged publicly when the course was designated as exploring "human values and existence"—one of the six required learning perspectives of Augustana's core curriculum. Moreover, I think the course experience is one of the key elements in preparing our pre-service teachers to use modern teaching techniques in their classrooms and to accept the role of authority for themselves appropriately. One student wrote that "we get to build everything ourselves . . . it was a pleasant transition from being taught to learning," indicating that he does have the skills and maturity to continue developing after college.

I have come to believe that the core goal of a college education, especially a liberal arts education, is to move students to the later positions of Perry's scheme because this is precisely what a person needs in order to become a mature, flexible generalist who will continue to develop after school. Because mathematics is so firmly rooted in proof, we are one of the disciplines that is best suited to helping students understand and internalize the habits of justification. Moreover, because the objects of our study are abstract, we are able to entertain hypotheticals more easily than most other disciplines. This work is exactly what makes it apparent to students that they must make assumptions and choose definitions. In other words, mathematics is perfectly situated to contribute to the work of the liberal arts.

Almost presciently, Perry himself noticed changes in the most common starting position of first-year students, which he connected to the marked changes in our national dialog and the role of authority therein. Analogously, I believe that Perry's scheme is going to be a key tool for understanding the waning public faith in higher education and for responding to the changing needs, perspectives, and skills of students entering college during the next decade, especially regarding the authority of hands-on parents and of information technology.

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14

Helping Students Develop Conscious Understanding of Axiomatizing

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> I don't really understand why do we have to show this. I know this [is true], why do I have to prove it, since I already know that?—Aerith (pseudonym)

> Though something may seem trivial now, it is helpful to have an understanding of what "common sense" ideas are provable and therefore usable to examine future behaviors and which are not.—Matthias (pseudonym)

These two quotes, taken from students toward the end of two proof-oriented university mathematics classes, portray a stark divergence in their understanding of the nature and purpose of proving. Aerith was a mathematics major who participated in my dissertation study on real analysis instruction. Even though she learned how to successfully prove from a very experienced and skillful instructor, she had not developed a conscious, metamathematical rationale for the game she played so successfully. Matthias was a mathematics education major who participated in my ongoing teaching experiments in axiomatic geometry[1]. In response to my experiences with students like Aerith, an express goal of these experiments is to simultaneously help students succeed in proving propositions rigorously and reflect on their proving to understand the nature and purpose of axiomatization. I value Matthias's explanation as some evidence of success for two reasons. First, he distinguishes "common sense" from what is "provable." Second, Matthias's explanation suggests that he is learning to apply mathematical skepticism [4] to his intuition, leading him to value proof over his sense that mathematical facts "seem trivial." These aspects of Matthias's metamathematical thinking are essential components of students' process of comprehending and valuing mathematical proof.

For historical, psychological, and empirical reasons that I shall lay forth in this chapter, I have found that neutral, axiomatic geometry is a fruitful context for achieving my goals of teaching students how to produce mathematically acceptable proofs and to understand mathematical proving. Euclid's *Elements* served for millennia as the most celebrated paradigm of axiomatics and rigorous proof. The advent of non-Euclidean geometries then deeply influenced the modern philosophies of mathematics. Psychologically, geometry's connections to perceptual and sensorimotor intuitions force students to confront and elaborate the relationship between their mathematical intuitions and

[1] My experiments include cycles of observing student thinking and reasoning about mathematical and metamathematical topics, designing curricula to address learning difficulties I identify, and teaching experiments to test the efficacy of the curricula with regard to my stated learning goals. This research agenda simultaneously serves purposes of scholarship in the science of mathematics education and of resource development for teaching practice (both that of my colleagues and myself).

mathematical formalization. Empirically, I observe in my experiments that, although spatial intuition can hinder students' sense of necessity for formalization, it can serve as an essential tool to guide students in reinventing axioms of planar geometry.

1 Instructional Context

In this chapter I set forth how I have been adapting my colleague Harvey Blau's geometry text [5] to explicitly foster students' metamathematical understandings of axiomatics. Many aspects of the method generalize to other mathematical content. So here I attempt to distinguish what is cognitively and mathematically essential for the process from what is particular to the body of theory in the text.

Northern Illinois University is a mid-sized, Ph.D.-granting institution with Carnegie classification "research-high." The book grew out of Blau's notes for teaching the course over many years here at NIU; as a result it sets forth appropriately challenging, but attainable, learning goals for our student population. We teach the course to advanced undergraduate mathematics majors, and it is only required for students specializing in mathematics education (i.e., preparing to teach high school mathematics). The course is required of those majors by state law because of geometry's presence in the secondary curriculum. I perceive it is doubly important for pre-service teachers to develop metamathematical understandings of proof because they have to help teach proof practices to their future students [18]. Their proof activity as teachers may not require the complexity and sophistication they encounter in undergraduate proof-oriented classes, but they need to understand the standards and values that underlie proving practice among mathematicians. Because our department does not require a specific transition to proof course (proofs are introduced in linear algebra), geometry is most students' first predominately proof-driven course.

2 The Challenge of Metamathematical Teaching and Learning

What does it mean to metamathematically understand axioms or definitions? I encourage the reader to ponder this question for a moment before reading further. For instance, how would you complete the sentence, "If students understood [axioms, definitions, proofs, etc.] properly, they would be able to"? Obviously such learning goals must be specified before they can be embedded into instruction. I have commonly heard advanced mathematics instructors complain that students do not understand one of these abstract processes, but they usually identify this in terms of token mistakes that students make in the course of their mathematical work. The problem with addressing these kinds of mathematical mistakes is that students likely were not thinking about the meta-concept of definition (for instance) when they solved a given problem. Rather, the student was thinking about lines, segments, or half-planes: the mathematical objects to which the definition refers. As a result, it is very hard to talk with students about the meta-theoretical matter until they themselves have reflected on the defining, axiomatizing, and proving processes. Simply telling students the rules ("use the definition" or "only using axioms") does not help them understand why the rules make sense or are important to doing mathematics. I perceive that a different approach is needed.

In my geometry class, I try to help students understand axioms, definitions, and proofs by guiding them to reinvent them. Then, guiding students in reflective conversations about their mathematical activity enables them to construct metamathematical understandings rooted in their mathematical experiences. The reader may have noticed that I intentionally frame axiomatizing, defining, and proving as gerunds (i.e., actions, processes, practices). I have two reasons for this. I already mentioned one reason pertinent to pre-service teachers: the guiding documents for K-12 mathematics instruction in the United States [18], [19] emphasize proof as a mathematical process through which all topics should be taught. While secondary teachers will not teach real analysis content in their classrooms, the processes of formalizing mathematics constitute the link between their university learning and their future instruction. The other reason I implement reinvention comes from the instructional tradition known as *Realistic Mathematics Education* [13], [14], [15]. I am strongly sympathetic to the arguments of Dutch mathematician Hans Freudenthal that students must be engaged in mathematical activity instead of being presented with the ready-made products of someone else's mathematical activity. While I cannot elaborate more fully on that body of instructional design and research work, it is important to acknowledge how it inspires and informs the current project.

3 Classical versus Modern Axiomatics

Thus, the goal I have set forth is for students to understand axiomatics through engaging in axiomatizing. What then is the nature of axiomatizing and how can it be done in a fruitful and efficient way in the classroom? De Villiers [9], following Freudenthal [13], articulates an important distinction between classical and modern interpretations of the nature and purpose of axioms. Classical axioms (à la Euclid) sought to clearly and precisely describe some known mathematical structure (the plane) in a hypothetical-deductive framework. Because axioms were descriptions or abstractions of familiar objects, they were generally assumed to articulate intuitively obvious statements. Axioms thus constituted, in this viewpoint, undeniably true claims. This process yields two primary results: conclusions derived from true claims are reliably and undoubtedly true (verification) and the axioms themselves articulate the properties fundamental to the original system (intuitive illumination). While spatial intuition provides people with a useful, gestalt way of reasoning geometrically, axioms in the classical sense break it up (literally *analyze*) by revealing its primitive, essential components. For example, consider how Euclid's fifth postulate formalizes the basic intuition of parallel lines in a plane. The statement identifies parallel lines via angle sums in transversals. The corpus of theorems proved from it reveals that many other planar phenomena depend upon or can be formalized by this property (e.g., angle sums of triangles). Another example is separation. Euclid made implicit use of separation without axiomatizing the notion, but the principle still holds that planar geometry, as we perceive it, entails the idea that a line separates the plane into two disconnected sides.

Modern axiomatics differ from this classical interpretation by emphasizing formal provability relationships rather than intuition. The key shift occurred when mathematicians like Gauss (1777–1855), Bolyai (1802–1860), and Lobachevsky (1793–1856) deduced the consequences of denying Euclid's fifth postulate and interpreted them as mathematically acceptable, though spatially counterintuitive. Though objects like the Beltrami-Klein model of hyperbolic space (which I use in my class) later provided visual representations of hyperbolic space, my experiences with students suggest that they find them spatially counterintuitive because distance does not correspond to the space between points. Based on the counterintuitive reformulation of geometry throughout the nineteenth century (alongside counterintuitive findings in analysis, set theory, etc.), mathematicians' perceptions of the nature and role of intuition in formal mathematics shifted. The intuitive criterion for creating and assessing axiomatic systems no longer seemed viable, so other criteria arose: independence, completeness, and consistency. Axioms no longer needed to be intuitive per se; axioms are valued and justified to the extent that they afford proofs of important results (or afford more efficient proofs thereof).

I set forth this brief history of axiomatization because I perceive an important parallel between the historical intellectual revolution and how experiences in axiomatic geometry help bring to the surface the same questions for students. The text I use has an initial chapter that traces a history of the parallel postulates and hyperbolic geometry. However, I find that students at that point are not ready to understand the significance of the story because they have underdeveloped conceptions of the role axioms play in mathematical theory (much less what it means to vary axioms). Thus, I attempt to engage students in the process of axiomatizing before reflecting on the history of axiomatization later on.

As I mentioned, there is a relationship between our perceptions of the nature of mathematics and the rules that we employ in formalizing mathematics. Guiding students to reflect on axiomatizing helps me as a teacher because it guides students sense-making, and because students' explanations about axioms reveal their underlying ways of reasoning about proof-oriented geometry itself. I provide examples of this toward the end of the chapter.

I even find students' conceptions of the axiomatic method correspond with their strengths and weaknesses in proving to some extent. For instance, students in my teaching experiments who interpret axioms as descriptions of the true geometry (Euclidean, spherical, or hyperbolic) tend to rely heavily on spatial intuition to construct proofs. This gives them some guiding insight and a sense of mathematical conviction, but they sometimes struggle to formalize their thinking appropriately ([2], [21], [22]) or to adhere to mathematical structure (numbers, sets, functions, etc.). Students who identify formal relations among statements in our axiomatic system (provability) tend to show more proficiency in precise proof construction and formal thinking, but at times lack the guiding insights of spatial intuition ([3], [1]). I shall elaborate more on these categories later, but this introduces a central hypothesis I am pursuing in my experiments: that there exist important relationships among students' interpretations of mathematical objects, their ability to produce valid and precise proofs, and their metamathematical understandings.

4 Instructional Materials

I am indebted to Harvey Blau for his work in developing the teaching materials I use in these experiments. What I describe in this section about the geometric body of theory reflects his instructional design work. Blau [5] sets forth 21 axioms of planar geometry over nine chapters. The text begins by introducing example planes (e.g., Euclidean, spherical, hyperbolic, Minkowski), though the text later introduces many more. Each plane is defined as a collection $(\mathbb{P}, \mathbb{L}, d, \omega)$ where $\mathbb{P}$ is the set of points, $\mathbb{L}$ is a set of subsets of $\mathbb{P}$ called lines, $d : \mathbb{P} \times \mathbb{P} \to \mathbb{R}$ is a real-valued distance function (the text uses the abbreviation AB for $d(A,B)$), and ω is the supremum of the set of all distances (called the diameter of the plane). The course usually culminates in proving a characterization theorem by which any plane satisfying all 21 axioms must be Euclidean, spherical, or hyperbolic. (ω is used to distinguish spherical planes where distances are bounded from others in which distances can be arbitrarily large.) The text acknowledges that more powerful axioms could reduce the absolute number of axioms, but the mathematical content of each axiom is kept manageable to portray the progressive introduction of structure to the system. The first axioms characterize distance, the relations between points and lines, and the *betweenness relations* among points on lines, which are defined in terms of distance equations: "B is between A and C," written "A-B-C," if the points are collinear and $AB + BC = AC$. Later axioms introduce separation, angular distance, and finally triangle congruence.

Finite examples can satisfy the first ten axioms, allowing students early in the semester to create their own planes (often consisting of just three to seven points) by specifying the sets defining lines and a tabular distance function. After the introduction of the first group of seven axioms, the text provides homework tasks of the form, "Construct a plane that satisfies Axioms 1 through n, but not Axiom $n + 1$." While the professor would view this as proof of axiomatic independence, students prior to instruction interpret the task in a variety of ways depending upon their perception of axiomatics. Activities of this type recur in the text when students can either create the required plane or identify it from among the examples provided.

The abundance of examples is important for students' reflection on axiomatization for several reasons. First, students are encouraged to interpret the meaning of the axioms across different planes. Like classical Euclidean axioms, I find that some students interpret axioms as descriptions of familiar planes (usually the Euclidean or spherical), especially if they have a propensity toward visual reasoning. While reasoning about familiar planes can be valuable (we often use them to motivate axioms), I notice that students with a strong descriptive interpretation of axioms struggle to distinguish the properties formalized by the axioms from other statements they perceive to be patently obvious. Second, students may recognize that axioms are not always true contrary to a common belief students hold about mathematical axioms (compatible with the classical view thereof).

Third, the example planes reveal the contribution of the axiom to the geometric system by showing what is possible when the axiom is absent or violated. Students interpret this in two complementary ways: the wonky examples motivate the need for the axiom to help the system describe the standard planes better and the introduction of the axiom rules out the idiosyncratic plane from the set of exemplars. These two viewpoints are compatible, but not identical. If students view axioms as being about familiar planes, strange planes show the need to form better descriptions of the standard examples. If students understand that the axiom system is intended to refer to a broad set of examples (a more sophisticated view), adding axioms simply reduces the set of exemplars like search criteria on an internet query. Finally, if students understand the set of axioms as an independent hypothetical-deductive entity, then the idiosyncratic examples exist to prove that new axioms cannot be proven from the previous set (consistent with the modern view of axioms). I have found it particularly effective to present the third interpretation to students by asking whether a statement is an axiom or a theorem. While both types of statements may be considered true, theorems are distinguished from axioms by being provable.

Blau's [5] overall pattern of axiomatic development could naturally be replicated in other mathematical domains (see [7] and [23] regarding algebra), but I find it important that the axioms appear over time and in dialogue with the set of example planes. One of the challenges of teaching geometric axioms is helping students balance the role and influence of visualization and diagrams when constructing proofs. While I try to keep students from abandoning their spatial intuitions, they also must understand when to reason spatially and when to suppress intuition (recall Matthias's quote). Students may have to suppress intuition when proving because some intuitive claims are not yet provable (and thus not yet true) with a given set of axioms. Students should call upon spatial reasoning when trying to produce new axioms, though spatial reasoning can guide proof production as well. I hypothesize that students who compare and relate their spatial reasoning and formalization will better connect their primitive intuitions to the class's body of theory. I anticipate this will help them gain greater control over their proving activity. So, by including many axioms in tandem with a wide range of examples, Blau's text provides rich fodder for engaging students in axiomatizing through the method of guided reinvention ([14], [15]). In what follows, I shall expand on my adaptations of his curriculum and provide further insights I have gained regarding students' learning.

5 Reinventing Axioms

I find it somewhat unfortunate that the tradition of axiomatic systems uses the term *model* to refer to any analytical representation of an axiomatically constructed plane (such as the Cartesian plane to Euclidean axioms, or the Klein-Beltrami model of the hyperbolic plane). This is almost the opposite of the use in mathematical applications, where we use more abstract mathematical tools (equations, functions, statistical methods) to model or formalize some experientially accessible phenomenon (physics, populations, economics)[2]. Psychologically, the Cartesian plane is very familiar and intuitive to American mathematics majors from their schooling. I think it makes more sense to consider the axiom system a hypothetical-deductive model of students' understanding of geometric phenomena. This approach of viewing axioms as a hypothetical-deductive description of geometric phenomena is consistent with the classical view of axiomatization.

To accomplish this, I begin the semester by introducing a few principal example planes as in the text. We explore them by asking similar questions to prime students to see some similarities as well as their differences. I initiate axiomatizing by asking students to formulate possible rules for how points, lines, and distances behave on all example planes. Students often rely on the Euclidean and spherical planes to observe patterns and articulate such rules, but they must quickly generalize their suggestions to see how and whether they hold elsewhere. They may test any of their proposals with the hyperbolic, Minkowski, and subway geometries (the last of which I introduced to the curriculum). As such, the axioms are not abstract for students in the beginning; rather, students identify more local and particular patterns before they engage in abstraction and generalization.

I developed the subway geometry in response to previous students' complaints that finite geometries didn't seem very geometric. The text introduces finite geometries, but does not provide spatial diagrams of them; instead it presents lines in set notation and distance functions in tables. I have found that even diagrammatic representations of finite planes invite problematic interpretations from students. Some of the (rather natural) problems students have include:

- assuming that lines must be straight,
- identifying locations with points, thus acting as though the set of points is dense, and
- assuming that distance must correspond to the "space between" points.

I thus provide them with a very simple subway map (actually a modification of the Fano plane) and ask them to geometrize it by identifying points, lines, and distances. I chose subways because subway lines need not be straight and it is natural to have two stations (points) that have no other station between them. Students also suggest distance functions other than linear distance such as number of stops or time of travel.

The first aspect of planar geometry I encourage students to axiomatize is distance. Working in groups of three to four, at least some students generally suggest that distance is positive ($AB \geq 0$), symmetric ($AB = BA$), definite ($AB = 0 \Leftrightarrow A = B$), and additive for collinear points ($AB + BC = AC$). The text's axioms formalize the properties, and I keep questions and activities ready in case the key idea does not arise easily. For example, I challenge their assumptions that distances along a line add appropriately by introducing an express train that skips a stop.

Other axioms also arise as students attempt proofs. Many instructors of proof-oriented classes are used to hearing students ask, "What can I assume and what can't I?" The reinvention setting affords a fruitful response to natural frustrations. When students identify that they want to use some fact in proving, I either ask the group to consider whether the claim holds on all our example planes, suggest the fact as a new theorem that they should prove, or articulate

[2] See [10] for a nice description of the notion of "model" in Tarski's logic, which is the usage I reject. I think Tarski's use of "model" is more consistent with the modern parlance of, say, a model train or model plane that allows one to touch and manipulate some less accessible object. I dislike the metaphor because it seems to imply that the axiomatic system is more definitive and the analytic system more primitive, and I find this inverts the psychological reality for students.

the fact as a new possible axiom to be included into the system. In this way, even if I do the work of articulating the axiom, the statement arises in response to students' felt needs in the proving process (similar to Lakatos' notion [16] of proof analysis).

6 Engaging in Multiple Mathematical Processes

In my class, the students' guided axiomatizing is interwoven with other key mathematical processes. For instance, axiomatizing distance also affords students the opportunity to generalize key patterns. Once students observe that collinear distances tend to be additive (i.e., one point is between the other two), they sometimes propose the following axiom based on the Euclidean situation: "For any three collinear points A, B, C, there exists some betweenness relation among them (one is between the other two)." While we use this criterion to rule out express train cases, there are also examples on the sphere for which this fails (as three points equally spaced along a great circle). The group works together to modify the proposed axiom to accommodate the spherical case using the parameter ω (similar to exception-barring as discussed in [16], [17]).

The reinvention process also invites students to engage in defining with concepts such as *between*, *segment*, *ray*, and *separation*. As I mentioned above, this is a valuable process because students' intuitions of "between" do not inherently depend upon distance. Students reason about betweenness spatially in ways that are hard for them to formalize or explicate [8] in terms of the body of theory in the text. Students begin by proposing the theorem: "For collinear points A, B, C the distances add. $AB + BC = AC$ whenever B is between A and C," where "between" is (for students) a primitive concept. Once I push them to define "between", students in my experiments shift toward adopting the equation as the definition. I think that this represents an important mathematical development away from defining mathematical concepts intuitively toward defining them formally[3].

7 Axioms versus Theorems

The most valuable opportunities that reinvention affords comes in the process of axiomatizing. The two key opportunities that arise in my experiments are when students propose a later theorem as an axiom and when students suspect that an axiom should be a theorem. In the first case, I let students agree to assume the axiom for a time, but later I prompt them to prove it. Once they have proven the statement, I try to initiate a metamathematical reflection by asking, "What happens when axioms are provable?" This is an example of what I described as students' reflections on metamathematics growing out of their proving activity. The students who proposed the axiom did so because it seemed intuitively and generally true (consistent with the classical view of axioms). However, once the axiom is proven, students in my experiments often suggest that it gets demoted or at least transferred into the theorem column. I find this can help shift students' focus from intuitive truth to formal relations of provability (reflective of modern mathematics).

Since students find this idea of relabeling the statement very helpful in thinking about axiomatizing, I sometimes introduce the concept more directly in class. I use an analogy between a mathematician's assumptions and an investor's money. My goal is to emphasize how each axiom should be useful to prove new theorems (draw a return) and we want to minimize the number of axioms. While the investor may have plenty of extra money, he chooses to spend it carefully. We could make assumptions liberally by going through the textbook and relabel every *theorem* an *axiom*, skipping proofs altogether. However, the aim is to get the most return for our investment. If we can possibly prove any claims based on previous assumptions, then we will save our assumption currency by rendering it a theorem rather than an axiom. While the analogy may not be perfect, I found that students' later responses to metamathematical questions often borrowed from this language, indicating that students drew on the analogy for their sense-making about the axiomatic method.

This money metaphor leads to the question that arises from the other important axiomatizing situation: when students suspect an axiom should be a theorem. How can one be sure whether spending an assumption is necessary? Students in my teaching experiments sometimes express feelings that certain axioms should be provable (much as

[3] A mathematical definition is *formal* in my usage if it is compatible with the intuitive category, attends to mathematical structure (number, function, set, etc.) and it affords valid mathematical proofs (what Dawkins [8] calls an *explication*, after Carnap [6]).

people felt about Euclid's fifth postulate for centuries). In other cases, they seem sure they cannot prove the axiom, but are unsure how one could prove it non-provable. This is why I value repeatedly posing and discussing activities of the form described above: "Construct a plane that satisfies Axioms 1 through n, but not Axiom $n + 1$." I find it helpful to discuss the relationship between this activity and the possible metatheorem, "Axioms 1 through $n \Rightarrow$ Axiom $n + 1$," because it is not immediately obvious to many students. It is further illuminating to expand the theorem to say, "In all planes in which Axioms 1 through n are true, Axiom $n + 1$ will also be true." As I mentioned, students often implicitly interpret axioms as referring to familiar planes rather than to all possible planes. The quantification that our standard mathematical parlance suppresses is important for students to recognize. It is more obvious in the expanded statement why the existence of one plane satisfying some, but not all, of the axioms proves the metatheorem false and proves that the axiom is independent of the previous ones. In this way, we make the implicit quantification explicit by stating metatheorems, which help students grasp the nature of axiomatic independence.

What is more, multiple iterations of the process seem to help students gain a working conception of the axiomatic approach. After students in a recent semester completed several such activities, I proposed a new axiom[4]. Matthias asked why it was an axiom and not a theorem because he felt they should be able to prove it. I acknowledged that they either needed to show it was not provable or prove it. Because I suggested it was an axiom, the group inferred that there must be some plane that satisfied all but this new axiom. Matthias anticipated that it would be some "wonky plane" and another student, Benjamin, added that the new axiom would "kick out" the strange example. In their dialogue, I recognized an emergent sense of how axiomatization progresses, similar to what I described above. New axioms are proposed from patterns of planar phenomena. To prove that it must be an axiom, one must produce a wonky plane that satisfies all but the new axiom. Then, once the axiom is admitted to the system, the wonky plane is removed from the set of exemplars. As the semester went on, the students began to anticipate this pattern with the introduction of new axioms allowing them to understand the mathematical importance of these idiosyncratic examples. It is the recognition of patterns in their mathematical activity that I call students' conscious metamathematical understanding of the axiomatic method. Students may explain the pattern in ways more compatible with classical axiomatics or with modern axiomatics, but I encourage all students to reflect on the process and come up with some self-consistent explanation.

8 Instructional Method

I should make clear that the reinvention process, though central, is not the only aspect of how I teach the course. I do not organize every class meeting in the same way, but have three frames for a class session, depending on what material I want students to learn. The first class structure generally facilitates the introduction of something new (a definition, axiom, or proof). I usually begin by providing students with an activity, which they complete in groups of three to four. Since axioms and theorems represent patterns of mathematical phenomena [20], I give students a sequence of questions meant to guide them to observe the pattern and articulate it clearly and precisely. For definitions, I usually begin by prompting students to consider various examples (or the objects on various planes) before attempting to formulate a definition and some basic theorems. For appropriately accessible proofs, I pose the task to students in small groups (or on homework) before letting them present their efforts to the whole class.

In each case, as students work in small groups, I circulate through the room and listen to the conversations, helping students who get stymied. I also monitor which groups articulate key ideas or reach common roadblocks so I can call on them to report once we reconvene as a class. I can thus control the order in which ideas appear and give struggling students more opportunities to share before others give the right answers. I find it very important to acknowledge common missteps and dead ends so students can begin to recognize why those intuitively enticing approaches are insufficient.

At the end of each activity meant to reinvent an axiom, definition, theorem, or proof, I provide students with closure through a reflective discussion. I use the recaps to ratify certain ideas; during the reinvention activity itself I allow some ambiguity, but to move forward students need points at which we achieve consensus. I also take these opportunities to highlight the mathematical processes that we used to solve the problem and pose the metamathematical questions

[4] For the sake of time and intellectual complexity I do not ask students to reinvent every axiom, definition, and proof. My goal is not that students completely reinvent the mathematical theory, but that they engage in enough of the process to understand how it works.

highlighted in this chapter. I also try to emphasize student contributions to the reinvention process, which I find gives students a sense of pride and efficacy in their mathematical activity.

A second way that I organize my class is around multiple choice activities. When I do not have time to let students develop new ideas on their own, I sometimes present students with varying examples of possible definitions, axioms, or theorems that I produced. Some of the options articulate common student misconceptions and others are intended to address students' troubles with mathematical language and notation. The students must interpret the various statements and decide which are most appropriate for the class's body of theory. These activities accomplish several goals. They encourage student input while introducing ideas much faster than open reinvention activities. Also, they challenge students to interpret mathematical language appropriately. One of the axioms that I present in this way is multiply quantified ("For every. . . , there exists. . . "). Mathematics education research indicates that students do not interpret such statements the way mathematicians do [12]. When I present my class with various permutations of a statement ($\forall X \exists Y$; $\forall Y \exists X$; $\exists X \forall Y$; $\exists Y \forall X$), they work to identify the differences between the statements. The activities allow students to evaluate the theoretical importance of the various statements (two are false, one is trivial, the fourth is the axiom). Most proof-oriented classes only present important statements to students, which fails to give them opportunities to determine whether a statement is theoretically significant or not.

Third, I often lecture when I present more difficult proofs of important theorems. While I may invite student contributions, some proofs are simply too long or sophisticated to expect students to reproduce in a reasonable amount of time. It is my hope that implementing reinvention activities around key axioms and definitions helps students better understand the more significant proof presentations. I do not maintain the common convention that we prove every theorem that we use, because many proofs are available in the text.

9 Assessing Metamathematical Learning

As I mentioned at the beginning, my goals for students' metamathematical learning operate in tandem with my more conventional goal that students become able to produce valid proofs based on the class's axioms, definitions, and theorems. The bulk of the assessment I administer has focused on the latter goal because it is much easier to pose a proof task to students than a metamathematical task. When I began teaching the course, I included metamathematical reflection questions in homework and tests, but only as completion grades (i.e., responses were not evaluated for appropriateness) or for bonus points. In many ways the questions served as formative assessment that gave me insight into how my students reasoned about metamathematical topics so as to inform and improve my future instruction.

As I gain more experience interpreting students' ways of thinking about axiomatics and posing questions that elicit meaningful responses, I intend to include more summative assessment of students' metamathematical understandings. In the next section I present some examples of questions or tasks for this purpose and explain how they relate to my teaching goals and the patterns in student thinking I have observed thus far. I provide some examples of student responses to the tasks for two reasons. First, I want to demonstrate how the questions provide interesting insights into how mathematics students interpret axiomatic geometry. Sometimes students display very sophisticated conceptions, but I think it is important to recognize the limitations of other students' responses because they justify the need for metamathematical instruction as laid forth in this chapter. Second, I want to further categorize the students' response types in light of the basic framework I developed in this chapter for the various views of axiomatizing. The categories serve as my operational definition of what it means for students to develop metamathematical understanding relative to the mathematical activity in this geometry course.

10 Four Types of Questions about Axiomatizing

The most obvious, but unfortunately least useful type of metamathematical questions are straightforward:

What are axioms and what is their purpose in advanced mathematics?

How is an axiom different from a definition or a theorem?

They generally elicit vague and incoherent responses because they require that students have some stable metamathematical understanding and that they have mastery over an appropriate language for discussing them. This direct approach to metamathematics is antithetical to my interest in helping students see metamathematics as a way of organizing their thinking about the particular mathematics they learn in my class (what I called reflection on their own mathematical activity). However, there are ways to phrase the questions that provide students with more opportunities for sense-making and thus afford more meaningful responses. They include some of the examples I provided earlier in the chapter:

> What is the significance of creating an example plane that satisfies Axioms 1 through n, but not Axiom $n + 1$? (I provide students with an instantiation of this question, not the generalized form.)
>
> How would we modify our system of geometric theory if we found a proof for one of our axioms based on the other axioms?
>
> Which of our example planes are in G[1-7], G[1-8], G[1-10], etc.? (Here G[1-n] denotes the set of all geometries satisfying Axioms 1 through n.)

The first two questions correspond to the main two teaching opportunities that I target in the course: questioning whether a given statement is an axiom or a theorem and proving a statement that had previously been called an axiom. I only ask them about independence and non-duplication of axioms after students have completed related tasks on which they can reflect to make sense of these metatheoretic properties.

A second type of metatheoretic question probes the relationship between the axiomatic body of theory and the example planes. As I mentioned, students often interpret abstract statements with reference to prototypical example planes, which can be helpful or problematic. I find it important to guide students' reflection about the issue. Sample questions include

> Early in the course we examined plane X. Later we proved theorem Y. Theorem Y is false on plane X, so how could we have proven it?
>
> We recently proved theorem Y. On which planes do we know it must be true? When can we use particular example planes to prove or disprove a theorem?
>
> We proved theorem Y from Axioms 1 through n. Is it possible to find a plane where theorem Y is true, but one of those axioms is false? Does this contradict our proof of the theorem?
>
> What is the role of our three main example geometries (Euclidean, spherical, and hyperbolic) in the development of our system of axioms?
>
> What is the role of our other planes (gap, Minkowski, subway, $\mathbb{R}^3$) in the development of our system of axioms?

A third type of question invites students to reflect on the hypothetical aspect of the implications they prove. Students often interpret the proof of a theorem to establish that it is simply true, rather than that it is true whenever certain conditions hold [11]. Students need encouragement to understand the genuinely conditional nature of theorems, and the course provides a rich context for doing so. Useful prompts include

> Identify some theorems that would not be true or provable if we did not assume Axiom X.
>
> If I presented you with a new example plane that you had never seen before, how could you verify that Theorem Y was true on it without testing Theorem Y directly?

The way the class progressively introduces structure through new axioms lends itself to reflection on how new theorems depend on assumptions (because they were often shown to be not provable prior to some pertinent axiom). One anecdote that justifies why this may be helpful to students comes from a sequence of interviews I conducted with two students taking the class under another instructor. Oren was confused about a theorem in the book because it was almost identical to a theorem in the previous chapter, but was non-trivially stronger. Oren proffered the explanation that, "there must have been put a constraint on proposition 7.5 to get 8.3, right?" Oren had not recognized how the introduction of a new axiom between the two theorems allowed the proof of a stronger claim (8.3). In mathematics textbooks, the epistemic shifts are implicitly marked by theorems' order of appearance in the text relative to the axioms, but students may not always track the progression or the important theoretical shifts it entails. For this reason, I hope that explicit reflection on the implication structure of the body of theory (which theorems depend upon which assumptions) will promote more sophisticated conceptions of the hypothetically dependent nature of truth in a mathematical body of theory.

The final category of question invites students to consider how the axiomatic system formalizes (and differs from) their geometric intuitions. I think geometric proof always involves some coordination between the spatial reasoning or experience from which it grows with the analytic and syntactic representations mathematicians have adopted for it. I want students to know how to segue their spatial reasoning into rigorous proof, and I believe reinvention helps in this process by guiding students to appropriate the various sources of information. This relates closely to my interpretation of Matthias's quote at the beginning of the chapter in which I claim he differentiated common sense from what was provable. Some questions that I pose to students to engender reflection on this topic are

> It seems obvious that if B is between A and C, then B is also between C and A. Why do we prove the theorem "A-B-$C \Rightarrow C$-B-A"?
>
> Which of our axioms formalizes the notion that you cannot get from one side of a line to the other without crossing the line? Name two theorems that depend on formalizing this idea.
>
> Which axiom guarantees that points in space are dense? How did introducing it affect our set of example planes?

11 Categories of Student Responses

In my research several trends in student thinking continue to emerge. I have identified three patterns of students' explanations about the importance of activities meant to prove axiomatic independence, each of which I shall define below: pedagogical, descriptive (classical axiomatic), and formal (modern axiomatic). None of these ways of reasoning is completely inappropriate, but some are more indicative of metamathematical understanding. In this section I provide a few examples of responses and my appraisal of their mathematical value relative to my instructional goals. Most of the responses I provide, if not otherwise stated, are in response to questions of the type "Why do we identify example planes that satisfy Axioms 1 through n, but not Axiom $n + 1$?"

Pedagogical responses emphasize the relation of student understanding to mathematical curriculum. For instance, two students in my class responded to such a task early in the semester by saying

> The point of the activity... is to gain a better understanding of the axioms of distance and incidence. By creating a model where six of the seven axioms hold true, you are illustrating that you understand the underlying concepts of each axiom.
>
> To gain a better understanding of the axioms.

These students' explanations are correct. The task I posed is a pedagogical exercise and should help them learn what the axioms say. The explanations are incomplete, though, because the students did not seem to notice that the exercise enables us to learn more about the axioms' relationship to each other, to our intuitions, or to the example planes. I find it interesting that students offering this kind of explanation performed poorly in my course.

Descriptive responses emphasize the relationship of the axioms to the prototypical planes or of the prototypes to the idiosyncratic example planes. This view is valuable because it encourages students to connect their spatial intuitions (Euclidean or spherical) to formal theory, and it provides motivation for new axioms to improve the compatibility between the hypothetical-deductive model and the prototypes. This compatibility is then formalized in the characterization theorem at the end of the course, which is a rich mathematical achievement. Some example responses in this category include:

> If one of the axioms fails, then impossible geometries could be possible.
>
> [Axiom] BP allows us to say betweenness must exist in certain situations. It allows us to talk about line segments and rays more meaningfully. I feel like it moves us closer to infinite geometries, or more recognizable plane geometries. It fits with our intuition that we should be able to call something between others.
>
> This shows us what kind of behavior is being removed from our model of a plane. These "bad planes" have characteristics that are contrary to our understanding of things like betweenness. Removing them by forming subsequent axioms returns our focus to the kinds of planes we want to work with, Euclidean, spherical, hyperbolic, etc.

While all the responses fit within the frame of moving the hypothetical-deductive model toward better descriptions of the prototypical planes, they vary in their mathematical appropriateness. Explanations like the first response reject any geometric meaning attributed to the idiosyncratic examples (especially finite planes), meaning that such students failed to abstract their understanding of a plane to assimilate these exemplars. As such, they likely view spatial Euclidean or spherical planes as real geometry rather than their analytical instantiations in terms of undefined terms, sets, distance functions, etc. These problematic notions bore themselves out in the students' performance. They often performed poorly in the class overall (i.e., on more traditional proof tasks). I call such responses *descriptive-exclusive* because they reject consideration of idiosyncratic planes.

The other two responses similarly emphasize the importance of the prototype planes, but in a way fitting for the hypothetical-deductive model being constructed. The key mathematical habit that I would like students to develop is considering every example plane as a valid exemplar until we have some axiom to exclude it, unlike the previous type who viewed finite geometries as impossible or as making no sense. Mathematicians often value idiosyncratic examples over prototypes because they provide instructive counterexamples to plausible theorems, but I think this stance reflects the advanced abstraction of their understanding. Mathematicians would say that the integers are an example of a group because this embeds the example in a much richer body of knowledge for the expert. The opposite is true for students. Integers carry much more meaning on their first introduction to group structure. At some point, I intend that students make the switch to the more formal stance, but I think the descriptive interpretations of axiomatizing represent a healthy developmental step from prototypes to abstract mathematical classes. Eventually, I would like students to recognize that finite geometries are valid mathematical entities that actually open new avenues of mathematical inquiry and insight, but I am aware this entails a difficult process of abstraction on their part. Some of my students' responses show movement in this direction, though:

> Essentially a new geometry could be possible based on a different set of axioms.

Responses like the three described in this paragraph I refer to as *descriptive-inclusive*, because they accommodate alternative possible geometries.

Student responses within a formal viewpoint recognize relations among statements in the body of theory (axioms, definitions, and theorems) and may reveal students' understanding of metamathematical ideas such as independence and efficiency in an axiomatic system. They most often reflect provability relations among formal statements. In many ways, they represent the most sophisticated interpretations of the given mathematical tasks, and the students who give them are often top performers in my class overall. An example explanation of this type is:

> To show that all the axioms do not imply the next. That in fact each axiom has its own unique way of describing a given plane. The reason they are axioms is exactly the point that they don't imply one another.

So, while I put value on these sophisticated interpretations of axiomatizing, I find that there is some mathematical value in a range of different frames. Even the *formal responses* tend to include some element of modeling the prototypes (the descriptive viewpoint). Thus the categories are not mutually exclusive. What is important for me as an instructor is that students develop coherent and self-consistent interpretations of axiomatizing that can help them to make sense of the proving process as they engage in it. I observe in my classes that while I cannot expect every student to reach the same level of abstraction in their understanding of what a plane is, some views are much more helpful to their ability to construct proofs than others. I hope that through guided reinvention and reflection, more students will be able to make these fruitful abstractions.

12 Summary

In this chapter I set forth an argument and method for helping students develop metamathematical understanding of axiomatics in tandem with learning a particular body of geometric theory. Such learning activities are valuable because they help students make sense of advanced mathematics and they help instructors understand their students' reasoning. I endorse the method of guided reinvention to engage students in active axiomatizing, defining, conjecturing, proving, abstracting, and generalizing. For this to work, it is essential that students have access to a rich body of examples, and that the instructor provides them with carefully designed activities to support their progress and reflection. I provided several categories of assessment activities intended to foster and reveal students' understanding of the axiomatic process. Finally, based on historical and empirical analysis of axiomatic interpretations, I demonstrated how students conceptualize axiomatizing in four main ways (which can co-exist): pedagogically, descriptive-exclusive, descriptive-inclusive, and formally. The categories are in order of progressive mathematical sophistication and seem to correspond somewhat with students' overall performance in the course. I hope that this chapter contributes to advanced mathematics instructors' ability to incorporate explicit metamathematical goals into their instruction and their ability to help students meet those learning goals as part of their apprenticeship into the mathematical community.

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15

The Philosophical and Pedagogical Implications of a Computerized Diagrammatic System for Euclidean Geometry

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1 Introduction

This chapter briefly describes the computer proof system CDEG, version 2.0, a computerized formal system for giving diagrammatic proofs in Euclidean Geometry. The existence of such a computer proof system shows that geometric arguments that rely on diagrams can be made rigorous, and this fact has important philosophical implications for how we understand and teach geometry. The full details of the computer system and its underlying logic are complicated and are discussed elsewhere; this chapter instead gives a brief example of how the computer system can be used to prove a theorem of Euclidean geometry, and then discusses some of the philosophical implications of such a system.

The chapter starts with a brief history of the use of diagrams in geometry, and explains why we might be interested in such a formal system. It then gives an introduction to what constitutes a geometric diagram in this context, and how such diagrams are manipulated by this computer system. As an example, it shows how Euclid's first proposition from Book I of the *Elements* [1] can be derived in this system. CDEG is meant to formalize the proof methods found in Euclid's *Elements*, and the following sections of this chapter discuss the correspondence between these two proof systems, the question of why we would want to develop a computer system that mimics Euclid's proof practice, and other related philosophical issues. Finally, the last section of this chapter discusses some ways these philosophical issues might bear on the way we teach geometry courses.

CDEG stands for "Computerized Diagrammatic Euclidean Geometry." This computer proof system implements a diagrammatic formal system for giving diagram-based proofs of theorems of Euclidean geometry that are similar to the informal proofs found in Euclid's *Elements*. It is based on the diagrammatic formal system FG, which is described in detail in my book, *Euclid and his Twentieth Century Rivals: Diagrams in the Logic of Euclidean Geometry* [7]. The book also describes an earlier version of CDEG; however, CDEG has evolved significantly since its publication, and is now publicly available in a beta version.

Euclid's *Elements* was written around 300 B.C. and was considered the gold standard of careful reasoning and mathematical rigor for roughly the next 2000 years. However, this changed with the development of the idea of a formal proof system in the late 1800s. A formal proof system is one in which all the allowable rules are set out carefully in advance in a way that can be followed mechanically, so as to leave no room for human error. Most formal proof systems are *sentential*, meaning that they manipulate sentences that are strings of characters in some formal language. The notion of a formal proof system is, in a way, the logical culmination of Euclid's proof method, in which a set of definitions, postulates, and common notions are set out in advance as assumptions, and proofs rely on them and on previously proven propositions. However, it has been commonly assumed that Euclid's *Elements* could not form the basis for a formal proof system, because his proofs make crucial use of diagrams as part of their arguments. The rules for using diagrams in proofs have not been well understood, and this made it seemingly impossible to incorporate these kinds of proofs into a traditional sentential formal system. Thus, in the twentieth century, the most commonly held view became that Euclid's proofs, as well as all other proofs using diagrams, were inherently informal and could not be made rigorous. The comments made by Henry Forder in *The Foundations of Euclidean Geometry* in 1927 are typical of this view: "Theoretically, figures are unnecessary; actually they are needed as a prop to human infirmity. Their sole function is to help the reader to follow the reasoning; in the reasoning itself they must play no part." [2, p. 42].

During the twentieth century, two approaches were used to give a rigorous foundation for Euclidean geometry. The first was to create sentential axiom systems for geometry that contained many more axioms than Euclid's. Several people proposed sets of axioms, the most famous of which is Hilbert's [3]. This approach gives rise to what is commonly referred to as "synthetic geometry." Another approach is to encode geometry within the theory of the real numbers via the Cartesian plane. This was the approach taken by Tarski [13], who showed that this provides a decision procedure for the parts of geometry that can be encoded this way—that is, a mechanical procedure that can determine if a statement is true or false (although it make take an impractical amount of time to actually come to a decision). This approach is generally referred to as "analytic geometry." Both approaches yield completely formal ways to prove theorems of Euclidean geometry. (See [7, Section 1.1] for a more extensive history of geometry and the use of formal systems in geometry.)

However, the proofs produced by the approaches generally look nothing like Euclid's proofs. So, while they may formalize Euclid's theorems, they do not formalize his proof methods. They therefore shed no light on the question of whether or not Euclid's proof methods can be made formal. In [7, Section 1.2], I propose the *formality hypothesis*:

Formality Hypothesis An informal proof method is sound if and only if it is possible to give a formal system with the property that informal proofs using the informal methods can always be translated into equivalent correct proofs in the formal system.

CDEG is designed as a diagrammatic computer proof system that can mimic Euclid's proofs in a completely rigorous way, and can show, via the formality hypothesis, that his proof methods that employ diagrams are valid modes of informal reasoning.

When we say that CDEG is a diagrammatic computer proof system, it means that it allows its user to give geometric proofs using diagrams. It is based on a precisely defined syntax and semantics of Euclidean diagrams. To say that it has a precisely defined syntax means that all the rules of what constitutes a diagram and how we can move from one diagram to another have been completely specified. The fact that the rules are completely specified is perhaps obvious if you are using the formal system on a computer, since computers can only operate with precisely defined rules. However, it was commonly thought for many years that it was not possible to give Euclidean diagrams a precise syntax, and that the rules governing their use were inherently informal.

To say that the system has a precisely defined semantics means that the meaning of each diagram has also been precisely specified. In general, one diagram drawn by CDEG can actually represent many different possible collections of lines and circles in the plane. What they all share, and share with the diagram that represents them, is that they all have the same topology. This means that any one can be moved and stretched into any other, staying in the plane. So, for example, a diagram containing a single line segment represents all possible single line segments in the plane, since any such line segment can be stretched into any other. See [7] for more details concerning the syntax and semantics of Euclidean diagrams.

2 CDEG Diagrams

A sample CDEG diagram is shown in Figure 1. It occurs in the proof of Euclid's first proposition and represents a line segment along with a circle drawn with that line segment as its radius.

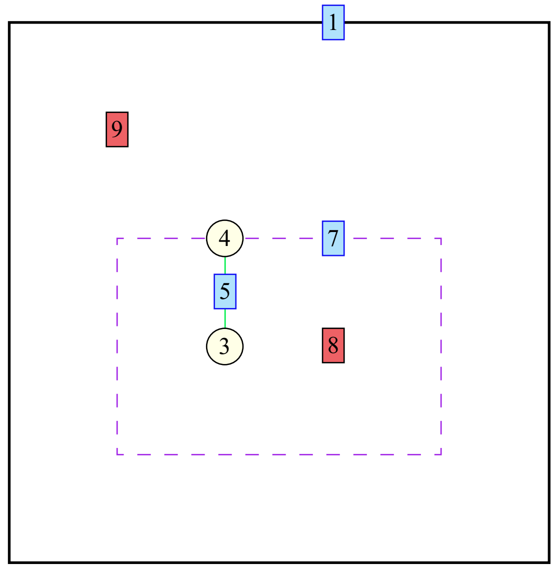

Figure 1. A CDEG diagram showing the second step in the proof of Euclid's first proposition.

CDEG diagrams contain two kinds of objects: *dots* and *segments*. The dots, which are shown as light yellow circles, represent points in the plane, while the segments represent pieces of lines and circles. There are actually two kinds of segments that can occur in a diagram: *solid segments*, which represent pieces of lines, and *dotted segments*, which represent pieces of circles. The dots and segments of a diagram are enclosed by a bold line called the *frame*. The dots and segments of a diagram, along with the frame, break the diagram into a collection of *regions*. Every dot, segment, region, and piece of the frame in a CDEG diagram is labeled with a number, so that we can refer to it. Dots are labeled with a number inside the yellow circle; segments are labeled with a number in a blue box along the segment, and regions are labeled with a number in a red box somewhere in the midst of the region.

The segments in a diagram are part of diagrammatic lines and circles. Each line and circle is assigned a different color, so that all the segments that make up a line or circle will be the same color. Lines may continue to the frame, in which case we consider them to be infinite in that direction. Thus, infinite lines intersect the frame in two places, rays intersect it in one place, and line segments don't intersect it at all.

Each CDEG diagram represents all the possible equivalent collections of points, lines, and circles with the same topology as the diagram. However, it is often the case that when we perform some geometric operation on a diagram, such as connecting two points with a line, there may be multiple possible outcomes depending on which of the equivalent collections you start with. CDEG allows for this possibility through the use of *diagram arrays*. A diagram array is a collection of several possible diagrams showing different possible outcomes.

CDEG allows you to manipulate diagrams through the use of construction and inference rules. These rules are meant to be *sound*. This means that if you start with a diagram D that represents a collection $\mathcal{C}$ of points, lines, and circles in the plane, and you apply a rule to D, then at least one of the diagrams in the resulting diagram array should still represent $\mathcal{C}$ (or, in the case of a construction rule that added a line or circle to the diagram, $\mathcal{C}$ with the appropriate line or circle added). The definition of soundness is the diagrammatic analogue of the normal logical definition of soundness, which is that a formal system is sound if the outputs of its rules only produce statements that are logical consequences of its inputs, so that the system can only produce true statements. For further details, see [7].

3 A Sample CDEG Session

As a brief example, this section shows how to reproduce the proof of Euclid's Proposition 1 from Book I of the *Elements*. A more detailed version is given as a tutorial in the *CDEG User's Manual* [5]. Euclid's Proposition 1 shows that an equilateral triangle can be constructed on any given base.

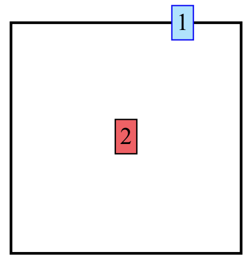

Figure 2. The empty primitive diagram as drawn by CDEG.

First we start CDEG and ask it what commands are available:

```
Welcome to CDEG!
(Type h for help.)
CDEG(1/1)% h
Options are:
<a>dd dot to segment, draw <c>ircle,
<d>elete objects, <e>rase diagram, get <h>elp,
apply <m>arker inference rules, con<n>ect dots,
extend segment in <o>ne direction,
<p>rint diagram as text, <q>uit,
add dot to <r>egion, <s>ave/load diagrams,
se<t> pd, or e<x>tend segment.
CDEG(1/1)%
```

The prompt here (CDEG(1/1)%) tells us that we are currently working with the first diagram in a diagram array that contains one diagram. Since we have just started the program, this is the empty diagram. This is the initial diagram that is displayed. It is shown in Figure 2. It contains a single region bounded by the frame; CDEG has assigned this region the number 2. CDEG assigns each object in a diagram a unique number by which it can be identified. Next, we use the “r” command (“add dot to <r>egion”) to add two new dots to this region:

```
CDEG(1/1)% r
Enter region number: 2
CDEG(1/1)% r
Enter region number: 2
```

CDEG now displays the resulting diagram, which adds two more dots, numbered 3 and 4. We can connect them using the con<n>ect dots command.

```
CDEG(1/1)% n
Enter first dot’s number: 3
Enter second dot’s number: 4
```

The resulting diagram is shown in Figure 3; this is the starting diagram for our proof of Euclid’s Proposition 1. Next, we will draw a <c>ircle centered at dot 3 and going through dot 4.

```
CDEG(1/1)% c
Enter center dot’s number: 3
Enter radius dot’s number: 4
```

The resulting diagram is the one that was shown in Figure 1 in Section 2. The diagrammatic circle in the diagram looks rectangular rather than circular, but all we care about here is the topology of the diagram.

Next, we want to draw another circle centered at dot 4 and going through dot 3. We will then form a triangle by connecting the endpoints of the segment to one of the points, dot number 12, on the intersection of the two circles.

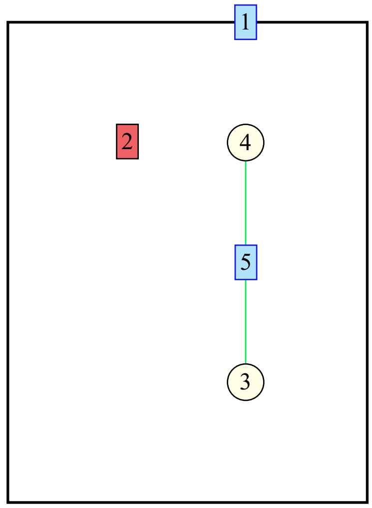

Figure 3. A CDEG diagram showing a single line segment.

```
CDEG(1/1)% c
Enter center dot's number: 4
Enter radius dot's number: 3
CDEG(1/1)% n
Enter first dot's number: 3
Enter second dot's number: 12
CDEG(1/1)% n
Enter first dot's number: 4
Enter second dot's number: 12
```

The resulting diagram is shown in Figure 4.

Next, we can mark segment 5 congruent to segment 25, because they are both radii of the purple circle in Figure 4, and, likewise, we can mark segment 5 congruent to segment 29, because they are both radii of the red circle. We can then mark all three of the segments congruent to one another using the `<c>ombine markers` command that is available as part of the `apply <m>arker inference rules` submenu. For more details of how this works, see the extended version of this example given in the user manual [5]. The markings of the diagram are actually given as text accompanying the diagram. Representations that combine text with diagrams are sometimes referred to as heterogeneous representations; thus, this is an example of a heterogeneous representation.

Finally, we may want to clean up the diagram by erasing the superfluous pieces added in the course of our construction. We can do this using the `<d>elete objects` command, leaving us with the diagram shown in Figure 5, which shows an equilateral triangle on our original base.

Thus, we have shown how to construct an equilateral triangle on a given base, duplicating Euclid's Proposition 1.

4 CDEG vs. Euclid's *Elements*

CDEG is designed so that there is a direct correspondence between the construction and inference rules that it uses and the postulates, common notions, and definitions that Euclid uses in the *Elements*. For example, the CDEG proof of Euclid's Proposition 1 that was given in Section 3 directly follows Euclid's proof of the proposition. For an extended discussion of all of CDEG's commands, explanations of how each is used, and how they correspond to Euclid's postulates, common notions, and definitions, see the *CDEG User's Manual* [5].

CDEG does not contain many inference rules other than those set out in Euclid's postulates, common notions, and definitions. It is commonly asserted that Euclid's *Elements* contains many subtle gaps that can only be repaired by adding further postulates in the manner, for example, of Hilbert's *Foundations of Geometry* [3]. The added postulates typically have to do with issues of betweenness, continuity, and the intersections of geometric objects. For example,

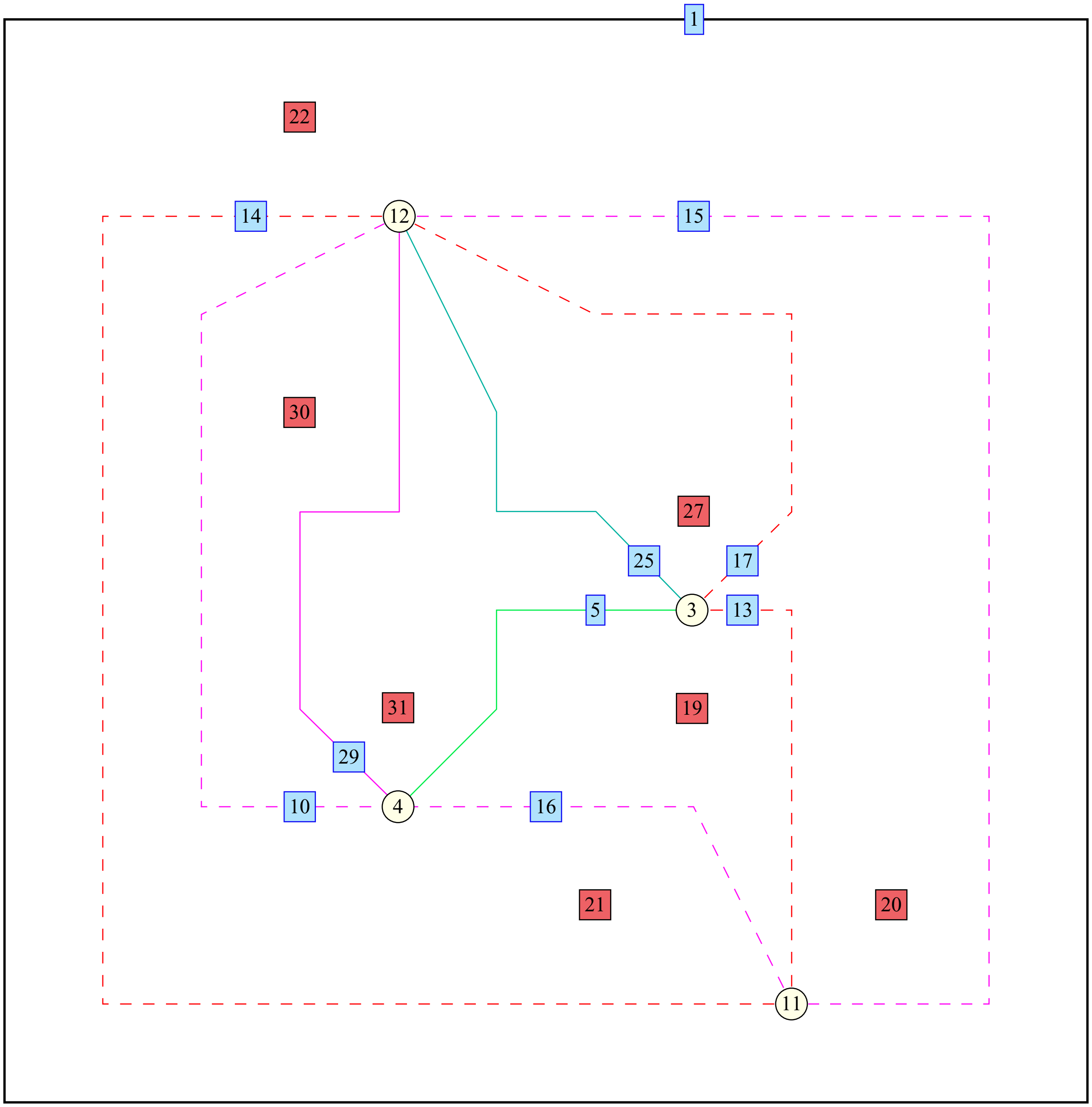

Figure 4. A CDEG diagram showing the fourth step in the proof of Euclid's first proposition

in the proof of Euclid's first postulate, Euclid assumes without apparent justification that the two circles constructed in the course of the proof must intersect. However, in CDEG this issue is solved by the underlying diagrammatic machinery. After we add the second circle, the single diagram shown in Figure 4 is produced, and in it, the two circles do, indeed, intersect, giving us two intersection points (dots 11 and 12) that we can use in the rest of the derivation. Similarly, most of the other gaps in Euclid's reasoning are taken care of by the diagrammatic machinery. Thus, we can view them as being part of an unarticulated diagrammatic process rather than as flaws in Euclid's arguments.

One particular way that CDEG differs from Euclid's *Elements* is in its adoption of the side-angle-side and side-side-side triangle congruence criteria as primitive rules rather than as propositions to be derived. Euclid derives the rules using transformations and the principle of superposition. While it is possible to formally mimic the derivations using transformation rules, as is done in FG [7, Section 3.3], for the computer system it was simpler to just adopt the triangle congruence rules directly.

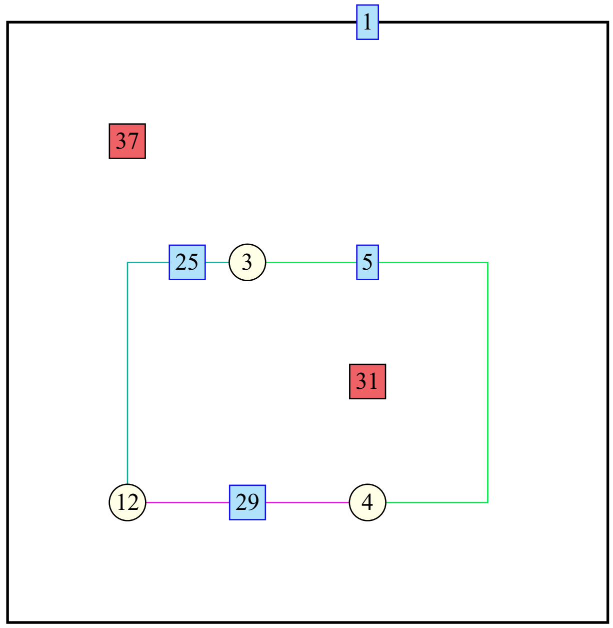

Figure 5. A CDEG diagram showing the triangle obtained in the final step of the proof of Euclid's first proposition.

Because it duplicates all of Euclid's proof methods, CDEG should be theoretically able to prove versions of all of Euclid's propositions from the first four books of the *Elements*, which is the part that deals purely with planar geometry.

However, anyone who tries to actually use CDEG to duplicate the books will quickly realize that in practice it will be difficult to use CDEG to duplicate all of Euclid's proofs. One issue is that as the diagrams become more complicated, the amount of computer time required for one step in the proof can grow exponentially. In particular, the commands that draw circles and lines can take an amount of time that is exponential in the number of objects in a diagram. Thus, some computations may take impractically long. Furthermore, the result of applying a construction rule to a single primitive diagram is a diagram array containing all of the topologically distinct possible diagrams that could occur when the newly constructed object is added, and the number of new diagrams in such an array can also be exponential in the number of objects in the original diagram. So the number of cases that need to be considered can also grow very quickly. See [7, Section 4.1] for more discussion of this phenomenon.

A related issue is that of unsatisfiable diagrams. Ideally, we would want CDEG 's construction rules to return as few diagrams as possible, in order to minimize the number of cases that need to be considered. Unfortunately, the rules sometimes produce diagrams that represent arrangements of circles and lines that cannot be physically realized with actual straight lines and circles. It turns out that this is unavoidable in practice. In [6], it is shown that the question of determining which diagrams that result from applying a construction rule are physically realizable is at least NP-hard, which means that it is not practically computable in a reasonable amount of time.

Another issue that exacerbates the problem of an exploding number of cases is the lack of lemma incorporation in CDEG. Lemma incorporation refers to the use of previously derived propositions and lemmas in proving new theorems. Most proof systems include the ability to do this, but CDEG does not, because it is technically harder to implement lemma incorporation in a diagrammatic setting. Of course, previously derived propositions and lemmas can always be rederived in the course of a proof. However, the additional objects in the diagram in the course of a later proof normally necessitate considering even more cases. Thus, the lack of lemma incorporation can lead to a huge blowup in the length of a given proof. For a more extensive discussion of lemma incorporation, see [7, Section 4.1]. Lemma incorporation will hopefully be included in some future version of CDEG.

In the meantime, one way to use CDEG is to try to duplicate one of Euclid's proofs, but to only complete the proof for one branch of the many possible cases that arise. This allows the user to avoid tediously looking at many different cases that are all essentially similar, while still seeing the essence of the proof. This might be one way to try to use CDEG with students. In this case, we haven't actually proven the theorem in general, but have rather given an illustration of how we would prove it. Interestingly, this actually mirrors Euclid's normal practice. In the *Elements*, he normally only gives a proof for one single case, the hardest one, leaving the other cases for the reader to fill in. For discussion of this practice, see Heath's commentary on Proposition 2 in his edition of the *Elements* [1]. This is one of the aspects of Euclid's practice that probably contributed to the twentieth-century impression that his proofs were inherently informal, since without a formal system like the one provided by CDEG, it wasn't clear that it was possible to give a principled account of how many cases really had to be considered.

5 Why a Computer System?

CDEG is essentially a computer implementation of the formal system FG described in [7]. Actually implementing the formal system on a computer was a highly non-trivial matter that took several years' work. Why would we want to implement an existing formal system on a computer?

The first reason that we might want a computer implementation is to demonstrate that the system really is completely formal: that the diagrams that are being manipulated are, indeed, completely specified as formal objects, and that the rules of the system are completely specified on them. With traditional, sentential formal systems, we do this by writing our axioms in a formal language, and then carefully writing rules of inference as typographical manipulations of its sentences. However, when our formal objects are diagrams, it is difficult to achieve this level of specificity without a computer implementation. Diagrams are complicated formal objects, and we have strong informal intuitions about how they should work that may cloud our ability to judge if our rules have been completely formally specified.

Furthermore, even if our rules are completely formally specified, without a computer implementation, it would be quite difficult to play with the formal system to see what derivations are like, and to make sure that they really work the way that we think they will. This is particularly true in geometry, where constructions can lead to case branching, with a large number of cases that are virtually impossible to keep track of without using a computer. Thus, we may not be able to prove everything we think we can.

This worry is not just academic. Several other diagrammatic formal systems have been proposed by other researchers and have appeared in print but have later turned out to have rules that were ill-defined or unsound, in the technical sense that they could derive conclusions that didn't logically follow from their hypotheses. For example, Isabel Luengo's formal system DS1, described in a chapter of the book *Logical Reasoning with Diagrams* [4], is unsound, as explained in [7, Appendix C]; likewise, John Mumma's Eu, described in [11], [12], and elsewhere, is also unsound, as described in [10]. Because neither system was implemented as a computer system, the way they worked was not completely specified concretely, and so they were not fully understood. This helps to explain why neither the designers of the systems nor the many referees who looked at their work before it was published noticed their significant problems. Thus, we should approach a proposed diagrammatic formal system with a certain amount of healthy skepticism. A working computer system is one way to allay some of the skepticism.

Secondly, a computer system is the only way to make a formal system widely available. Many potential users will not be able to make sense out of a formal system that is just specified mathematically, but will be able to try out a computer implementation.

The third reason for a computer implementation is to be able explore exactly what the formal system is able to prove. Above, I claim that CDEG should be able to duplicate the first four books of Euclid's *Elements*. The only way to verify this claim is to systematically go through each of Euclid's proofs, and to see how to duplicate it within CDEG. To date, I have done this with many proofs from Euclid's Book I, but have not yet gone systematically through all of Euclid's proofs. Doing so is a possible future project that would be essentially impossible without the computer implementation.

One small example of the way that the computer implementation sheds light on the formal system has to do with the way that CDEG handles subtraction of segments. Euclid's common notion 3 states that "If equals be subtracted from equals, the remainders are equal." I originally thought that this rule should be derivable from the rule for addition of segments, and so it wasn't included in CDEG version 1.0. It was only when I actually tried to do the derivation within CDEG that I discovered that the proof I had in mind wouldn't work because it relied on Euclid's Proposition 2, and one case that arose in the proof of Proposition 2 could only be proven using common notion 3. Thus, segment subtraction is now included in CDEG as a primitive rule. I don't think I would have found this mistake without using the computer system itself, and it was a significant omission, since without this change, even Euclid's Proposition 2 would not have been derivable! I thought that I had found something that Euclid missed, but it turned out that, as usual, he was one step ahead of me.

I hope that readers of this chapter will be interested in trying CDEG for themselves. It can be downloaded from www.unco.edu/NHS/mathsci/facstaff/Miller/personal/CDEG/. However, the version of CDEG that is now available is a beta version and most likely still contains bugs. If you try out CDEG and discover any bugs, please let me know by sending me an email. I can be reached at nat@alumni.princeton.edu.

6 Philosophical Implications for Teaching

I have not used CDEG with students in any geometry class that I have taught. However, I think that the philosophical issues it raises are directly relevant to how we teach geometry, and that they arise naturally in many geometry courses.

Geometry classes tend to be a venue in which ideas about proof and proving come up in our curricula. So they are a good place to discuss philosophical questions about proof: What does it mean to prove something? Why do we prove things? What constitutes a convincing argument in mathematics, and how are they different from convincing arguments in other disciplines? Where do definitions come from, and what is their role in proof? How do we pick what assumptions we are going to make, and then how do we use them? The geometry classes that I teach are generally taught in an inquiry-based format, and these sorts of questions arise naturally all the time. (For detailed descriptions of some of my inquiry-based geometry classes, see [8] and [9].) They are questions whose answers mathematicians tend to take for granted, but students who are asked to write definitions and proofs without a specific template for the first time invariably struggle with them.

As, indeed, they should. The answers to the questions are by no means as simple as they might seem, and their difficulties are reflected in many geometry curricula. For two thousand years, Euclid's *Elements* was considered the gold standard of logical reasoning, and was itself the basis for most geometry curricula. With the rise of formalism in the philosophy of mathematics, however, the new view was that Euclid's reasoning was not rigorous, and that a lot of the lack of rigor came from Euclid's use of diagrams. This gave rise to formal sentential axiomatizations of geometry, like that found in [3]. However, unlike Euclid's *Elements*, the formalizations of geometry don't reflect informal practice in giving proofs in geometry very well, and they generally don't work well as a starting point for students studying geometry for the first time. So we get geometry textbooks—high school geometry textbooks, especially—that don't really want to follow either kind of approach, and end up muddled somewhere in the middle, trying to follow an axiomatic system more complicated than Euclid's, adding new axioms whenever a proof would be too hard, but still don't make clear precisely what role diagrams play in their proofs.

CDEG and its underlying formal system are intended to show geometry teachers that diagrammatic proofs in the style of Euclid can be made perfectly rigorous, so that teachers don't need to shy away from these kinds of proofs. For the most part, they probably won't show CDEG to their students, any more than they would show them a fully formal sentential presentation of geometry. Instead, I would hope that teachers would share with their students informal geometric proofs using diagrams in the style of Euclid, would ask their students to be able to produce their own proofs in this style, and that, along the way, they would facilitate wide-ranging class discussions of the basic philosophical questions of what makes these proofs convincing or unconvincing.

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VII

Other Upper-level or Capstone Courses for Mathematics Majors

The advantage of discussing issues in the philosophy of mathematics in upper-level or capstone courses for mathematics majors is that students at that stage are sophisticated enough to appreciate many of the issues that arose during the foundational period. They bring their own experiences with mathematical objects to epistemological and ontological discussions. On the other hand, in most upper-division courses (except capstone or senior seminar courses) there is so much mathematics one wants to include that it is hard to make time for philosophical considerations. Yet sometimes finding that time improves what students remember long after the course is over.

Chapter 16: Sally Cockburn, in her “Senior Seminar in Philosophical Foundations of Mathematics,” discusses a wide variety of topics in the philosophy of mathematics. The course begins with students studying transfinite numbers as a mathematical topic. In the second half of the semester, the discussion turns philosophical. Among issues considered are the ontological status of infinite numbers: are they mathematical objects in the real world or social constructions? Students are led to discover the set-theoretical paradoxes, and then read of the responses to them: logicism, formalism (axiomatics) and intuitionism, and Gödel’s theorems. They also read about the problems with infinity earlier in the history of mathematics, as well as other earlier topics (such as Kant’s classification of knowledge) in the philosophy of mathematics.

Chapter 17: Nathan Moyer, in “Connecting Mathematics Students to Philosophy,” reports on a three-week project (one that uses little class time) to provide students with an overview of philosophical questions. He uses it in a real analysis course, but it would work equally well in other upper-level proof courses (such as number theory, abstract algebra, or senior seminars). To consider “what is mathematics?” students learn about logicism, intuitionism, formalism, and platonism. They also consider how they are connected with the four ways of knowing (intuition, empirical senses, innate reason, and authority). The project uses some of what students had read earlier in required university core classes and involves three assignments. For the first two, they read several articles and write a short response paper; the final assignment is a paper discussing their answer to one of the questions raised, and how their view of mathematics affects how they do mathematics.

Because Moyer’s university has specific core requirements that are rare at non-religiously-affiliated institutions, significant modifications would be necessary to adapt Moyer’s project elsewhere—but many institutions have some core requirements. The idea of connecting some of the knowledge and understanding that students have developed in core courses to philosophical issues in mathematics is worth considering at a range of schools.

16

Senior Seminar in Philosophical Foundations of Mathematics

Sally Cockburn
Hamilton College

1 Introduction

At Hamilton College, the small liberal arts college at which I teach, every concentration must culminate in a capstone experience, undertaken by students in their senior year. In the mathematics department, it takes the form of a seminar, capped at ten students, whose defining characteristic is that class time is devoted almost entirely to student presentation and discussion, with the instructor mostly playing the role of moderator. We offer seminars in a number of different areas, including dynamical systems, statistics, topology, and graph theory. In 2008, I developed one focusing on the famous set paradoxes, which I somewhat grandly titled Philosophical Foundations of Mathematics. Since then, it has been offered four times, and each time I have received a very positive reaction from students, who tell me that they appreciated the new mathematical content and the opportunity to discuss the broader nature of the discipline as a human intellectual endeavor.

2 Description of the Course

To recruit students, I circulate the following to juniors pondering their senior seminar options.

> In the first few weeks of real analysis, you learned that there are different types of infinite sets, namely countable and uncountable sets. Can we treat the cardinalities of infinite sets as actual numbers? Georg Cantor thought so, and in the late nineteenth century, he developed an extensive theory of what he called transfinite numbers, including
>
> how to tell when two transfinite numbers are different;
>
> how to compare the sizes of two different transfinite numbers;
>
> how to add, multiply and exponentiate transfinite numbers;
>
> how many different transfinite numbers there are.
>
> At first, Cantor's work met with considerable resistance from the mathematical community, but by the turn of the century, the world's foremost mathematician, David Hilbert, declared, "no one will drive us

> from the paradise that Cantor has created for us"! Then, disaster struck. Various mathematicians, including Cantor himself, started finding troublesome paradoxes in the theory of infinite sets. Serious mayhem in the foundations of mathematics ensued. Many issues remain unresolved to this day.

The seminar meets for two 75-minute classes a week, and is divided into two quite different parts. During the first nine and half weeks of the semester, students learn the basics of naive and axiomatic set theory. The last four and half weeks of the course are devoted to reading and discussing a selection of papers on some of the philosophical issues raised by this material.

2.1 Set Theory

I use the Moore method for the first portion of the course, inspired by my colleague Richard Bedient, who has been using it for years in his topology senior seminar. (See his recent article in PRIMUS [1].) Students are collectively responsible for fleshing out the skeleton of a textbook on set theory; it contains definitions as well as lemmas, propositions, and theorems, but the students must supply examples and proofs. In writing the skeletal textbook, I borrowed very heavily from Seymour Lipschutz's volume in the *Schaum's Outlines* series [7] for material on naive set theory, and from Robert Wolf's *A Tour Through Mathematical Logic* [12] and Mary Tiles' *The Philosophy of Set Theory: An Historical Introduction to Cantor's Paradise* [11] for axiomatic set theory. A list of the chapter titles indicates the topics covered.

1. Naive Set Theory: Basic Definitions
2. Cardinal Numbers
3. Ordered Sets
4. Well-Ordered Sets
5. Ordinal Numbers
6. Set Paradoxes
7. Axiomatic Set Theory
8. Transfinite Numbers in Zermelo-Fraenkel Set Theory
9. The Cumulative Hierarchy of Sets

As a sample of what the students are given, see the current version of Chapter 1 in the appendix to this chapter, which we cover in the first 75-minute class meeting. TEX files for all chapters are available on my home page (people.hamilton/edu/scockbur). An additional facet of the course is that students learn how to typeset mathematics using TEX. For this reason, we usually refer to the textbook as the TEXbook.

Students are expected to come to class having read and filled in the shaded boxes of the section(s) assigned for that day. During class time, students take turns presenting their examples and proofs at the board. The audience is expected to respond actively, by constructively pointing out errors, asking for clarification, or offering alternative approaches. I tell the class that I also welcome feedback, and every year students have made suggestions for reorganizing, adding to or deleting from the TEXbook to improve it for future incarnations of the seminar. At each class meeting, a different student acts as the scribe, responsible for recording the presented material and inserting it in the appropriate sections of the textbook.

For this portion of the course, I grade students not only on the homework they bring to class, but also on their willingness to present their work at the board and to respond to their fellow students' presentations. Students tend to be quite shy initially, but I do my best to engender an atmosphere that is informal and collaborative, rather than competitive. Once they see that everyone makes mistakes at the board, even the best students, they relax and eventually are jostling to present what they see as the choice proofs. I give two midterms on the material, because that is the incentive that math majors are accustomed to for keeping up with terminology and theory.

As students work through the TEXbook, they experience increasing disorientation. Most have taken real analysis with me as juniors, and after some resistance eventually made their peace with the distinction between countably and uncountably infinite sets. They get a thrill from learning how Cantor defined addition, multiplication, and exponentiation of his transfinite cardinal numbers, but tend to view this work as they would fantastical stories of fictional characters. (Tellingly, the fall 2012 seminar students designed a cover for the TEXbook featuring characters from the recent film adaptation of J. R. R. Tolkien's *The Hobbit*, and inserted as an epigram the following quote from J. K. Rowling's *Harry Potter and the Deathly Hollows*: "Of course it is happening inside your head, Harry, but why on earth should that mean it is not real?" [9]). This prompts discussion on the ontological status of infinite numbers. I prod them for their opinion on the ontological status of finite numbers, or indeed triangles, functions, vector spaces, and other mathematical objects. This leads naturally to the question of whether mathematics is somehow out there in the real world (the platonist position), or is simply a social construction. However, I limit the discussion to about ten minutes or so, saying that we will return to the debate in the second portion of the course. My aim at this point is merely to raise philosophical issues, while maintaining as neutral a stance as possible. I believe students need some time to think through their own positions, and the opportunity to reevaluate them as we work through the technical material.

A theorem students are asked to prove in Chapter 5 asserts that every ordinal number has an immediate successor. I can usually count on the person presenting the proof to argue that given a well-ordered set A, we can let x be something that is not in A, and then A with x adjoined at the end will be a well-ordered set of greater ordinality. I can also usually count on someone in the class asking, "What if A is the set of everything (i.e., the universal set)?" Again, I suppress debate after about ten minutes of discussion, essentially sweeping the problem under the rug. Students start to sense that we are rowing a boat that is springing leaks, replicating the growing unease the mathematical community of the late nineteenth century experienced. Serendipitously, we cover the famous set paradoxes the day before the October fall recess, so students have a long weekend to ponder their implications. Until this point, most of the math majors, either consciously or unconsciously, have viewed their subject as a bastion of Absolute Truth. They have never had occasion to call into question an entire body of material that had seemingly been proved using the same trusted logical techniques they have seen throughout their college careers.

When students get back to campus and we progress to axiomatic set theory, they tend be suspicious of the entire enterprise. The axioms of foundation and replacement seem far from self-evident. The subset axiom and power set axiom seem to be included only to preserve Cantor's theorem, which, to many students' minds, seems to be the main source of the problem. The distinction between Zermelo-Fraenkel sets and classes seems artificial. They bristle at the line in the TEXbook that states, "like the primitive terms and axioms, the laws of logic are simply accepted without explanation or proof of validity." As we work though the chapter on transfinite numbers in ZF, students are able to prove propositions, but struggle to gain an intuitive understanding of transitive sets, let alone ordinals and cardinals, paving the way for their appreciation of the tenets of formalism. The final chapter of the TEXbook covers the cumulative hierarchy of sets, and to help prepare students for this material, I have them read George Boolos's essay "The iterative concept of set" [2]. This serves as a nice transition to the second phase of the course.

2.2 Readings

For this portion of the course, students are required to come to class with a one-page response to the day's reading(s) that includes a concise summary of the main points and a list of questions or issues they want to discuss. I appoint someone to start off the class by presenting a response for the first fifteen or twenty minutes; the remainder of the class is a free-wheeling discussion on whatever students found most intriguing or controversial. I assign a letter grade to the response papers as well as on their overall participation in class discussions.

By this point, students are bursting to engage in the philosophical debates that have been percolating under the surface for nine weeks. Before getting to papers dealing directly with the set paradoxes, however, I assign three readings providing some historical context. We spend one class each on

1. A. W. Moore, *The Infinite*; Introduction, Chapters 1 and 2 [8]
2. G. Berkeley, "The Analyst" [3]
3. I. Kant, *Critique of Pure Reason*: Introduction, Transcendental Aesthetic, sections 1–7 [6].

The selections from Moore's book summarize classical Greek approaches to the infinite, including a summary of Zeno's paradoxes, a comparison between metaphysical and mathematical infinity, and Aristotle's distinction between actual and potential infinity. I include Berkeley's paper so that we can discuss how a prior crisis in mathematics, also involving the infinite, was resolved by replacing the notion of infinitesimals with the notion of limits (essentially, replacing a concept based on actual infinity with one based on potential infinity). The Kant reading introduces students to his classification of knowledge as *a priori* or *a posteriori*, analytic or synthetic. We also compare Kant's views on the relative merits of mathematics and metaphysics with those of Berkeley. This usually prompts a general discussion of faith in mathematics, particularly faith in axioms and faith in the laws of logical inference.

The readings for the next class are

4. (a) G. Cantor, "Grundlagen" sections 1–8; Late Correspondence with Dedekind and Hilbert [4]
   (b) D. Hilbert, "On the infinite" [2].

In section 8 of "Grundlagen," Cantor draws a distinction between what he calls the immanent and transient reality of numbers, which allows students to return to a discussion of the ontological status of numbers. Cantor also makes a poignant argument in this section for intellectual freedom, suggesting that new mathematical concepts be judged purely on their usefulness; those that are "fruitless or unsuited for their purpose . . . will be abandoned for lack of success." [4]. Students are surprised to learn that issues of censorship could arise in so apparently objective a discipline as mathematics, but are also leery of Cantor's seemingly market-driven approach to mathematical truth. Hilbert's seminal paper serves as a linchpin for the course. In it, Hilbert decries the chaos produced by the set paradoxes, sets as a goal the preservation of as much of Cantor's approach to transfinite numbers as possible, lays out the basic principles of formalism and describes his famous program.

The next class is devoted to examining the schools of logicism, formalism, and intuitionism. The readings I assign are

5. (a) A. Heyting, "Disputation" [2]
   (b) B. Russell, "Introduction to mathematical philosophy" [2]
   (c) L. E. J. Brouwer, "Intuitionism and formalism" [2].

I include the amusing paper by Heyting because it lays out some broad outlines that can help students navigate the more nuanced papers by Russell and Brouwer. I have found that student reactions to intuitionism vary greatly. Some years I am the only one willing to defend the intuititionist approach and other years I am the only one willing to criticize it.

At this point in the fall semester, we have reached our week-long Thanksgiving break. In preparation for our coverage of Gödel, I assign Rebecca Goldstein's highly readable book *Incompleteness: The Proof and Paradox of Kurt Gödel* [5] over the holiday. I treat this as background material only, telling students we will not spend any class time discussing it. To my surprise, they still read it. Our week on Gödel is as follows:

6. K. Gödel, "What is Cantor's continuum problem?" [2]
7. *Gödel's Incompleteness Theorems* (worksheet available at people.hamilton/edu/scockbur).

The paper by Gödel includes his eloquent defense of platonism. After stating his belief that the continuum hypothesis must be either true or false, he muses on the likelihood of it being one or the other. Students are surprised that he also refers to the fruitfulness criterion mentioned by Cantor. I have to repeatedly remind them that Gödel is only using it as one tool in making an educated guess. There is obviously not enough time in the course to give a very thorough treatment of Gödel's incompleteness theorems, but my worksheet goes into somewhat more detail than Goldstein's book. This is one instance where I stray from the seminar format and give a traditional lecture, the only other instances being lectures in the first portion of the course on the Cantor-Schröder-Bernstein theorem and the well-ordering theorem.

For the last week of the semester, the readings are

8. A. J. Ayer, "The *a priori*" [2]
9. Max Tegmark, "The mathematical universe" [10].

Goldstein's book tends to demonize logical positivism so I believe it is important to assign a reading by someone who can defend it. In various incarnations of the course, I have used for this purpose Carnap's "Empiricism, semantics and ontology" [2], Quine's *Two Dogmas of Empiricism*, or extracts from Wittgenstein's *Tractatus* (the last two available online), but I have found students more receptive to Ayer's arguments. Discussion often centers around Ayer's contention that Kant provides both semantic and psychological criteria for distinguishing between analytic and synthetic knowledge. Ayer argues that a sufficiently intelligent being would know all of mathematics immediately upon being informed of definitions of the terms involved. Mathematics is simply one vast tautology that tells us nothing about the empirical world. Max Tegmark's article provides a sharp contrast, with its central thesis being that the universe *is* a mathematical structure.

At the end of this portion of the course, students write a six- to eight-page final paper. The assignment is to select a paper in the philosophy of mathematics that is not covered in class, summarize the main points of the paper, put the paper in the context of the material we have read, and explain their own areas of agreement and disagreement with the paper. Roughly a week before the paper is due, I have the class over to my house for dinner, where they must each give a ten-minute informal oral presentation on the work they have done on their paper so far. I have found this step invaluable in helping students focus their ideas and anticipate possible counterarguments to their positions.

3 Student Reaction

The best student evaluations I have received have been from students in this course.

> [T]his is the best course I have taken to Hamilton to date. The combination of technical mathematics, introduction to respected mathematical writing, and discussion of mathematical philosophy gave rise to an enormous development in thinking skills and an enormous expansion of our (students) knowledge of set theory at a much higher level than I would have expected in the undergraduate curriculum.

> I thought the class was great. I really enjoyed both parts of the course: the TEXbook and reading philosophical papers. I thought the two halves went very well together.

> This is the most intellectually challenging class I have ever taken. I found it very challenging at first to write proofs on the board in front of an audience of peers, but I felt that I acclimated to this environment later in the semester. Class discussions were extremely engaging.

> The seminar style, student-based discussion and lecture is an excellent way to learn mathematics. Hearing other students present proofs and their method of thinking is invaluable to developing one's own mathematical knowledge. The discussion in the philosophy third of the class was extraordinarily interesting, engaging and intellectual.

> The course was extremely interesting and challenged everyone's opinions and views. I cannot imagine a better way to stretch myself academically. As a primarily student-run course, everyone in the class took pride in the work they did. It was the most open and rigorous learning environment I've ever been in.

> I thought this class was exceptional—some really mind-bending material that probed at the very heart of mathematics.

It is clear from the comments that the success of the course owes as much to the seminar format as to the content. Because the class size is small (twice there have been ten students, twice there have been only five), a lot of bonding goes on, both inside and outside the classroom. Another major factor in the success of the course, in my opinion, is the presence of a large number of students pursuing double majors. For example, students who are also concentrating in history have provided valuable insights on the intellectual environments in which Aristotle, Berkeley, and Kant worked. In particular, they caution their classmates against the arrogance of dismissing thinkers who were unaware of, say, non-Euclidean geometry. Students also concentrating in psychology and neuroscience tell others in the class about experiments that suggest that animals such as pigeons and rats subitize (demonstrate primitive arithmetical abilities), enriching our discussions about whether mathematics exists inside or outside human beings. Students also concentrating in physics contribute their knowledge of quantum theory and astrophysics to our discussions of whether infinity exists in the physical world. They can offer a modern response to Berkeley's query "whether there be any need of considering Quantities either infinitely great or infinitely small?" [3] Double concentrators in computer science relate Gödel's work on incompleteness to computability and the halting problem.

4 Final Thoughts

I believe that this course gives students the opportunity to step outside the inner workings of mathematics and get some sense of the discipline as a whole. As senior math concentrators, they have been exposed to a number of courses in pure and applied mathematics, and as liberal arts students, they have been trained in critical thinking as well as written and oral communication skills. In the foundations seminar, they can bring all their intellectual tools to the table and forge a memorable capstone experience.

5 Appendix

The following is the first chapter of the TEXbook, which we cover in the first 75-minute class meeting. Students are told that each shaded box requires some action from them, whether it be providing a formal proof or merely fleshing out the details of a definition, example or note.

Naive Set Theory: Basic Definitions

Definition. A *set* is a collection of objects, called the *elements* or *members* of the set.

Note. For now, the terms "set" and "collection" will be treated as synonymous.

The empty set is a collection of no objects, denoted either $\emptyset$ or $\{\}$. (Be careful: $\{\emptyset\}$ denotes. . . .)

Definition. Set B is a *subset* of set A iff . . . . . The *power set* of a set A, denoted $\wp(A)$, is. . . .

Note. To show that two sets A and B are equal, ...

Definition. The *union* of sets A and B is given by $A \cup B = \ldots$, and the *intersection* is $A \cap B = \ldots$ The *relative complement* of B in A is $A \backslash B = \ldots$

Definition. The *Cartesian product* of two sets A and B is given by

$$A \times B = \{(a, b) \,|\, a \in A, b \in B\}.$$

More generally, for a finite collection of sets $A_1, A_2, \ldots, A_n$,

$$\prod_{i=1}^{n} A_i = A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n)\} \,|\, a_i \in A_i\}.$$

Definition. If A and B are sets, a *relation from A to B* is a subset R of $A \times B$.

Notation. For all $a \in A, b \in B, a\, R\, b \iff (a, b) \in R$.

Example 5.1 Let $A = \{1, 2, 3, 4\}$ and $B = \{10, 11, 12, 13\}$ and define a relation by $a\, D\, b \iff a$ divides b. Then $D = \ldots$

Definition. An *equivalence relation* is a relation R from a set S to itself that is reflexive, symmetric and transitive; that is, ...

Definition. If R is an equivalence relation on a set S and $x \in S$, then *the equivalence class of x in S under R* is given by $[x] = \{y \in S \,|\, (x, y) \in R\} = \{y \in S \,|\, x\, R\, y\}$.

Example 5.2 Let $S = \mathbb{Z}$, and define $x\ R\ y \iff x - y$ is divisible by 3; that is, $x - y = 3k$ for some $k \in \mathbb{Z}$. Then R is an equivalence relation on $\mathbb{Z}$ because ... Moreover, $[0] = \ldots$.

Theorem 5.1 An equivalence relation R partitions a set S into equivalence classes.

Proof. We must show that every $x \in S$ belongs to one and only one equivalence class. Since R is reflexive, $x \in [x]$. Now suppose that we also have $x \in [y]$ for some $y \in S$. Then we must show that $[x] = [y]$ ...... □

Definition. A *function* $f : A \to B$ is a relation $f \subseteq A \times B$ such that for each $a \in A$, there exists exactly one ordered pair in f whose first coordinate is a. An *injection* or *one-to-one* function is...; a *surjection* or *onto* function is ....; and a *bijection* is ... [Phrase these in terms of this new definition of a function as a set of ordered pairs.]

Note. If $A \neq \emptyset$, [are there any functions $A \to \emptyset$ or $\emptyset \to A$? What about $\emptyset \to \emptyset$?]

Notation. If $f : A \to B$ is a function, then for all $a \in A$ and $b \in B$, $f(a) = b \iff (a, b) \in f$.

Definition. If $f : A \to B$ is a function and $C \subset A$, then the *restriction of f to C* is the function $g : C \to B$ defined by $g(c) = f(c)$ for all $c \in C$.

Notation. The restriction of f to C is sometimes denoted $f|_C$.

Note. Any restriction of an injective function is injective. This is not the case for surjectivity.

Note. If $f : A \to B$ is a function, then $f : A \to f(A)$ will be a surjective function.

Definition. Let $\{A_i \mid i \in I\}$ be an arbitrary collection of sets. Then the union and intersection of the collection are respectively

$$\bigcup\{A_i \mid i \in I\} = \{x \mid x \in A_i \text{ for some } i \in I\},$$
$$\bigcap\{A_i \mid i \in I\} = \{x \mid x \in A_i \text{ for all } i \in I\}.$$

Definition. Let $\{A_i \mid i \in I\}$ be a nonempty collection of nonempty sets; that is, the index set I could be finite, countably infinite, or uncountable. A *choice function* is any function $f : \{A_i \mid i \in I\} \to \bigcup\{A_i \mid i \in I\}$ satisfying $f(A_i) \in A_i$. That is, a choice function chooses exactly one element from each set in the collection.

Definition. The *Cartesian product* of a nonempty collection of nonempty sets $\{A_i \mid i \in I\}$, denoted by $\prod\{A_i \mid i \in I\}$, is the set of all choice functions defined on $\{A_i \mid i \in I\}$.

Note. This generalizes the definition of Cartesian products of finitely many sets because...

AXIOM OF CHOICE. *The Cartesian product of a nonempty collection of nonempty sets is nonempty. Equivalently, there exists a choice function for any nonempty collection of nonempty sets.*

Note. This is an *axiom*, not a theorem. We do not *prove* it is true, we *accept* it as true.

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Connecting Mathematics Students to Philosophy

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1 Introduction

Exploring the rich supply of philosophical questions that arise in each academic discipline is certainly a worthwhile calling, but also brings challenges. This is especially true of mathematics. Undergraduate mathematics majors will often complete their four-year degrees without once being exposed to the philosophical assumptions and underpinnings of mathematics. They focus on the question of how to do mathematics, through calculations and procedures, and justifying them through an introduction to rigorous proof. Yet rarely do they have an opportunity in the classroom to ask the simple question: what is mathematics?

This chapter describes an upper-division mathematics class project that helps students wrestle with the deep philosophical questions of mathematics in a relevant way. It will discuss the ways in which students are able to explore historical views of the philosophy of mathematics, including the four main philosophies of logicism, formalism, intuitionism, and platonism. The role of the instructor is primarily to facilitate discussion, encourage the students to think deeper about the issues, and help foster connections to a student's worldview. The goal of the project is to help students establish their own responses to philosophical questions that are grounded in their personal worldviews.

This project was carried out in the first semester of a two-semester sequence of real analysis consisting of 25 junior and senior mathematics majors. While many of the students were planning on attending graduate school in a mathematics related field, others were future teachers, and still others were industry-bound. The project is suitable for any upper-division mathematics course, yet it would be recommended for a proof-based course such as analysis, abstract algebra, number theory, or even a senior capstone.

Whitworth University is a private Christian liberal-arts institution. The undergraduate enrolment is about 2,300 students. Whitworth has a worldview studies program that is part of the general education curriculum. It requires that students take a sequence of three Core courses that furnish them with the basic categories of worldview thinking, including the nature of God, the nature of humanity, how we know, the nature of reality, and how we should live individually and corporately. The students are also equipped to explore the parameters of their own worldviews. I utilized Whitworth's Core program to provide a foundation for each student to view the deeper questions of mathematics.

2 Class Project

The project itself is not intended to be an exhaustive in-depth study of the philosophy of mathematics. Its purpose, however, is to provided students an overview of philosophical questions to stimulate conversation and reflection. It spanned roughly three weeks of the semester, but only half of two class days were taken up with discussion time. Thus, I was able to minimize the amount of class time taken away from covering the required topics of the course.

There were three primary goals I sought to achieve through this project. By the end of the semester I wanted students to

- be exposed to deeper questions about the nature and foundational assumptions of mathematics,
- gain a basic understanding of the four main philosophies of mathematics from a historical perspective, and
- reflect on these questions in light of their own personal worldview.

The first goal is intended to simply broaden the students' view of mathematics. That is, to help them see and appreciate an area of mathematical understanding beyond the standard undergraduate curriculum. I believe that wrestling with these deep questions is a worthwhile goal, especially at a liberal arts institution where a high value is placed on developing students with strong intellectual capacity and curiosity.

The second goal is necessary in order for students to grasp how others throughout history have addressed these questions. They have the opportunity to examine the mathematical philosophies of logicism, intuitionism, formalism, and platonism as well as the mathematicians who espoused them. This provides students a framework of terminology and concepts to help process many of the deeper issues addressed.

The third goal allows students the opportunity to ground their philosophical viewpoints of mathematics on their own worldview or deep convictions. Prior to this class each student had already spent some time defining and reflecting on his or her own personal worldview through Whitworth's Core program.

The project itself is broken up into three assignments. The first two consist of article readings, a short response paper, a time of small group discussion, and a brief whole class discussion. The third assignment, a culminating paper, requires students to personally reflect on the first two assignments.

Assignment 1

Students read "Mathematics as an objective science" [5], *Mathematics: A Concise History and Philosophy* [1], chapter 39, and "Mathematics and the myth of neutrality" [3]. The first two provide a historical context for the development of philosophies regarding the foundations of mathematics. They explore how formalism, intuitionism, logicism, and platonism developed, define their core tenets, and discuss the inadequacies of each in providing a comprehensive foundation for all mathematical thought. The third article's thesis is that beliefs are integral to mathematics and that ultimately mathematics is not neutral, but shaped by worldviews. This viewpoint is articulated in the article by the following quote from Paul Ernest [3]:

> Mathematical truth ultimately depends on an irreducible set of assumptions, which are adopted without demonstration. But to qualify as true knowledge, the assumptions require a warrant for their assertion. There is no valid warrant for mathematical knowledge other than demonstration or proof. Therefore assumptions are beliefs, not knowledge, and remain open to doubt.

After reading the articles, the students wrote a short response in which they summarized and briefly reflected on the content presented. Then, they had time during class to get in groups of three or four to discuss what they had read. To guide the discussion forward they were given a list of thought-provoking and reflective questions:

- Do you agree or disagree with Ernest's quote above? Explain.
- To which of the four philosophies would a statistician, geometer, or physicist likely belong?
- How would each of these philosophies likely view the fundamental theorem of calculus?
- We have discussed the four ways of knowing: intuition, empirical senses, innate reason, and authority. To what extent has each of these played in your understanding of mathematics?
- Consider the definition of a function given in your textbook. List all the definitions and operations that must be known prior to understanding the concept of a function. How are they known?

One of the challenging aspects of developing this project was coming up with good questions that stimulate reflection and discussion. For this assignment, the questions were instrumental in forcing the students to think deeply about their understanding of mathematics. When confronted with the fourth question, many students initially indicated that authority was their primary mode of understanding mathematics. The teacher and the book tell us what mathematics is, and we believe it. Yet, the other epistemologies begin to take a more significant role as the students begin to wrestle with their dependence on intuition and reason.

My rationale for posing the second question was to have students think about how one's philosophical view of mathematics may impact or reflect the ways in which one uses mathematics. For example, many of my students felt that a geometer would more likely be a formalist than a intuitionist based upon the nature of his or her mathematical work. Similarly, a physicist would most likely hold to a platonist viewpoint because of his or her use of mathematics in describing the physical universe. For a physicist's work to have meaning, there must be a belief in the existence of mathematical objects independent of our human language and thoughts. There were no right or wrong answers, but it helped students to solidify their understanding of each philosophy in a meaningful context.

The purpose of this assignment was to begin to move students into a deeper conversation about mathematics, one which many students had never taken part. It addressed the goal of exposing students to the larger philosophical nature of mathematics. It forced students to confront the idea that one's philosophical view of mathematics may be shaped by a particular set of beliefs and have an impact on the ways in which one engages with mathematical concepts. Several students commented that this philosophical discussion was entirely new to them. It served to enlarge their view of mathematics beyond simply calculations and proofs.

Assignment 2

This assignment is primarily designed to explore how the work of St. Augustine of Hippo can inform this philosophical discussion. Augustine was an early Christian theologian whose writings were very influential in the development of Western philosophy. With this understanding, the students read an article in the *Journal of the Association of Christians in the Mathematical Sciences* entitled "An Augustinian perspective on the philosophy of mathematics" [4]. The article examines how Augustine addresses four basic philosophical questions of mathematics based upon his view of God. What is the nature of mathematical objects? How do we obtain knowledge of them? What is the meaning of "truth" in mathematics? How do we account for the effectiveness of mathematics in describing the physical universe? Augustine's views are presented in a very accessible way for students to grasp. The article goes on to describe the historical shifts in the philosophy of mathematics, including contributions from Descartes, Kant, and Russell.

Similar to Assignment 1, the students are given class time to converse about the articles in small groups. Here is a list of the discussion questions posed:

What can you infer about the worldview of Descartes, Kant, Russell, and Augustine based on their view of mathematics?

Would these philosophers agree that all knowledge is contingent upon faith? Do you agree with this statement?

What was the relationship between theology and mathematics before the Renaissance and Scientific Revolution? After?

What do you think about the statement "God created the integers and all else was the work of man"?

The students seemed quite engaged during this discussion time as I observed more lively debates and open conversations. Obviously my students came into the class with a variety of beliefs, some of which led to disagreements with Augustine's assertions. This served to increase the quality of the discussion as each student's perspective was heard. The goal of the next assignment is to convert that perspective from the verbal discussions into a reflection paper.

Assignment 3

The project culminates in a two-part paper that brings together many of the concepts developed in the articles and discussions. In a two-page paper the students were to articulate a personal answer to one or several of the philosophical questions introduced in class regarding the nature of mathematics. Their response was not simply to come from their casual, surface-level observations, but from their deep convictions about the nature of God, humanity, and the world. Toward that aim, they were prompted continually to seek out connections between their view of mathematics and their personal worldview. I also made available some supplementary articles that were not required reading but suggested as additional resources. The articles, given in the list of references, include [2] and [6].

The paper prompt was intentionally vague in order to provide freedom for the students to explore what they wanted. I assured my students that there were no right or wrong answers. These questions had been pondered for centuries with no tangible, concrete conclusions. Students were not being graded heavily on content, but rather on the depth of reflection and the connections between their worldview and mathematics.

While engaging the question of whether mathematics is created or discovered, one student wrote:

> I believe that only the language of mathematics is created by humans. That is, the physical notation we use to write out, solve, and analyze problems. The heart of mathematics exists outside of a person's created understanding of it. We do not create the properties that will then govern effectively every area of our lives. We do our best to understand mathematics, creating definitions that most accurately describe something we are discovering. The mathematics that govern such things as nature did so before humans attached any labels. Flowers did not begin to exhibit mathematical patterns once we "created" the Fibonacci sequence; rather we discovered the pattern, realizing that it had existed in our world all along.

The second part of the paper was designed to bring some practical application to the philosophical rumination. I wanted students to discuss how any of this makes a difference in their lives. Thus, they wrote a one-page paper in response to the question: how does your view of mathematics, as described in part one of the paper, affect the way you do mathematics? Within this framework, they were to reflect on their motivation for being a mathematics major.

This portion of the paper was primarily intended to move students away from the standard response of "I like math because I'm good at it." After Whitworth, many of the students would go on to graduate school, teach, or work in industry in some field related to mathematics. This required them to dig deeper and examine the role of mathematics in their future vocation. Many students enjoyed the opportunity to write about mathematics on a more personal level.

Pondering the connections between her view of mathematics and her future career, one student wrote:

> My view of mathematics will be very important to my career. As a teacher, I will have the opportunity to model mathematics as more than just symbols and formulas, but as a framework which the world lies in. My worldview of math should be evident in the way I instruct my students about math. It gives purpose to the field of study in schools because it is a subject that is practical, applicable, and useful in understanding the world, rather than being merely a subject that a student must take to pass a state required exam.

3 Student Learning

I believe that the philosophy of mathematics can be a tremendous way to introduce a new dimension of mathematics into the classroom. Understanding the underlying assumptions of mathematics in light of a worldview framework provides a depth of education that a standard undergraduate curriculum may not afford. This serves to educate the entire person, mind and heart, as is consistent with the mission of Whitworth and many liberal arts institutions.

At a minimum, the project served to expand students' view of mathematics. It provided a categorical framework for them to define a view of mathematics that they had never considered. It confronted them with the prospect that their beliefs may meaningfully interact with their view of mathematics. It required them to consider introspectively their motivation and calling as it relates to engagement with mathematics.

Although I felt that my students were prepared to take part in the project, I was realistic in what I could expect from their papers. I was not looking for fully developed philosophical arguments about the nature of mathematics. Since this was the first time my students had ever confronted these issues, I was satisfied with evidence of self-reflection, critical thinking, and an effort to make worldview connections. The questions that arise from a deep philosophical examination of mathematics may never have satisfactory answers. Yet, we should not let that deter our students from wrestling with them. Their search for answers has the potential to yield much fruit, including an awareness and clarity of personal vocation. This is vital in moving students beyond the idea that mathematics is simply a tool necessary for a successful career. It also helps students view their mathematical skills and abilities for greater purposes in light of their calling.

4 Student Feedback

At the beginning of the semester I asked students if they had previously given any thought to the philosophy or origins of mathematics. All but one student said that they had given very little or no thought to this subject. So the material covered in the article readings and discussion time was uncharted territory for nearly all of the students. Overall, they responded very positively to this project. I received feedback from the students both informally through conversations and formally through a survey I conducted at the end of the semester. Following are some of their responses:

> It was surprising that there were so many different views of the origins of mathematics. Before this assignment I believed that everyone had the same view about where mathematics came from.
>
> I feel as if I have more tools to articulate my view of mathematics. My view hasn't changed, but I feel better prepared to explain and defend my view. It is more defined for me now.
>
> Reading those articles really caused me to think more about what I agree with and what I disagree with and how I would fit into different categories.
>
> I do mathematics because it's something I enjoy and it's a way for me to achieve my higher purpose of reaching the youth by building personal relationships in the classroom and showing them the beauty of mathematics. I will present some of these same things we talked about to my high school classes to get them thinking about what math really is.
>
> I don't think my worldview impacts how I do math at all. However, it definitely impacts why. The reasons I do math derive straight from my reasons for doing anything which boil down to worldview.

5 Discussion

From my experience with the project, students are hungry to discuss foundational areas of mathematics given the opportunity. In fact one of the most frequent comments I received from students was the lack of time committed to discussion. They wanted to hear the viewpoints of others in class beyond their small discussion groups. Although I allotted the last five minutes of the period to a full class conversation, it was not enough. In the future, I will definitely provide more scheduled time for a large group discussion. In addition, I will have students engage with several small groups to allow for more sharing of a diversity of opinions. I believe that the additional discussion time is needed for students to fully appreciate different viewpoints as well as feel like their voice is being understood by others. Because this time may infringe upon the delivery of other course content, another idea is utilizing an online discussion forum where students are encouraged to post their thoughts.

Obviously, there are many other recent philosophies that could serve to enlarge these discussions, such as structuralism and social constructivism. As the project was meant to be an introductory overview of the philosophy of mathematics, I felt that bringing them into the project added more complexity than we needed. One recommendation for improving the project would be to include a list of supplementary articles that include newer philosophical ideas. Then students would have an optional way to pursue those ideas that interested them.

As mentioned earlier, the project relied heavily on the previous work students had done reflecting on their personal worldviews in Whitworth's Core classes. It is vital for them to know exactly what they believe and why they believe it. Yet this project can function well at institutions that do not have a similar general education requirement. I would suggest spending a day in class or assigning some additional outside reading that introduces students to the basic categories of worldview thinking, including a basic introduction to epistemology. Once students have spent some time considering their personal worldviews, they will be better prepared to approach the philosophical questions of mathematics.

6 Conclusion

I believe there is currently a lack of awareness and acknowledgement of the philosophy of mathematics among undergraduate mathematics majors. Gaining insight into philosophical questions may not help students get into graduate school or find a job, but it certainly enlarges their view of mathematics. It also provides a means of deepening the important discussion of the interactions of worldview and mathematics. At Whitworth University, and many other liberal arts institutions, this type of discussion is highly valued as a key component of a larger goal of educating the whole person.

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VIII

History or Philosophy of Mathematics Courses

Chapter 18: Thomas Drucker, in "History Without Philosophy is Mute," describes how, throughout a history of mathematics course, some consideration of philosophical issues significantly enhances student understanding of the changes that were taking place. This can be especially important when the course is taken by future teachers, for whom an understanding of how mathematics has developed will inform their later teaching. He offers ways to incorporate philosophy of mathematics no matter whether the instructor is taking a chronological or a thematic approach.

Chapter 19: Reuben Hersh was an early advocate of teaching a course in the philosophy of mathematics that goes beyond the early twentieth-century foundational schools. In "Let's Teach Philosophy of Mathematics" reprinted from the *College Mathematics Journal* from 1990, he advocates teaching a philosophy of mathematics course from a humanistic perspective. The course discusses the nature of mathematical reality, the nature and meaning of infinity, and the relation between mathematics and the physical world. It studies the development of mathematics, such as the Dedekind-Cantor construction of the real numbers, from both historical and philosophical perspectives, as a solution of problems going back to the origins of the calculus. The course includes the foundational period, the problems that led to it and the outcomes as well as new trends in the philosophy of mathematics since Gödel's results.

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History Without Philosophy Is Mute

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1 Introduction

Departments of mathematics at many universities offer a history of mathematics course, at least in part to meet national requirements for accreditation for teacher certification (see [23], for example). However, most programs do not have space to include a course in the philosophy of mathematics. This chapter advocates the inclusion of some philosophy of mathematics in a course in history of mathematics, in part to fill the curricular gap.

More than that, however, the case can be made that it is impossible to teach the history of the subject fairly without covering relevant philosophical material. As I discuss below, material that is now considered more philosophical than mathematical used to be included as part of the curriculum for students of mathematics. The fundamental question of what a number is may not be raised in any class for prospective teachers. Understanding how mathematics has developed will be easier and recognized as more important if the developments are seen against a philosophical background. The changes in mathematics over the millennia are hard to miss, but philosophical changes with regard to mathematics have been perhaps even larger.

The University of Wisconsin-Whitewater offers a history of mathematics class each fall semester to students majoring in elementary education who minor in mathematics. They are required to have taken pre-calculus (or to have placed out of it), but a fairly large proportion of the students have not taken calculus. Since the university is one of the main sources for teachers in the state of Wisconsin, the requirements and the additional features of the courses required of would-be teachers have long-term repercussions.

History classes can easily take on a lecturing tone, not the best form of communication with a live audience. It can be difficult to get students to express opinions about mathematics, by contrast with getting them to answer explicit questions within mathematics. However, raising philosophical issues encourages such discussions.

2 Chronological Approach

One of the ways in which history of mathematics courses are typically arranged is chronologically. If that arrangement is used, then a question that is bound to arise is when, how, and why mathematics appears in early civilizations. It is impossible to answer that question without determining what mathematics is, and that is an essentially philosophical question. Moreover, this approach is consistent with many introductory science courses, which begin with a definition of the subject to be studied.

After the initial question of trying to identify mathematics, discussion of early mathematics (Egyptian and Babylonian, for example) is usually followed by trying to understand what led to the evolution of Greek mathematics. Among all the issues there is room for discussion of the Pythagorean view that all is number. Students are generally willing to believe this if it is taken to mean that mathematics helps us to understand the physical world. One way to pursue that side of the issue is to start the discussion of whether the applicability of mathematics reveals something about the world or something about our brains. Another side of the issue is to suggest that the early Pythagoreans meant something more by "all is number" than simply that mathematics is useful for science. It can be worth raising the point of whether the universe somehow is made up of numbers and what that might mean. The issue of how numbers make up the universe has continued to be discussed through the millennia, e.g., Galileo's line, "The Book of Nature is written in the language of mathematics." (See [8].)

The Platonic dialogues offer serious philosophical meat in their discussions of mathematics. In particular, passages like the divided line in *The Republic* [24, 509d-513e] suggest a particular role for mathematical objects in the Platonic representation of the universe. Mathematical objects are themselves an abstraction of particulars, but they are not the last word in abstraction (a position held by the Forms). The use of the term "mathematical platonism" in contemporary philosophy might suggest a connection with Plato, but reading Plato's texts has encouraged various commentators to conclude that Plato was not a platonist – at least, of the mathematical stripe. Plato's ambivalence about the practice of mathematics is made explicit in the *Republic* (ibid, 527a).

The Euclidean tradition is always going to receive plenty of time in a history of mathematics course, and the structure of *The Elements* can be used as a guide to some of the philosophical questions that were being asked at the time. The primacy of proof emerges with Euclid, and it is an interesting mixture of history and philosophy to try to figure out why. It may be connected with the paradoxes of Zeno against which Aristotle argues and in the light of which he created formal logic. The question of whether logic led to *The Elements* or whether studying the structure of *The Elements* led to logic may not be one that can be resolved, but it is an interesting example of how mathematics and philosophy interact. The discussion in Jonathan Barnes' *Truth, etc.* [2] forces the student to confront what Aristotle was trying to do, while the first chapter of Mary Tiles' *Mathematics and the Image of Reason* [27] traces the Euclidean influence on logic.[1]

The discussion of Pythagorean number theory can be carried forward to the medieval period. One of the residues of the Pythagorean approach is the association of nonmathematical characteristics with the whole numbers. Tables of opposites suggest that numbers were associated with characteristics that they have lost in our mathematics, although they may still be present in our mysticism, superstition, and folklore. Medieval texts introduced the positive integers by associating them with theological or empirical trappings. After readings from books of number theory from the medieval period, students can be encouraged to ask whether numerology was a necessary precursor of modern ideas about numbers, just as alchemy was a precursor of chemistry.[2]

The emergence of new ideas in the work of Descartes about how mathematics ought to be done can be connected with his views about the nature of mathematical truth. (For the text of Descartes in readable form, see [6]. Gaukroger supplies much matter for students to think about with regard to the Descartes texts in [12]. The best short discussion of how Descartes's views of mathematics differed from his predecessors is in [18].) Students can be asked whether the truths of mathematics constrain even God or whether God could have created a different set of mathematical truths. These kinds of discussions are appropriate even in a secular university, although they do not typically take place in a mathematics course.

Even though some students in a class aimed at elementary education majors may not have seen calculus, it is worth explaining the ways in which Newton (see [13]) and Leibniz tried to lay foundations for the idea of derivative. (An older source for Leibniz, still worth consulting, is [17].) In particular, the notion of numbers being infinitely close looks useful, but the students can be brought to see that these are not numbers with which they are familiar. This offers

[1] *Truth, etc.* [2] is the best book for making ancient and medieval logic relevant to students of history of mathematics. [27] surveys the historical influence of the philosophical motivations for Euclid. For something only eight centuries after Euclid, see [25], where Glenn Morrow's translation and notes mix history and philosophy in good measure.

[2] D. P. Henry [16] does more with the formal aspects than the mathematical content of mathematics of the period. What I do at an institution where the students are not quite up to the linguistic challenges of Latin, much less Hebrew, is to supplement this more formal material with my translation of some of the passages of Abraham ibn Ezra's *Sefer ha-mispar* that are especially accessible (and Pythagorean).

the chance to see Berkeley's criticism of the foundations of calculus as a philosophical objection to a mathematical development.

The arrival of non-Euclidean geometry in the early nineteenth century poses questions about the relationship between mathematics and the world which had earlier been taken for granted. After the development of the vast extent of Euclidean geometry, the mere existence of non-Euclidean geometry is already a challenge. How it was answered may be illustrated historically, but the question of what sort of mathematics genuinely describes the world around us can be taken as a philosophical one. (An older combination of philosophy and history is [4]. More focused and contemporary is [28].) In the aftermath of general relativity, however, it may rather be seen as a physical one, and students might be encouraged to investigate whether there is any room for further philosophical considerations once a physical theory has weighed in on the question.

In another sense, non-Euclidean geometry goes together with the new algebraic structures that were being introduced at the same time. Mathematicians now define what a number is, not by metaphysical discussions, but by specifying what sorts of laws it satisfies. Since there are different laws satisfied by different sets of objects, the question of what a number is may no longer have a unique answer. Since there is a way of uniting the various geometries (by Klein's Erlanger Program, say), one can similarly look at attempts to lay a foundation for all the different kinds of numbers there are. The use of different laws for Boolean algebra than for standard arithmetic might be taken to suggest that our understanding of truth in mathematics needs to go beyond the arithmetic of ordinary positive integers.

Frege's foundational efforts are an excellent example of what can happen when one tries to translate a particular philosophical vision into a mathematical system. Leibniz had put forward the notion of a perfect language ("lingua universalis") in which reasoning ("logica characteristica") could be carried out unerringly. Frege translated this vision into both the notation of his *Begriffsschrift*[3] and the definitions and theorems of his subsequent work. Fortunately, mastering Frege's notation in the *Begriffsschrift* is not required to assess or to explain the success of his enterprise. There is a large secondary literature that enables students to appreciate Frege's program without having to work through all the details themselves. (Examples of the literature on Frege from diverse points of view are found in [1], [9], [26], and [30]. The stream of publications continues in the twenty-first century.) At the time Russell's paradox appeared to require abandoning Frege's project, but the later twentieth century saw a resurgence of interest in what has come to be called a neo-Fregean approach.[4]

By contrast with Frege's project, the effort by Brouwer and his intuitionist disciples to rewrite mathematics involved even more fundamental changes. When Euclidean geometry was rewritten by Bolyai and Lobachevsky, they changed the axioms but retained the logic. For intuitionism it was not enough to change the axioms; the logic itself had to be changed, which had the effect of altering what consequences could be inferred from a set of axioms. There is a serious question about what, if anything, remains the same when the logic is changed. Studying the subsequent development of alternative logics helps students to appreciate the extent to which mathematical reasoning from Euclid onwards depends on a logic previously taken for granted. There is no doubt that Brouwer is tough reading as a primary source. Description of the mathematics can be found in [3]. For the philosophical roots of the technical approach, see [10], and, for even more on the historical issues that served as background, see [29].

John Stuart Mill's approach to mathematics grew out of British empiricism and used generalization instead of abstraction. While this may not seem to go far enough to cover mathematics beyond arithmetic, it is the core of Hilbert's picture of intuitive mathematics. Hilbert then developed a mathematical universe by analogy with playing a game. Students feel that they have a good sense of what constitutes a game, but they may not be able to see how the mathematics they have learned can be reduced to a game with rules. A game becomes unplayable when there is an inconsistency in the rules. Some of Hilbert's program can be carried out provided that there is no inconsistency in the rules of mathematics (see [7]). The importance of consistency proofs is analogous to having a definitive answer when checking whether a move in chess is legal. Most chess players are willing to accept that the rules of their game are consistent (on the basis of centuries of experience), but some mathematicians are not so easily satisfied.

[3] Part of the translation of the *Begriffsschrift* can be found in [11]. Heck argues in [15] that there is material of both philosophical and historical interest in Frege's later writing that is illuminated by considering the scheme that Frege uses in his "concept notation."

[4] See [14] for a survey of the term neo-Fregean.

The response of the mathematical community to the foundational crisis at the end of the nineteenth century was that it was something less than a catastrophe. This helps students to understand how the practice of mathematics can go on in the face of what might look like an insuperable objection. This is a good moment in a course to draw the comparison with the criticism of the foundations of calculus by Berkeley and others. The mathematicians of the eighteenth century continued to do mathematics without losing sleep over the difficulties of interpreting the fundamental operations in the new system. Similarly, most mathematicians in the early years of the twentieth century did not allow the paradoxes from Cantorian set theory to deter them from doing algebra or analysis. They figured that, just as Weierstrass was ultimately able to come up with an explanation of the notion of limit that drew its fangs, so mathematicians would find a way of resolving the new paradoxes. Russell and Whitehead provided in *Principia Mathematica* a foundation for mathematics that sought to avoid the paradoxes to which Frege's system had fallen victim. (There are few people inclined these days to make their way through the pages of the three volumes of Russell and Whitehead. However, the introduction and first few sections give insight into what they are trying to accomplish. It also helps to consult Russell's other (and shorter) treatments in *Principles of Mathematics* and *Introduction to Mathematical Philosophy*[5].) They imitated Frege by going back to basic philosophical considerations to start their system. By contrast, Zermelo's axiomatization of set theory did not come with a full-fledged philosophical underpinning. That may be one of the factors explaining the longevity of Zermelo's set theory.

The proof of Gödel's incompleteness theorems can be seen as the culmination of the mathematicization of philosophy. Translating statements about mathematics into mathematics, as Gödel did in his incompleteness theorems, is an exercise the students can practice (see [21, pp. 190-206]), even if they do not follow Gödel's argument in detail to its conclusion. The secondary literature on Gödel's work has grown considerably even since the publication of John Dawson's *Logical Dilemmas* [5] in 1997. Dawson attests to the slow rate at which the mathematical community assimilated Gödel's results and the enthusiasm with which the philosophical community jumped on the results to support a particular philosophical position. The relationship between truth and proof returns the students to some of the questions that came up in looking at Euclid, especially when both truth and proof may need to be reexamined in the light of modern developments.

3 Thematic Approach

Another approach to the history of mathematics is the thematic. As an example, one can follow the topic of infinity from the time it first emerges in mathematics to the current preoccupation with the study of infinite sets. The unbounded was a negative category in Greek thought going back to the Pythagoreans. It was seen as giving rise to paradoxes in Plato (in the *Sophist*) and Aristotle's criticisms of Zeno. There is perhaps more medieval material connected with the infinite that is still worth reading than with most of the standard themes of mathematics. The origins of modern schools of the philosophy of mathematics can be traced to attitudes toward the infinite, and Cantor's work offers concrete statements about the topic that are not just a matter of philosophical preconceptions. The notion of infinity puzzled mathematicians in the early nineteenth century trying to figure out, say, the limit of the cosine function as the argument approaches infinity. This leads to an observation that is a valuable lesson for these students: not every question about the infinite that appears to be mathematically precise has an answer meaningful either mathematically or philosophically.

The theme of the connection of mathematics with the real world can take the students in various directions. For Plato, after all, the reality of the external world was a sham. In the early days of probability and statistics, students can see questions about the nature of the link between mathematics and the external world being answered by technical developments, such as an axiomatization of probability.

Another theme is the separation of the profession of mathematician over the millennia. We do not know of Greek mathematicians who had no philosophical interests. The term "philosopher" continued to be applied to mathematicians and scientists into the nineteenth century. The mathematical societies of earlier centuries were likely to have "philosophical" as part of their name.

[5] Both have appeared in many editions.

4 Using Philosophical Material

Fortunately, there is a rich literature connected with most of the philosophical issues associated with the history of mathematics. (For examples of philosophically well-informed history, see the titles in the further reading section. The trend of good historical scholarship in the philosophy of mathematics goes back at least to Imre Lakatos [20] and Philip Kitcher [19].) Students can be introduced to writers like Plato and Descartes, although they usually need to have their attention focused on fairly brief texts. For example, in the *Meno* Socrates claims to extract a geometrical result from his interlocutor who has never studied geometry. Students can be asked whether this is a model of teaching and what sort of learning might be going on. In a setting where discussion is important, it is a helpful reminder that Plato did not present even mathematics without the give-and-take of conversation. It is important early on to encourage students to express their own ideas rather than to assume that what was just said by the previous speaker, any more than what Aristotle claimed, is irrefutable. Almost any history class will include Newton's observation about "standing on the shoulders of giants" (as explored, at length, in Robert K. Merton's book of that title, [22]). If a discussion starts by contrasting different views of mathematics (say, Plato's and Aristotle's), students will have to learn how to decide on whose shoulders they wish to stand.

An effective way to evaluate student progress in the philosophical material is by having them write essays. Typically, they will be asked to expand on classroom discussions, although the papers usually need to be kept short. Assignments can ask students to explain the distinction between numbers and numerals. Students can be asked to tie logic to mathematics in contexts all the way from Euclid to the twentieth century. They can consider whether the existence of paradoxes is bound to be a feature of any discipline. When students are given the option of writing about a philosophical topic in a longer paper, they almost invariably prefer to stick to a topic that is less contentious (such as the life of a mathematician).

The study of the philosophy behind mathematics helps students to understand why mathematical practices change. We tend to assume that the numbers we talk about are the same as the numbers of the Greeks. This is not obvious, since we talk about them in different ways. If the students have been looking at and producing proofs in their mathematics courses, they might learn why this rather arcane activity is pursued with so much enthusiasm. Changing standards of rigor may turn what was once accepted as a proof into something that does not make the grade. To support the view that mathematics is the language of science, the developments in the former can be tied to those of the latter as a way of explaining the connection. However, we know that certain ideas from sciences of centuries past (like alchemy and astrology) have disappeared from modern science. While mathematical practices also change, mathematical and scientific practices do not necessarily change in synchrony.

It is never easy to get students to think about mathematics rather than simply to do mathematics. They may well have had some experience thinking about philosophical issues connected with popular topics like medical issues. They also all have a sense of what mathematical objects are. After a class like this, they should be able to encourage their students to speculate about the nature of mathematics rather than just impart technical skills. Even middle school students discuss when life begins and ends. They can also think about why division by zero is forbidden.

The history of mathematics by itself already can create some disquiet in students who were under the impression that the theorems and rules they have learned come with some sort of eternal guarantee. Adding discussion of basic philosophical issues can increase that effect. Nevertheless, reminding students that there are different ways of thinking about an issue is probably a useful contribution to their being able to appreciate that there are different ways of solving a problem. The students who do not like the philosophical content behind the history are liable to complain that they are not just solving problems. One can give examples of situations where it is easier to solve a problem when the problem is properly understood. The purpose of the philosophical content is to help students see alternative perspectives (always important for teachers) and have a deeper understanding of what may underlie mathematics.

Students who have a more extensive technical background in mathematics than those in the Whitewater course described above can be brought to consider issues in more advanced areas. There are also plenty of topics connected with probability and statistics that conceal philosophical disagreement. The mainstream history of mathematics is already enough to bring out philosophical issues that students are likely never to have investigated. Students may come into a history of mathematics course with some of the same assumptions that they bring into history courses in general. The material is expected to be a sequence of names and dates, possibly tied together by a narrative. Looking at the philosophical underpinnings of questions that arise in the history of mathematics should help students realize that the history of mathematics is not just a matter of who invented what when.

Further Reading

This brief selection of philosophically well-informed volumes on the history of mathematics is not meant to be exhaustive. It is simply a constructive existence proof of such work.

Anglin, W.S., *Mathematics: A Concise History and Philosophy*, Springer Verlag, New York, 1994.

Aspray, William, and Kitcher, Philip, eds., *History and Philosophy of Modern Mathematics*, University of Minnesota Press, Minneapolis, 1988.

Dawson, John W., Jr., *Logical Dilemmas: The Life and Work of Kurt Gödel*, A K Peters, Wellesley, 1997.

Graham, Loren, and Kantor, Jean-Michel, *Naming Infinity*, Harvard University Press, Cambridge, 2009.

Gray, Jeremy, *Plato's Ghost*, Princeton University Press, Princeton, 2008.

Netz, Reviel, *Ludic Proof*, Cambridge University Press, Cambridge, 2009.

Stillwell, John, *Mathematics and Its History*, Springer Verlag, New York, 2002.

Bibliography

[1] Baker, G.P. and Hacker, P.M.S., *Frege: Logical Excavations*, Oxford University Press, New York, 1984.

[2] Barnes, Jonathan, *Truth, etc.*, Oxford University Press, Oxford, 2007.

[3] Beth, Evert W., *The Foundations of Mathematics*, Harper & Row, New York, 1966.

[4] Bonola, Roberto, *Non-euclidean Geometry*, Dover, Mineola, 1955.

[5] Dawson, John W., Jr., *Logical Dilemmas: The Life and Work of Kurt Gödel*, A K Peters, Wellesley, 1997.

[6] Descartes, René, *The World and Other Writings*, translated by S. Gaukroger, Cambridge University Press, Cambridge, 1998.

[7] Detlefsen, Michael, *Hilbert's Program*, Reidel, Dordrecht, 1986.

[8] Drake, Stillman, *Galileo*, Oxford University Press, Oxford, 1980.

[9] Dummett, Michael, *Frege: Philosophy of Mathematics*, Duckworth, London, 1991.

[10] Dummett, Michael, *Elements of Intuitionism*, Oxford University Press, Oxford, 2000.

[11] Frege, Gottlob, *Begriffsschrift*, in Peter Geach and Max Black, eds., *Translations from the Philosophical Writings of Gottlob Frege*, Blackwell, Oxford, 1966.

[12] Gaukroger, Stephen, *Descartes: An Intellectual Biography*, Oxford University Press, Oxford, 1997.

[13] Guicciardini, Niccolò, *Isaac Newton on Mathematical Certainty and Method*, MIT Press, Cambridge, 2009.

[14] Hale, Bob and Wright, Crispin, *The Reason's Proper Study*, Oxford University Press, Oxford, 2001.

[15] Heck, Richard G., *Reading Frege's Grundgesetze*, Oxford University Press, Oxford, 2012.

[16] Henry, D. P., *Medieval Logic and Metaphysics*, Hutchinson, London, 1972.

[17] Ishiguro, Hide, *Leibniz's Philosophy of Logic and Language* 2nd ed., Cambridge University Press, Cambridge, 1990.

[18] Kenny, Anthony, *The God of the Philosophers*, Oxford University Press, Oxford, 1979.

[19] Kitcher, Philip, *The Nature of Mathematical Knowledge*, Oxford University Press, Oxford, 1983.

[20] Lakatos, Imre, *Proofs and Refutations*, Cambridge University Press, Cambridge, 1976.

[21] Mendelson, Elliott, *Introduction to Mathematical Logic*, 4th ed., Chapman & Hall, London, 1997.

[22] Merton, Robert K., *On the Shoulders of Giants*, The University of Chicago Press, Chicago, 1993.

[23] NCTM-CAEP Standards, NCTM CAEP mathematical content for secondary, addendum to the NCTM CAEP standards 2012, available at www.nctm.org/Standards-and-Positions/CAEP-Standards/ (accessed July 29, 2015).

[24] Plato, *Republic*, translated by T. Griffith, Cambridge University Press, Cambridge, 2000.

[25] Proclus, *A Commentary on the First Book of Euclid's Elements*, translated by G. Morrow, Princeton University Press, Princeton, 1970.

[26] Sluga, Hans D., *Gottlob Frege*, Routledge and Kegan Paul, London, 1980.

[27] Tiles, Mary, *Mathematics and the Image of Reason*, Routledge, London, 1991.

[28] Torretti, Roberto, *Relativity and Geometry*, Pergamon Press, Oxford, 1983.

[29] van Dalen, Dirk, *L.E.J. Brouwer*, Springer, London, 2013.

[30] Weiner, Joan, *Frege*, Oxford University Press, Oxford, 1999.

19

Let's Teach Philosophy of Mathematics

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Is mathematics an arcane technical specialty, unrelated to history, philosophy, literature, or art? Is each mathematical subject a self-contained, static, timeless structure, with no meaning or value outside itself? Are the axioms and deductive rules of mathematical systems simply "given," not to be evaluated, criticized, rejected, or changed?

Most readers will answer all of these questions with a resounding, "No!" Mathematics is not isolated; its different branches are intricately interwoven with each other and with all of human culture. The axioms of mathematical systems are not arbitrary; they evolved historically so as to yield the structural arrangement of that particular subject; many other axiom systems and structural arrangements are possible for each mathematical subject.

Yet, as we all know, the dogmatic style of textbooks and of teaching is pervasive and deeply rooted. Mathematical subjects appear as isolated and inhuman piles of axioms and algorithms. The student gets a brief, brutal message: "Here it is, swallow it down!" Many writers have urged that our teaching methods be changed, that we use a problem-centered, historically guided, humanistic approach to the teaching of mathematics. Still, dogmatic teaching will not disappear soon.

One step in the right direction would be to offer one course that takes an "open," humanistic approach—that concentrates on where mathematics comes from and tells the students about the difficulties to which our present theorems offer solutions. Such a course could be called, for example, "Philosophy of Mathematics." For fifteen years I have been offering such a course at the University of New Mexico. The purpose of this article is to recruit other teachers who would like to try something similar in their schools.

One source of misunderstanding should be cleared up right away. The course I am talking about is not the same as the "Philosophy of Mathematics" course that is offered today in some mathematics and philosophy departments. That familiar course is really a course on the "foundations of mathematics." (Indeed, it may be presented under either title, sometimes philosophy of mathematics, sometimes foundations of mathematics.) It covers elementary set theory, introductory logic, and the foundation or construction of the number systems.

That course, though titled philosophy, actually ignores all philosophical issues, even though its subject matter originated from philosophical concerns of the late 19th and early 20th centuries (the period of what Imre Lakatos called "foundationism"). As it is usually given, that "philosophy of mathematics" course simply presents one particular branch of technical mathematics. I am not attacking "foundations"; on the contrary, I regard it as a fascinating and important subject for research and for undergraduate education. But in addition to it, we need a course that really does present the philosophy of mathematics. Within it, foundations appear only as a part, important but by no means the whole story.

At this point some readers ask, "What else could you mean by philosophy of mathematics, if not foundations?"

One answer is to look at other academic subjects. Science, for instance, versus philosophy of science; art versus philosophy of art; law versus philosophy of law. No one thinks that a course in philosophy of science should be a course in science or that a course in philosophy of art should be a course in art. Rather, they take a philosophical approach to the problems of the scientist or artist. In this way, it is believed, they can be interesting and valuable to the scientist or artist, or to any person with an interest in science or art.

Similarly, a course in the philosophy of mathematics is *not* a course in technical mathematics, not even in technical logic or set theory. Only incidentally will it prove theorems, derive algorithms, or develop skill in solving problems. (The philosophy of science course develops no laboratory skills, and the philosophy of art, no studio techniques.) Instead, the philosophy of mathematics exposes and inspects the philosophical dilemmas of mathematics, both past and present: the nature of mathematical reality, the nature and meaning of infinity, the relation between mathematics and the physical world. It presents some answers that have been given to these dilemmas, and points out how the proposed answers fall short.

Such a course makes sense only for students who possess a certain minimal shared knowledge of the history of mathematics. Therefore, the course must include as much history as they will need to appreciate the philosophy. One must start with the Greeks (or even earlier). Acquaintance with Euclid is indispensible in understanding the mathematical philosophy of centuries earlier than our own. And, of course, one must know Euclid before one can talk about non-Euclidean geometry. The story of non-Euclidean geometry is decisive in the history of philosophy, not only in mathematics, but also in general epistemology. Yet this topic is completely absent in today's "Philosophy of Mathematics" course!

Violating the historical order for the sake of teaching convenience, my philosophy of mathematics course goes on from Euclidean and non-Euclidean geometry to the birth of the calculus, the notion of infinitesimal, and Bishop Berkeley's critique of Newton's (and Leibniz's) calculus. Then comes the explication of limit by means of ϵ, δ, and the Dedekind and Cantor constructions of the reals.

Isn't this just straight mathematics? Where is the "philosophy?" The philosophical content here is that the Dedekind-Cantor construction of the reals is presented in its historical-philosophical context, as the resolution of a centuries old confusion which goes back to Newton and Berkeley. We are studying mathematics as part of the "history of ideas." This focussing on the historical evolution of ideas is the crucial philosophical content of the course.

The construction of the reals, using as it does infinite sets of rationals, leads inevitably to Cantor, infinite sets, and the antinomies (contradictions) in informal set theory. Here we have arrived finally at the heart of the traditional foundations course. But we go more deeply into the philosophical aspects of foundations than does the usual foundations course. We present (in brief) the principal logical contributions of Frege and Russell, as well as their philosophical viewpoint of "logicism." We read Brouwer (and/or Bishop) and discuss in detail some examples in intuitionism. Then we talk about Hilbert's formalism, and its downfall by way of Gödel's incompleteness theorem. (A very nice, short proof of Gödel's theorem, using Turing machines, is presented in Martin Davis's article, "What Is a Computation?" [28].) We finish up with Turing machines, computability, and unsolvability, topics with obvious philosophical import and with a timely connection with computers.

In this historical sequence, the foundationist project has its place—an important place—as a response to the philosophical presuppositions of its time. Of these presuppositions, the foremost was that mathematics *must* have an unshakeable foundation, that it has a special role to play as provider to mankind of indubitable truths, of unshakeable certainty.

The assignment of that role to mathematics goes back very far, to Kant, to Descartes and Spinoza, to St. Augustine, to Plato, finally to the Pythagoreans. In this venerable tradition, the goal was to justify religion, the knowledge of God, by a linkage with mathematics. It was believed—and still is believed, by more than a few—that mathematics gives indubitable knowledge of truths independent of experience or the physical senses. This belief, for theologians and metaphysicians throughout the history of Christendom, backed up the claim that one could also have indubitable knowledge of God.

In the late 19th century, mathematics went through a crisis. First, the discovery of anti-intuitive curves—Peano's space-filler, or Riemann's nowhere-differentiable—discredited visual intuition. Then Russell's paradox and the other antinomies cast doubt on the new set-theoretic foundations. Because it was an article of faith that mathematical truth had to be absolutely solid, the restoration of foundations was seen as an urgent philosophical task by Russell, Brouwer, and Hilbert. This common goal justified Lakatos in denoting them all as "foundationists." From today's perspective, Brouwer's intuitionism, Hilbert's formalism, and Russell's logicism were all competing sects of one school (one which has by now dried up and withered away). This is the philosophical background that makes the content of the foundations course philosophically interesting.

It is even more interesting to bring the story up to date. The last topic in the usual foundations course is Gödel's theorem. That is fifty years old. What has happened since then? "Foundationism," as a philosophical program, has stagnated, although "foundations," as a mathematical subspecialty, has prospered.

That doesn't mean that the philosophy of mathematics is stagnant. New trends in the philosophy of mathematics are emerging, leaving foundationism behind. A witness to this is the anthology [29] of Thomas Tymoczko. It contains articles by 16 authors, all in one way or another challenging or departing from the foundationist paradigm. These trends are parallel to the upheaval in philosophy of science in recent decades, which was led by Karl Popper, Thomas Kuhn, and others. Parallel trends in the philosophy of mathematics are still at a very early stage. They have the following features: philosophy of science (or mathematics) is closely linked to its history and to the actual practice as reported by practitioners, rather than to *a priori* dogmas of what science (or mathematics) *should* be or *must* be. The subjects are seen as evolving and developing, not static; as interacting with culture and society, not isolated; as having truths that are conditional, not absolute.

The class attracts a variety of students, from freshmen to graduate students. Most are in science and engineering. There are a few mathematics majors, and some in fields such as economics, psychology, philosophy, and architecture. Very few drop out. The great majority express gratitude for the opportunity to spend a semester thinking about interesting ideas.

A reader who is totally habituated to the traditional mathematics course might ask, what do the students *do*? That is, if they aren't doing homework problems and taking quizzes, what else is there? Yet the university or the college is full of courses without mathematics homework or quizzes. The students read, they write papers, they listen, take notes, comment on and discuss the lectures. A substantial number give class presentations, sometimes on my assigned material, sometimes on their related interests (recent ones have been in theoretical computer science, foundations of quantum mechanics, and psychology of learning mathematics).

The course does not have a textbook. I use extracts and selections from a number of different sources. The books listed in the following annotated bibliography have all proved useful as reading or source material for lectures. I also include some sample topics for term papers and essay questions for the final exam.

Sample Term Paper Topics

(Two or three are assigned to each student.)

1. Since it has been shown (in lectures and class discussion) to be absurd to define geometry as the study of (a) physical objects, (b) formal axiom systems, (c) eternal non-human ideal objects, or (d) ideas in the minds of particular individuals, we must conclude that geometry is a risky and unreliable theory. [Criticize the viewpoint expressed here.]
2. Why is math worth doing and/or why isn't math worth doing?
3. Is math the same as computing? If not, what's the difference? If yes, why do people think they're different?
4. Could we know the physical world without math? Why or why not?
5. Is math good or bad for the social sciences?
6. Is math part of logic?
7. Is Euclidean geometry as true, truer, or less true than non-Euclidean geometry?
8. Why is Archimedes ranked with Newton and Gauss?
9. What does it mean to say a mathematical object exists? How can we know of such existence?
10. What was Berkeley's critique of Newton? Who was right?
11. Read and report on an article or a chapter from a book in the annotated bibliography.

Sample Final Exam

Tell all you know about any three or four of the following:

1. Pre-Euclidean, Euclidean, and Archimedian mathematics.
2. Non-Euclidean geometry; where it came from, what it is.
3. Numbers: natural, rational, real, and transfinite. Construction and properties.
4. Logicism, formalism, intuitionism (or constructivism). One or all.
5. Formal logic, axiomatic systems, Gödel's theorem.
6. Existence in mathematics. What does it mean? What should it mean?

Annotated Bibliography

1. Asger Aaboe, *Episodes from the Early History of Mathematics*, New Mathematical Library, Mathematical Association of America, 1964. Describes the archaeological work on the Babylonian cuneiform tablets, presents examples of sexagesimal notation and of solutions of quadratic equations with very large coefficients. This is followed by chapters on Euclid, Archimedes, and Ptolemy. I present Euclid's axioms and postulates, emphasizing the 5th (parallel). I also use Archimedes' famous determination of the relative volumes of the sphere, cone, and pyramid. Archimedes' use of "unorthodox" methods (infinitesimals, centers of gravity) is very clearly explained by Aaboe.
2. E. T. Bell, *Men of Mathematics*, Simon and Schuster, New York, 1937. This book is an unrivaled collection of novelizations of the lives of mathematicians. Students can enjoy Bell's narrative flair, while learning the lesson that not everything in print is gospel truth.
3. Paul Benacerraf and Hilary Putnam, *Philosophy of Mathematics*, 2nd ed., Cambridge University Press, Cambridge, 1983. This is a standard collection of articles in the philosophy of mathematics, overwhelmingly weighted toward the foundationist viewpoint.
4. E. Bishop, *Foundations of Constructive Analysis*, McGraw-Hill, New York, 1967. Most of this book is technical mathematics. The first chapter is an eloquent statement of the intuitionist-constructivist viewpoint.
5. E. Bishop, *Aspects of constructivism*, Lecture Notes, New Mexico State University, 1972.
6. C. Boyer, *History of the Calculus and Its Conceptual Development*, New York, 1949. Dover reprint, 1959. This book could well merit a course in itself. The leading theme is the struggle to clarify the notions of infinitesimal and limit. Boyer, writing before Robinson's nonstandard analysis had been created, takes it for granted that the necessary and desirable resolution was the abandonment of infinitesimals.
7. H. Curry, Some aspects of the problem of mathematical rigor, *Bulletin of the American Mathematical Society* 47 (1941), 221–241. Takes an extreme formalist stance.
8. Tobias Dantzig, *Number, The Language of Science*, Doubleday Anchor Books, Garden City, 1956. The story of number systems from natural to complex, written with the style of a poet and the depth of a philosopher.
9. Philip J. Davis and Reuben Hersh, *The Mathematical Experience*, Birkhäuser, Boston, 1981. One of my favorite books.
10. Philip J. Davis and Reuben Hersh, *Descartes' Dream*, Harcourt Brace Jovanovich, Boston, 1986. Describes the impact of mathematics and computing on today's society.
11. A. Dresden, Some philosophical aspects of mathematics, *Bulletin of the American Mathematical Society* 34 (1928), 438–452. In sympathy with Brouwer, but not as extreme.
12. Howard Eves and Carroll V. Newsom, *An Introduction to the Foundations and Fundamental Concepts of Mathematics*, Holt Rinehart and Winston, New York, 1965. This is the closest thing to a textbook for the course. It is a historically presented account of the main ideas and results of mathematics from the Greeks to Hilbert and Gödel. Beautifully written, with many interesting problems which unfortunately are too hard for students

who aren't mathematics or science majors. I use their chapters on Euclidean and on non-Euclidean geometry. I find their work on calculus and on 19th- and 20th-century mathematics needs to be supplemented.

13. Gottlob Frege, *The Foundations of Arithmetic*, Harper and Brothers, Harper Torchbooks, New York, 1960. Logicism from the horse's mouth. Written in a lively, polemical, sarcastic style that keeps my students awake.
14. K. Gödel, What is Cantor's continuum problem? *American Mathematical Monthly* 54 (1947), 515–525. Takes a Platonist stand.
15. Hans Hahn, The crisis in intuition, pp. 1956–1976 in J. R. Newman, ed., *The World of Mathematics*, Simon and Schuster, New York, 1956. Beautifully written, presents the "pathological" space-filling and nowhere-differentiable curves that undermined the authority of visual intuition in the 19th century.
16. G. H. Hardy, Mathematical proof, *Mind* 30 (1929), 1–25. "Typical" philosophy of the working mathematician.
17. Reuben Hersh, Some proposals for reviving the philosophy of mathematics, *Advances in Mathematics* 31 (1979), 31–56. My personal view on what's wrong and what to do about it.
18. M. Kline, *Mathematical Thought from Ancient to Modern Times*, Oxford University Press, Oxford, 1972. An amazing compendium, up through the early 20th century, carefully and reliably presented. By far the most complete and authoritative history of mathematics available today.
19. Philip Kitcher, *The Nature of Mathematical Knowledge*, Oxford University Press, New York, 1983. The most ambitious of the attempts to ground the philosophy of mathematics in its history and in the actual practice of mathematicians today. The later chapters, on history, are probably more accessible to students than the earlier chapters, which are more technical in the philosophical sense.
20. S. Körner, *The Philosophy of Mathematics*, Harper Torchbooks, New York, 1962. A useful compendium of the three foundationist dogmas (logicist, formalist, intuitionist) with critiques of each.
21. Imre Lakatos, *Proofs and Refutations*, Cambridge University Press, Cambridge, 1976. A landmark in 20th century philosophy of mathematics. A veritable geyser (or avalanche) of wit and learning. The nature of mathematics is expressed by means of a classroom dialog on the Euler formula $V - E + F = 2$.
22. Jean Piaget, *Genetic Epistemology*, Columbia University Press, New York, 1970. Piaget was involved in the philosophy of mathematics, not just its psychology and pedagogy. The philosophical importance of his ideas has yet to be recognized.
23. J. Pierpont, Mathematical rigor, past and present, *Bulletin of the American Mathematical Society* 34 (1928), 23–53. Describes the crisis in foundations with tongue firmly in cheek.
24. M. Polanyi, *Personal Knowledge: Towards a Post-Critical Philosophy*, University of Chicago Press, Chicago, 1960. This is mostly philosophy of science, not of mathematics, but it is a fascinating, exciting book that everyone should read. On mathematics, he offers the startling insight that in order to be accepted as part of mathematics, an idea or method has to be interesting.
25. G. Pólya, *How to Solve It*, Princeton University Press, Princeton, 1945. Like Piaget, Pólya's work on teaching and learning had deep philosophical import, which has yet to be recognized. This book is one of the intellectual master works of the 20th century.
26. Alfred Renyi, *Dialogues on Mathematics*, Holden Day, San Francisco, 1967. These dialogues can be read by anybody. Painlessly and accurately they answer, what is mathematics? What is applied mathematics? How is mathematics useful in science?
27. Rudy Rucker, *Infinity and the Mind*, Birkhäuser, Boston, 1982. This is a fun book, a melange of logic, mathematics, and mysticism. A special bonus is Rucker's personal recollections of conversations with Gödel.
28. Lynn Arthur Steen (ed.), *Mathematics Today*, Vintage Books (Random House), New York, 1980. A collection of articles widely varying in level, subject, and originality. I always use the article by Martin Davis ("What is a Computation?") to teach Turing machines and Gödel's theorem.
29. Thomas Tymoczko (ed.), *New Directions in the Philosophy of Mathematics*, Birkhäuser, Boston, 1986. An anthology of recent writings, most of which try to base philosophy on the actual practice and experience of mathematicians.

30. L. A. White, The locus of mathematical reality, *Philosophy of Science* 14 (1947), 289–303. Reprinted in *The World of Mathematics*, J. R. Newman (ed.), Vol. 4, Simon and Schuster, New York, 1956, pp. 2348–2364. A beautiful presentation, by a famous anthropologist, of the view that mathematics' "locus of reality" is in human culture, the shared consciousness of communities.

IX

Mathematical Education of Teachers and Teacher Educators

It is particularly important for future teachers and teacher educators to have an opportunity to consider a range of philosophical questions concerning mathematics, for exactly the same reason that this volume has been written for college-level mathematics faculty. Understanding some of the subtleties involved in our understanding of mathematical objects and processes will help them understand some of their students' confusions. Further, it will give them a deeper understanding of the material they will be presenting.

Chapter 20: Erin Moss discusses "Philosophical Aspects of Teaching Mathematics to Pre-service Elementary Teachers." Her goal is that her students come to the views that all people are capable of developing and verifying the mathematics they need to use, and that mathematics is a collaboratively developed discipline in which reason is paramount and in which multiple approaches are valuable. To acquire these views, students must be active participants in the development of their understanding. The course is inquiry-oriented and problem-based. After briefly introducing the topic of the day, students work in groups on tasks for which they have no ready-made algorithm, and which allow multiple entry points and multiple methods of solution. The focus is on conceptual understanding. Students explain solutions to their peers, decreasing reliance on the instructor.

Chapter 21: Elizabeth de Freitas, in "Pre-service Teachers Using Core Philosophical Questions to Analyze Mathematical Behavior," discusses a philosophy of mathematics course for pre-service teachers. It covers a broader range of modern philosophy of mathematics topics than any other discussed in this volume. They include realist, conceptualist, and nominalist views, and work by Corfield and Lakatos, which she refers to as descriptive epistemology, work by Lakoff and Núñez and others on embodied cognition, and work by Roth on material phenomenology. Students read primary and secondary philosophy of mathematics literature to open up discussions about why we have the mathematics we have, inviting consideration of how mathematics is embodied in particular material practices such as diagrammatic reasoning. Students design, carry out, and report on experiments, mathematical tasks, or interview questions on topics such as the nature of infinity or the role of diagrams in proofs. She discusses how answers to core philosophical questions (especially epistemological questions, but also a significant number of questions about ontology and applicability) are linked to pedagogical approaches.

Due to the very broad philosophical background de Freitas brings to the course, most mathematics faculty would need to do considerable additional reading to be able to follow her example. However, she provides a good introduction to the concepts she uses in the course, and her bibliography gives an excellent set of readings for faculty who are interested in looking further. Also, aspects of her course can be incorporated into other courses without including all the readings.

Chapter 22: Alfinio Flores, Amanda Jansen, Christine Phelps, and Laura Cline describe "A Mathematics Inquiry Course: Teaching Mathematics in a Humanistic Way." The philosophical viewpoint underlying the course starts with Alvin White's humanistic mathematics, a thread that was then taken up by Reuben Hersh and others and called social constructivism. From this viewpoint, mathematical praxis has various facets—it is not just the finished product—and social interactions are crucial to the process. The course, for graduate students in mathematics education, is inquiry-based, but in a very different sense from what is now called "IBL" (or Inquiry-Based Learning). Students gain an understanding of mathematical practice by engaging in mathematical research as mathematicians do, but into questions at a level at which they are capable making progress. Students work in groups in class on smaller inquiry projects and individually outside of class on their individual inquiry projects. They write up their findings and referee each other's papers. They learn problem posing by varying previously solved problems. The focus is thus on inquiry rather than particular mathematical content. The authors do not make their philosophy explicit during the course, although they discuss with their students how to adapt the course for (preservice or inservice) mathematics teachers, as many of their students will become university faculty members themselves.

20

Philosophical Aspects of Teaching Mathematics to Pre-service Elementary Teachers

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1 Introduction

The majority of Americans believe that mathematics is the domain of an elite few—the ultra-logical and intellectually brilliant—with no room for their own contributions [1]. A significant mechanism by which this negative perception is perpetuated is in the mathematics classroom via pedagogy. Teachers from kindergarten through the college level most commonly engage in a transmission-based pedagogy, with the instructor positioned as the sole mathematical authority in the classroom. To encourage higher rates of achievement and persistence in the American population and to build a more equitable society, everyone must feel a sense of ownership of mathematics. We must encourage the belief that all people are capable of developing mathematics for their personal use and verifying the mathematics they employ, and we must provide early opportunities for this to occur. In a classroom setting, this means allowing students to assume mathematical authority by co-creating mathematics with one another and with their teacher. In this chapter, I share my experiences teaching a mathematics course for future elementary teachers and describe the ways that my philosophy of mathematics guides my approach to the course. In particular, I discuss pedagogical strategies I use to achieve my aims of increasing students' responsibility for creating and communicating mathematics.

I am a mathematics professor at a four-year public institution with an enrollment of about 7,500 undergraduates and 1,000 graduate students. My focus is in mathematics education, so a great deal of my teaching load is devoted to preparing prospective preK-12 teachers in mathematics content and pedagogical methods. One course I commonly teach is a mathematics content course that primarily serves early childhood education majors (who receive certification to teach grades preK-4), along with some middle-level education majors (who receive subject-specific certification for grades 4–8). The course addresses topics such as sets, development and properties of the real number system, operations, traditional and alternative algorithms, problem-solving, reasoning and proof, pattern recognition, and algebraic thinking and representation. Only a few students in each 36-student class are studying for middle-level certification in mathematics. Therefore, the majority of students arrive in my class with high levels of math anxiety and low levels of interest.

Having taught this content for several years at two different institutions, I have had the opportunity to experiment with and adjust my pedagogy until it aligns with my philosophy of mathematics. Reflecting on my teaching, there are multiple contexts in which I make efforts to impart my philosophy to students. I discuss four of them in this chapter: lesson structure, task selection, classroom interactions, and assessment. Within each context, I describe what I do in the classroom and the aspects of my philosophy that I intend to transmit through my actions.

2 Philosophy

Before initiating a discussion of my teaching methods, I will explain in greater depth the philosophy of mathematics underlying my instruction. Questions that are important to characterizing a philosophy include: What is mathematics? Who develops mathematics and how is it developed? What does it mean to develop mathematical knowledge, and under what circumstances does this happen? My philosophy of mathematics incorporates elements of both content and learning, which I view as inseparable from one another. And because I am in the business of education, I consciously allow my philosophy of mathematics and learning to inform my approach to mathematical pedagogy.

The overwhelming majority of my students—similar to most Americans—believe that mathematics is the memorization and application of incomprehensible formulas by smart people very different from themselves. Further, they believe that school mathematics is a necessary torture during which they must recall formulas long enough to pass a test, at which point their anxiety will be alleviated and the formulas may be forgotten. This is a disturbing mind-set among adults who will be influencing the mathematical knowledge and attitudes of young children. Teachers who lack confidence with particular mathematics topics tend to utilize a rules-based approach to teaching them ([3], [6]). In part, this may be to spare themselves from potential discomfort by avoiding situations that would require them to respond to conceptual questions or novel solution methods [2]. Their students, relegated to a position of passive learning, then learn the topics superficially. Some of them, in turn, become teachers. Thus, the cycle continues.

For this reason, it is important that my students recognize that mathematics is a dynamic discipline of exploration in which common sense and reasoning are paramount. It involves inquiring into problems that are often complex and time-consuming, via a process that may require several false starts and multiple attempts. I want students to view mathematics as a collaborative human effort, wherein divergent thinking is valued and justification of ideas is essential. Students all must believe themselves to be capable of contributing to the solution of a problem, as there is no single right way to solve it, and sometimes even multiple solutions. This philosophy of mathematics gives an individual agency in his or her learning and helps make the act of doing mathematics accessible to everyone.

If I aim to promote the philosophy among my students that mathematics involves curiosity, creativity, and persistence, then it is hypocritical of me to lecture to them all period, demonstrate problems for them to copy, and immediately answer every question they pose as if I am the sole mathematical authority in the classroom. Over the course of the semester, I want my students to be increasingly empowered by their own mathematical discoveries, creations, and connections as they decrease their reliance on me. Therefore, my classroom must require them to be active participants in the construction of their understanding and in the mathematics that is communicated among the teacher and students.

The style of pedagogy that most aligns with what I strive to do in my own classroom is the reform-oriented pedagogy originally described by the National Council of Teachers of Mathematics (NCTM) in 1989 [4] and refined through various iterations, including the 2014 document *Principles to Actions: Ensuring Mathematical Success for All* [5]. The Conference Board of the Mathematical Sciences (CBMS) [1] was heavily influenced by earlier NCTM publications and the Common Core State Standards for Mathematics (CCSSM) in defining the university-level mathematical experiences necessary for prospective K-12 teachers. Although my classroom practice is similar to that described in the visions of NCTM, CCSSM, and CBMS, I frame my own teaching through a philosophical lens in this chapter. My goal is to help other mathematics educators be more mindful about aligning their instruction with the beliefs about mathematics and learning that they already likely hold.

3 Lesson Structure

In the structure of a traditional mathematics lesson, a teacher might present content, procedures, and examples for the majority of the class period and then request that students practice applying them to a series of exercises. In contrast to this, I begin a lesson with a brief launch. It involves introducing any definitions that are essential to completing the upcoming task or materials that students may be using in their investigations. Then I pose a task for students to work on in small groups of around three students, ideally seated at large tables. I circulate throughout the room, facilitating students' collaboration and encouraging them to have meaningful conversations about their work. Once most of my students have made significant progress, I have a few of them share their solutions with the class. Often, the presenters are chosen by me to represent a variety of strategies. The students' work drives our whole-class discussion of mathematics, and I allow the content we discuss to come out of their solutions. The variety in solutions allows for interesting connections among content domains to arise organically. It also provides instructional scaffolding for struggling students—in other words, intentionally exposing students to intuitive solutions first can help support their understanding of more formal mathematical approaches used by some of their peers. Since I am comfortable with the big ideas of the material and have experience directing this course, I know there will be ample opportunity to address them via the students' work. While intense examination of specific core mathematical ideas is always the end result, the direction by which we get there and the interesting, fruitful detours we take differ each semester.

The structure of my lessons communicates that common sense and reasoning are paramount to solving mathematics problems, and that with persistence, all students are capable of thinking mathematically. By having students work together and converse, both in small groups and as a large class, they see that mathematics involves the collaborative acts of listening carefully and sharing ideas. The fact that I talk very little to the entire class, and that I ensure that students' work drives instruction, encourages the transfer of mathematical authority from the teacher to the students. It is important to me that they develop independence and autonomy through the realization that everyone can discover, explore, and create mathematics—mathematics professors, elementary school teachers, and even elementary school students.

4 Task Selection

The mathematical tasks that I give students to work on tend to share three features: 1) They are more complex than "practice problems," which I think of as problems for which an algorithm is already known; 2) there are multiple entry points for each task, enabling students at different skill levels to at least make progress towards a solution; and 3) they may be solved in more than one way. For example, during a unit on algebraic representation, I provide my students with the simple sequence of figures in figures 1–4 made from square tiles and ask them to determine the number of white tiles in subsequent figures. How many tiles are in the fifth figure? The sixth? The seventeenth? The hundredth? Can you write an equation to calculate the number of tiles in the *n*th figure?

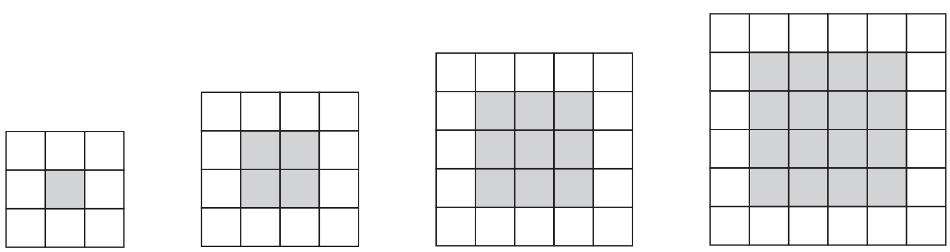

Figure 1 Figure 2 Figure 3 Figure 4

All students are able to represent the growing pattern with drawings or tables of values and articulate the order they see in the progression. Weaker students are able to determine the number of tiles in a particular figure (e.g., the seventeenth figure) even without seeing it, either by keeping track of the numbers in the algebraic sequence or by imagining the dimensions of the figure. At this point, I tend to pose follow-up questions that push these students toward deeper levels of understanding. For instance, once they have determined the number of tiles in the seventeenth figure, I ask them about the twenty-seventh figure to see if they can determine that they can simply add ten additional groups of four to their total. Often it only takes one or two follow-up questions before a student recognizes the multiplicative relationship between the figure number and the four tiles added each time. Stronger students are able to generalize the pattern in order to determine an algebraic expression giving the number of tiles in the *n*th figure of the sequence—some through trial and error, others by noticing and trying to represent patterns in the numbers they have organized into a table, and others by translating the figure's geometry into an algebraic expression.

While this is the typical format of most lessons, I also incorporate tasks where students are not directly involved in creating mathematics, but are instead learning content and pedagogy via an examination of others' mathematical thinking. Sometimes, I ask my students to investigate children's written work. For instance, I show my students a word problem necessitating the multiplication of two two-digit numbers along with three correct solutions written by elementary-age children. The problem reads, "There were 35 dogsleds. Each sled was pulled by 12 dogs. How many dogs were there in all?" [8, p. 239]. The first child added the number 35 twelve times. The second child calculated twelve times 30 and twelve times five and added the numbers together. The third child added the products of two and five, two and thirty, ten and five, and ten and thirty. My students have to collaborate with their tablemates to gain insight into each child's thinking. Why does a particular strategy work? What aspects of multiplication does each child understand? Which students demonstrate a high enough level of conceptual understanding that we might consider introducing them to common multiplication algorithms?

As another example, I have my students study video clips of elementary school children taking part in mathematical problem-solving interviews. The clips were developed and made public through the federally-funded Integrating Mathematics and Pedagogy (IMAP) project (www.sci.sdsu.edu/CRMSE/sdsu-pdc/nickerson/imap.htm) at San Diego State University. We watch a video in class when covering content relevant to it. I ask my students to describe what they notice, and I pose questions to get them to interpret and discuss children's reasoning. They are challenged to listen closely and open their minds to understand and articulate a problem from another person's point of view. One video shows a fifth-grade student named Rachel using a misremembered algorithm to incorrectly convert a mixed number to an improper fraction. Unprompted, she then offers an alternate solution method, sketching a picture of the quantities involved and explaining her reasoning. Rachel is momentarily stumped when her methods yield different answers, but over the course of the video clip, she is able to use her conceptual understanding to determine the mistake in her original algorithmic approach. She attributes her initial difficulty to the fact that her teacher—in a departure from her typical pedagogy—taught the procedure before the concept for this topic. For my students, hearing a fifth-grader articulate that she recalls procedures much better when she figures them out for herself is powerful support for the pedagogy I am promoting.

In addition to teaching mathematics content in meaningful ways, the aim of my tasks is to be consistent with aspects of my philosophy. I want my students to internalize the message that thinking mathematically does not equate to rehearsing procedures, but instead involves a great deal of creativity and persistence when encountering obstacles. I also want to take them out of the mindset that there is only one right way to solve a problem. Studying different solution methods is useful for two main reasons. First, investigating a problem from different perspectives can help to strengthen a student's mathematical understanding by encouraging her to make connections across content areas. Also, my students' future profession will require them to understand mathematics from their elementary and middle school students' perspectives, which may be very different from their own.

5 Classroom Interactions

Whether students are working in small groups or are all involved in a class discussion, my interactions with them are similar. I try my hardest to support their thinking without giving them the answers or forcing them to solve a problem my way. To support students who are struggling to make progress with a problem, I ask them questions such as: When you started this problem, what did you try first? What does your diagram represent? For students who believe they are finished, I don't validate the work they have done. Instead, I ask even more questions. How did you get your answer, and how do you know it is correct? Have you found all possible solutions to the problem, and how do you know? What questions do you have for [a tablemate] about her solution method? Can you find another way to solve this problem? I try to demonstrate flexibility, occasionally letting a group pursue an avenue of exploration that is tangential to the problem I originally posed when I feel it could lead to an exciting discovery. I look for opportunities to compliment originality in their thinking with comments like: I've never thought of the problem this way before! Cool diagram! What an interesting approach! Let's explore that connection you hinted at in your solution! I want students to learn that success in mathematics comes mostly from curiosity, resourcefulness, and perseverance, not necessarily from innate ability. And I want them to decrease their reliance on me for validation and start to develop the propensity to question mathematics themselves in ways that foster their independence.

In a traditional classroom, students are expected not to interact with peers for the most part, since they are supposed to be listening to their teacher. This would be contradictory to my philosophy that collaboration is important in mathematics and exploring multiple approaches is valuable. In my classroom, I expect students to encourage, support, question, argue, and even resolve mathematical conflicts. I am clear with my expectations that for collaborative work to be productive, a group needs every member's input on every problem, and it must be a safe space for sharing ideas that may or may not pan out. In the beginning stages of group work, I ask my more extroverted students to be sensitive to students who initially hesitate to make contributions. They should consistently check in with a quieter classmate, asking for her ideas if she does not volunteer them. As when students talk to me, I also expect students to justify their thinking to one another. They must explain to their peers how they arrived at a solution and why it works. Their audience must ask clarifying questions until they understand the mathematical reasoning involved in the solution. If students disagree with one another, they are obligated to (politely and respectfully!) articulate the disagreement and provide support for their argument. If a disagreement happens during the whole class discussion and the class seems unsure of which argument is correct, I give students a few minutes to talk through the issues in their small groups. This extra time with tablemates results in most students having greater understanding of the situation and several additional volunteers ready to justify their conclusion in front of the class.

6 Assessment

When I finally assess students via a quiz, test, or project, I give them non-routine mathematics problems, forcing them to apply knowledge rather than regurgitate procedures. Other questions might require them to solve a problem in two significantly different ways. Sometimes, I have students respond to an unusual solution method. Does the method work in this case? Why or why not? Does it work in every case? Study this child's solution to the problem, and predict how the child will solve a new problem I have provided. I do assessments in this way to reinforce the value of conceptual understanding in doing mathematics: simply memorizing procedures is not enough to be successful in my class or as a teacher. This also communicates the value of multiple solution methods and connections across content areas.

7 Effects on Student Learning and Understanding

My natural style when first beginning to teach was to lecture with a conceptual focus, probably subconsciously modeled after my favorite high school mathematics teacher, who had a knack for helping me make sense of mathematics. In his lectures, he was always certain to tell why a formula or procedure worked. He could explain a difficult idea in multiple ways, using everyday language. And he drew pictures illustrating concepts whenever possible.

Because this conceptual focus helped me finally understand mathematics that I had just been memorizing up to that point and awakened my curiosity in the subject, it was natural that I taught that way too. My teaching evaluations suggested my conceptual focus was both helpful and appealing to some of my students, but I was by no means reaching a majority of them. In fact, in my initial years of teaching this course, 25% − 30% of students received a grade that was not sufficient for their degree program. Now, after increasing my skills at using an inquiry-oriented approach, less than 10% of students do not successfully complete the course. Reviewing my assessment materials, I see that my exams have actually gotten more challenging through the years, yet students overall are doing much better than when I simply lectured. The numbers don't reflect the scope of the change, however. Truly comprehending the effect of a completely transformed pedagogy on student learning requires a qualitative examination of the knowledge that has been gained over the semester.

Students learn much more than rote procedures that may quickly be forgotten. They develop conceptual understanding of ideas that they can then use in different situations. Students learn to support their thinking with sound mathematical justification, borne from this understanding. They also become more proficient in creating representations for problem scenarios, such as organizing information in a drawing or table to understand a problem better or to communicate their thinking to others.

Besides witnessing significant gains in conceptual understanding, I see important changes in the ways many students approach mathematics problems. In the first weeks of the semester, after assigning a problem to groups, several hands are quickly thrust into the air to announce "We don't know where to start." "I can't tell what this problem is asking." "We tried something and it didn't work, so now we are stuck." By the end of the semester, hands are raised very little if at all. Instead, students stay busy approaching a problem from multiple directions, persisting when they encounter roadblocks, and turning to one another to get their questions answered or test their ideas out. When I first notice this change during group work, about midway through the semester, I point it out to students: "Do you remember a couple of months ago, when I would give you a problem to work on, and you'd stare at it and then almost immediately ask me for help? Now I'm walking around and nobody needs me—you are coming up with your own ideas and showing perseverance and using your own reasoning skills to figure out what to do. This is exactly what you want your own students to do when you become a teacher. But how do you get students to act this way? It's not easy. How do I talk to you when you're doing group work?" One student usually breaks the ice by saying in a mock-accusing tone that I never tell them whether their answer is right or wrong, and then others recall the types of questions I ask and the ways I redirect their questions back to group members. I mention that elementary students respond very well to consistency, and if they can establish similar norms in their classrooms, their students will amaze them with what they are able to accomplish.

8 Overcoming Obstacles

The problem-based pedagogy I describe was very difficult for me to embody fully. Although I was introduced to it through readings a decade ago during my doctoral studies in mathematics education, it took several years of gradual modifications to get to a point where I had truly transformed my teaching. A major barrier to transformation was the difficulty I had counteracting the influence of my own schooling. Like most people, I had primarily experienced mathematics delivered in a lecture format—from elementary school through graduate school. And like most mathematics majors, I had been successful in that environment.

For all my difficulties envisioning a different type of pedagogy, it was even more challenging to embody it. When my students struggled to understand a concept or gave up too early on a task I had posed, I had to fight the urge to jump in and simply tell them how to do the problem. I can typically refocus myself with the guiding question, "How do I turn more of the mathematical thinking over to the students?"

Student resistance can be a problem in two very different ways. The first type of resistance is typical of most students in my class: I am asking them to actively participate in a subject where they have intense anxiety, to take risks, and to mathematically justify their work. The second type of resistance is more common of the few students who were strong math students in the past. They tend to have been successful in previous math courses because they excelled at memorizing procedures. From their perspective, I am asking them to give up on a way of learning that they were successful at, in favor of an approach that requires deeper thinking to confront problems for which they don't immediately have a strategy.

Student resistance persists for several weeks, and then it melts away in nearly all cases. I have found that it helps tremendously to be explicit about what I am trying to accomplish with my pedagogy. We repeatedly compare our in-class approach to a topic to the way most of us learned about it in grade school. Students are more than happy to discuss the difficulties they had with a concept, and most of this can be traced to a prior focus on learning algorithms and procedures without conceptual understanding. Many students have shared with me verbally or in course evaluations that they wish they had learned mathematics this way in the first place, because now they are able to understand it and they feel more confident encountering novel problems. Creating a safe space for students to share ideas without fear of ridicule is also critical. I keep the mood in my class lighthearted and energetic, and I demonstrate enthusiasm for the subject and for students' work. I regularly describe the growth I notice in students' curiosity, creativity, and persistence, and as a result, students start taking pride in their in-class contributions, creating a loop of positive feedback.

Another way of countering student resistance is to demonstrate that unlike prior teachers, I am concerned with much more than students simply getting the correct answer. I never praise students for a correct answer, and I show no interest in it if it is unsupported. Instead, I praise students' mathematical thinking—an interesting connection they made to another mathematical topic, a unique approach to solving a problem, a representation that is helpful in illustrating an aspect of the problem. I will occasionally have students present work to the class on the document camera even if their group has gotten stuck or if the overall result is an incorrect answer. The whole class collaborates to determine which parts of their thinking are mathematically sound and to complete the solution that was started.

A final way that I combat resistance is to continually link the pedagogical choices I am making to the possibilities for elementary school teachers. In my experience, although students realize that they are taking a course in mathematics content, they are eager to start planning for and imagining themselves in their own classrooms. I share anecdotes from my work in elementary schools to demonstrate how traditional pedagogy often squelches the creativity of younger students, changing them from creative and open-minded first graders to algorithmically-focused fifth graders who struggle to solve nonroutine problems. I describe lessons I witnessed and we talk about how a teacher could use students' responses as a springboard to discussing topics in more depth. I make sure there are frequent opportunities for my students to reflect on their education and plan for the ways they will improve on it for the next generation.

I now feel confident that I embrace a problem-based pedagogy that is consistent with my philosophy of mathematics as a dynamic, inquiry-oriented discipline. This has yielded overwhelmingly positive results for my students and for me. It is admittedly challenging to maintain an intense focus on student thinking and determine how to connect their ideas to one another when I do not know ahead of time which will be developed. The fact that every class is different, however, keeps my teaching fresh and informs other courses I teach. I learn a great deal of mathematics from my students—even about topics that I previously thought trivial. This helps me maintain genuine enthusiasm for my profession.

For further reading, Smith and Stein's book *Five Practices for Orchestrating Productive Mathematical Discussions* [7] provides a practical framework for structuring a lesson to effectively incorporate students' mathematical ideas into your teaching.

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21

Pre-service Teachers Using Core Philosophical Questions to Analyze Mathematical Behavior

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1 Course Overview

In this chapter, I discuss a course in the philosophy of mathematics designed to help future high school mathematics teachers develop an understanding of philosophical questions about mathematics. My aim was to equip the future teachers with philosophical skills for analyzing mathematical behavior. Teachers who can analyze their students' mathematical activity for how concepts are at work can get beyond simple evaluative responses to students—typically assessing their performance as right or wrong—and can begin to explore their students' mathematical thinking. This is a difficult skill to develop, and philosophy offers one way of doing so. Philosophical questions about mathematics open up discussions about why we have the mathematics we have, inviting consideration of how mathematics is embodied in material practices.

The pre-service teachers in the course read and discuss primary and secondary literature in the philosophy of mathematics, compose a formal argument in support of a position on a core philosophical question, and design experiments where they collect empirical data from research participants.[1] The experiments usually consist of a set of mathematical tasks or interview questions that focus on some core philosophical question concerned with the nature of mathematical infinity or the role of diagrams in proofs, or some other topic. The pre-service teachers then select five to ten people to engage in the activity—for example, mathematicians, siblings, or peers. They record the responses using video or audio, or in some cases just observation, and also collect any written artifacts. They then analyze the data they've collected, through the lens of their core question, and reflect on how their findings inform their understanding of mathematics and mathematics teaching.

Throughout the course, our discussions link core philosophical questions to pedagogical approaches to mathematics education. Thus the course mixes traditional philosophy of mathematics with the study of lived experience and everyday embodied habits, approaches more often associated with phenomenology and other continental traditions of philosophy.[2] Such a mixture is extremely challenging, in part because so much of the philosophy of mathematics

[1] During the last decade, *design experiments* have become the dominant paradigm in educational research methodology. See the seminal paper by Cobb et al. [4] for details. In general, design experiments entail engineering a particular task for one person or a small group of people to engage with, and then observing or videotaping the activity for how it sheds light on learning theories.

[2] For a quick introduction to phenomenology, see plato.stanford.edu/entries/phenomenology/ (accessed April 20, 2014).

has historically framed its questions in abstract analytic terms, and in part because the approach demands applying theoretical tools to the study of everyday practice. Corfield [5] calls this *descriptive epistemology* because it entails interpreting mathematical activity for how it reflects philosophical assumptions about the nature of mathematics.

Students at Adelphi University enroll in the course often not knowing anything about philosophy, let alone the philosophy of mathematics. They are usually in their fifth year of a combined bachelor degree in mathematics and master's degree in education, and are just beginning to go into high school mathematics classrooms and practice teach. Many of them are returning to high school classrooms where procedural drill is the norm for instruction, which runs counter to the inquiry methods they've been learning about in their teaching methods courses as well as the problem-based learning methods they tend to see in their undergraduate mathematics courses at our college. I open the course with a set of statements that Brown [3] calls the common "image" of mathematics. We go through each statement, and discuss whether they agree or disagree with it. This is often an opportunity to unpack some of the vocabulary within the philosophy of mathematics, and I use it as a way to model traditional philosophical arguments, where initial work must be done to define the terms before arguing one's position. The statements are

1. Mathematical results are certain.
2. Mathematics is objective.
3. Proofs are essential.
4. Diagrams are psychologically useful, but prove nothing.
5. Diagrams can be misleading.
6. Mathematics is wedded to classical logic.
7. Mathematics is independent of sense experience.
8. The history of mathematics is cumulative.
9. Computer proofs are merely long and complicated regular proofs.
10. Some mathematical problems are unsolvable in principle.

As we move through the list, I contest any easy collective consensus, and put forward alternative views. I have taught the course for three years, and each time students tend to agree with #1-5 without much deliberation. I believe their agreement with the first five is due, in part, to the fact that the statements seem compelling and commonsensical, and the students haven't been trained to query or interrogate the hidden assumptions of such statements. Their hesitation with #6 reflects their belief that logic is one kind of mathematics, studied in a course on logic, and not related to the rest of their mathematics courses. This strange disconnect seems to be prevalent among the many pre-service teachers I have taught, as well as many high school teachers that I've worked with. As we move to the last four statements, they become less confident about their responses. They often initially don't know what it would mean for the history of mathematics to be or not be cumulative, nor have they ever compared computer proofs with regular proofs in terms of their structure or epistemic claims. Unsolvability in principle is not something they have ever encountered. So the list and the kind of dissonance it produces is an effective way to introduce some of the core issues in the philosophy of mathematics. The one statement that they strongly disagree with is #7, and they cite their own learning experiences as evidence of how it is incorrect. I ask them to elaborate, to reveal how the response might be linked to philosophical assumptions about the role of the body and the material world in mathematics. I also introduce into the mix a few references to famous events in the history of mathematics, to help them begin to think about these statements from a more global perspective.

2 The Core Questions

Students are exposed to the following set of core questions on the first day of class. They drive many of the discussions and course assignments.

1. Can a diagram function as a mathematical proof?
2. What is the nature of proof? How has mathematical proof changed over history?

3. Is there such a thing as mathematical intuition? Where is it? Is it innate?
4. Is mathematics indispensable to science? Could science work without math?
5. What should the relationship be between logic and mathematics?
6. What is the status of axioms? Are they grounded in reality?
7. Are mathematical propositions necessarily true (or false) (rather than culturally or contingently true or false)?
8. Is mathematics a language (a system of symbolic signs and not a part of the physical world)?
9. Can we speak about actual infinity (or just potential infinity)? How has the concept of infinity influenced the development of mathematics?
10. What is the role of the body in doing mathematics? How is mathematical knowledge embodied?
11. Is mathematics discovered or invented? What are the ontological implications of your answer?
12. Is mathematics objective and certain (rather than subjective and open to revision)?

Another essential discussion on the first day of class focuses on the difference between epistemic and ontological concerns. The need to keep the concerns separate while understanding their relationship helps considerably as students go on to formulate arguments to support their positions on the core questions. Perhaps because they are education students, they seem more at home with epistemological questions (How do we come to know the concept of number?) and are initially baffled by ontological questions (What is number?).[3] I have learned to motivate the latter by suggesting that their future students will ask of mathematics "what does this have to do with anything?" and that they should treat this as a philosophical question rather than offer platitudes or lies that students see through ("You need math in life."). I suggest that they can unpack the question posed by their own students—the question of relevance—by discussing how the question has fueled the entire history of the philosophy of mathematics. Rather than dismissing these students as bored and not motivated to learn, I suggest that they help them build their instinctual response (to "How is this relevant?") into a philosophical engagement with the nature of mathematical knowledge. They can tell their students about the various schools of thought that developed as a means of answering it. They could even, I suggest, introduce one of the core questions into their lessons, as a motivator for all those astute students who have asked this difficult question.

As a way of integrating the history of mathematics with the philosophical questions, pre-service teachers research and present a ten-minute slideshow on a discussion topic each week. Sample topics are Zeno's paradoxes, the parallel postulate, fractal geometry, and zero. Presentations focus on the history of the topic, and as instructor I introduce links between the topics and the core philosophical questions that structure the course. For instance, a presentation about the parallel postulate might simply recount attempts to prove it and mention developments of non-Euclidean geometry without linking the developments to our readings about Kant and his claims about the *a priori* synthetic nature of geometry. Further links need to be made to help the students grasp how this topic is related to questions about the certainty and objectivity of mathematics, and its relation to science. It has always intrigued me that these students, despite being immersed in mathematics, a field known for its careful deductive methods, struggle so much in composing a formal philosophical argument. Many of them confess to having selected mathematics because they don't enjoy reading and writing. However, I feel strongly that, as future teachers, they need to become excellent communicators, and I treat the course as an opportunity to build that skill as well. I have designed guidelines to help them structure their assignments, and I work with various draft versions of their papers to help them improve this skill.

3 General Philosophical Themes

The distinction between ontology and epistemology helps us narrow in on students' assumptions about mathematical practices, as we discuss how platonism and other schools of thought consist of an ontological claim and an epistemological claim. In this we follow Bostock [2]. We ask: in what sense can universals (redness or beauty or triangles

[3] Epistemological questions concern the nature of knowing, while ontological questions concern the nature of being. Students are trained to parse questions according to these two concerns. For instance, the question "How do we become aware that numbers don't necessarily have to be numbers of something?" is both epistemological insofar as it asks about how we are "aware" and also ontological in that it asks about the nature of number.

or numbers) be said to "exist"? This, as Bostock reminds us, is a question about the ontological status of universals. Most students don't quite know how to engage with the question, although they are more than ready to grant universality (generality) to geometric figures or arithmetic entities like numbers. They tend to think of this generality as cross-cultural, and I discuss with them how the question also pertains to the metaphysical. I offer them some choices: if universals exist, do they exist outside the mind, or simply as mental entities? If they exist outside the mind, are they corporeal or incorporeal? If they exist outside the mind, do they exist in the things that are perceptible by the senses or are they separate (or independent) from them? To further support and scaffold their exploration of these questions, I offer three schools of thought, each with a different answer to these questions, and I ask the students to decide who they most identify with. I am really forcing their hand in this, in that I hope to show them that the responses do not actually exhaust the possible answers to the ontological question. In the next section, I discuss how new directions in the philosophy of mathematics offer different choices. But the choices first given, drawn from those used by Bostock [2], are simplifications so that they can begin to engage in debate. As in all such sorting and labeling, we can query whether a particular mathematician or philosopher is a good example of a particular philosophical paradigm, and I am careful to tell the students that they will debate these issues later, after reading more primary texts:

- The *realist* (Plato, Frege, Gödel) claims that universals exist outside the mind and are independent of all human thought.
- The *conceptualist* (Descartes, Kant) claims that they exist in the mind and that they are created by the mind. Some claim that we create universals based on sense perception and some say they are innate and do not require perceptual stimulation.
- The *nominalist* (Hilbert, Field) claims that they do not exist at all. Some claim that the words and symbols we use are mere shorthand for longer ways of expressing the same idea and some claim that statements with such terms are simply untrue in the sense that they refer to nothing.

The assignment of the names to the schools is not perfect, but it works as a starting point. Gold [8] points out that the way one speaks reveals in part which school one aligns with:

> Simply to say a certain mathematical statement is true involves taking a philosophical position. If you are a formalist, you say, rather than that "this theorem is true," that "it is a theorem within a given axiom system." For a substantial collection of philosophers of mathematics (nominalists, fictionalists), there are no mathematical truths, because there are no mathematical objects for them to be true about [8, p. 153].

One of the difficulties in starting with the main schools of thought, and then trying to tease out the subtle differences and ways in which these philosophers' claims are not perfectly aligned with the school, is that the students are not yet ready to delve deeply into historical subtleties. For instance, it might seem a travesty to put Descartes and Kant together, since Kant pushed past Descartes' claim that mathematical truths are innate, clear, and distinct ideas, so that he might attend to the synthetic nature of mathematical judgment. According to Kant, space and time are the mind's contribution to experience. Space and time are the form of experience, a form imposed by us on the raw data of experience. Historians of philosophy usually oppose Descartes (the rationalist) against Locke and Hume (empiricists). Bostock [2], however, claims that Locke, Hume, and Descartes, despite their differences, share the same beliefs about the ontological status of mathematical objects (they are ideas or mental entities), and differ in how they think we acquire mathematical ideas. One might then associate Kant with this approach as well, since, as Brown suggests, according to Kant, "Our *a priori* knowledge of geometric truths stems from the fact that space is our own creation." [3, p. 119]. Similarly, arithmetic is connected to time and the fact that time is also a form we impose on the world. This conceptualist approach seems to have saturated many of the later treatments of the philosophy of mathematics, seeping into the realist and nominalist camps as well. Brown indicates that Frege (a platonist) embraced Kant's view on geometry, Hilbert (the formalist or nominalist) embraced Kant's view on arithmetic, and even Russell (the logicist) can be characterized as Kantian. One might also argue that the conceptualist approach has saturated theories of learning, and has become full-fledged in cognitive psychology and its dominant image of learning. This image assumes that learning entails an acquiring of a set of cognitive schemas. Pre-service teachers need to be aware of this history so that they might become empowered to identify and critique the theories of learning that structure the curriculum policy they are meant to adopt in their classrooms.

The pre-service teachers are shown how much of the philosophy of mathematics since the nineteenth century has been contending with the Kantian assertion that mathematical truths are *a priori* and *synthetic*. Kant claimed that if a proposition is necessary and universal then it is an *a priori* judgment. He defined a proposition to be *analytic* if it could be judged simply by analyzing the terms of the proposition. The proposition "All bachelors are unmarried" is analytic because the concept bachelor contains, as Kant would say, the concept of being unmarried—and such containment is reciprocal, in that being unmarried and male contains bachelor. But Kant argued that mathematics was not analytic, and instead *synthetic*. The proposition $12 = 5 + 7$ is synthetic, claimed Kant, because the concept of 12 does not contain the concept of $5 + 7$. He then asserted that all mathematical truths were synthetic *a priori*. How can that be possible? This is a perennial question in the philosophy of mathematics, the question as to how pure mathematics is possible. Hacking [9] claims that one has to look closely at applications of mathematics if one is to address—or contest—this question of purity. By examining the activity of mathematics one begins to see that application is everywhere, whether it be conventional applications of mathematics to the physical sciences, or one set of mathematical concepts applied to another mathematical field, or simply the fact that the statement $2 + 3 = 5$ seems to say something about the world that is necessarily true. In other words, the distinction between pure and applied is tenuous at best.

Corfield [5] argues that the philosophy of mathematics has spent far too much time on the foundational ideas of the 1880–1930 period, and neglected the thinking and doing of real mathematicians both before and after that period. Corfied believes that a philosophy of mathematics should

> concern itself with what leading mathematicians of their day have achieved, how their styles of reasoning evolve, how they justify the course along which they steer their programmes, what constitute obstacles to these programmes, how they come to view a domain as worthy of study and how their ideas shape and are shaped by the concerns of physicists and other scientists [5, p. 10].

He names this approach *descriptive epistemology* and defines it as the "philosophical analysis of the workings of a knowledge-acquiring practice." [5, p. 233]. Imre Lakatos [11] is often taken as inspiration in this approach to the philosophy of mathematics. He examined the process of meaning-making in mathematics by studying the historical evolution of concepts and procedures, and offered insight into the form of deliberation that characterized creativity in the work of mathematicians. He was interested less in the so-called foundational issues in mathematics, and more in the empirical and material making of mathematics, an approach he called "critical fallibilism":

> It will take more than the paradoxes and Gödel's results to prompt philosophers to take the empirical aspects of mathematics seriously, and to elaborate a philosophy of critical fallibilism, which takes inspiration not from the so-called foundations but from the growth of mathematical knowledge [12, p. 42].

Hersh [10] characterizes Lakatos as a philosopher of mathematics who was committed to studying the social and humanist aspects of doing mathematics. For Hersh, Lakatos was a humanist because he celebrated the specificity of informal reasoning found in the work of mathematicians, rather than or in addition to the generality of its truth claims. For Lakatos, the examples of informal reasoning are not simply unfinished formal proofs, in which the pertinent axioms and logical rules of inference are suppressed, but rather a significantly different mode of inquiry, a non-axiomatic argument that has its own trajectory and its own becoming.

Despite the significance of this more humanist perspective on the philosophy of mathematics, which values the study of informal and unfinished mathematical activity by experts, we still lack philosophical insight into the experiences of those—students for instance—who, for the most part, do mathematics from an outsider or fringe position. We have yet to grapple philosophically with the embodied, affective, and symbolic aspects of these kinds of mathematical encounters. What would it look like if we borrowed Corfield's descriptive epistemology or Lakatos's critical fallibilism to study everyday mathematical behavior? This materialist approach, with its emphasis on various kinds of embodied activity (diagramming, gesturing), and its move away from a purely cognitive or brain-based model, lends itself to the study of mathematics teaching and learning in everyday contexts. In the next section, I describe how my course focuses on the role of diagram and gesture in mathematical activity.

4 Diagrams, Movement, and the Mathematical Body

Questions about the status of diagrams in proofs are easy for students to connect with, and link directly to the opening readings by and about Plato. Students are drawn to the compelling distinction that Plato draws between the physical world and the realm of mathematics. We discuss the theory of ideal forms, and how Plato was motivated by the gap between the ideas we can conceive and the physical world around us. Some students see in the proposal of an ideal realm a way of reconciling their belief in the universality of mathematics with the messiness of learning, but more often than not they are drawn to a conceptualist approach, perhaps Kantian, whereby mathematics is considered an invention that aligns with the physical universe. Thus they tend to ascribe to the human mind a consciousness or intuition that is capable of bringing together the ideal forms (triangles, numbers, etc.) that are unchanging and eternal (the realm of being or essence) with the physical realm (the realm of becoming or change). We discuss how there is a strong dualism (between mind and body) at work in this approach, and how it plays out in different pedagogies. The vast majority of pre-service teachers split mind from body, arguing that we grasp the ideal forms only through mental reflection, while we understand the physical world through the senses, just as Plato might say. Most of the contemporary philosophy of mathematics we read in the course questions the validity of the dualism, and we discuss the main criticisms of platonism that were formulated centuries ago.

Diagrams figure prominently in this discussion, as they have, since Plato, if not before, bridged the dualism in ways that trouble its claim to a clean distinction [6]. In small groups, the students are given a set of visual proofs from [14], in some cases without the corresponding algebraic expression or equation, and asked "What does this diagram prove?" I use this question to provoke debate, as it gets to the heart of concerns about what constitutes a legitimate proof in mathematics. Brown supplies a number of interesting examples as well, in particular a set of diagrams that might be considered to be proofs of an infinite pattern. Consider, for instance, the diagram in Figure 1.

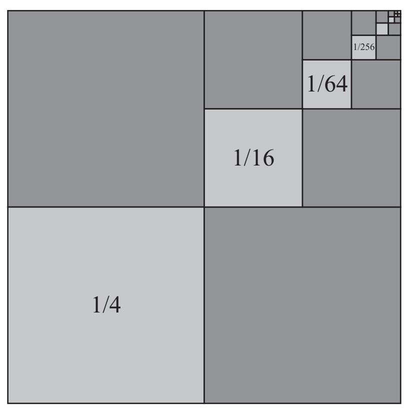

Figure 1. What does this diagram prove?

The students discuss what is entailed in using this diagram to demonstrate that

$$\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\frac{1}{256}+\cdots=\frac{1}{3}$$

is true. We discuss to what extent the diagram might function as a proof of the statement. When we compare the diagram to a formal proof by induction, the students tend to allow the latter greater mathematical certainty. But what is notable is that over half of the students regularly claim that induction proofs are entirely meaningless to them, and that mathematical induction conveys no certainty at the level of personal belief or conviction. This seems to reinforce the dualism of mind and body, where the diagram stands in for the body, and the formal deductive logic of inductive proof stands in for the disembodied cognitive acts of the mind. My aim is to trouble this distinction so that they might begin to look at the embodied acts of mathematics, in this case the diagram, as exactly where the mathematics is happening.

We read excerpts from Plato (*Meno, Theatetus, Republic*) that help ground the core questions in historical contexts. Although they tend to find Socrates overbearing in the *Meno*, they begin to grasp how the Socratic method emerges from philosophical assumptions about the nature of mathematical diagrams and concepts. We compare the method to the kind of questioning sequences they see in their observations in classrooms. For Plato, geometrical knowledge is obtained by pure thought and divorced from sensory observation, which seems to go against what many of the students experience in mathematics classrooms. This is when they become somewhat unhappy with their platonism. As Brown [3] explains, Plato considered the diagram as merely a heuristic to help us access the pure forms of mathematics. In principle, a good mathematics student would grasp the ideal form of the circle without the need for a diagram, since these were always tainted by the corruptions of the physical world. Plato is rather disparaging of all this talk of diagrams and gestures:

> The science of geometry is in direct contradiction with the language employed by its adepts . . . their language is most ludicrous . . . for they speak as if they were doing something and as if all their words were directed toward action . . . [they talk] of squaring and applying and adding and the like . . . whereas in fact the real object of the entire subject is . . . knowledge . . . of what eternally exists, not of anything that comes to be this or that at some time and ceases to be. (Plato, cited in [22, p. 21].)

> Don't you also know that they use visible forms besides and make their arguments about them, not thinking about them but about those others that they are like? They make the arguments for the sake of the square itself and the diagonal itself, not for the sake of the diagonal they draw, and likewise with the rest. These things themselves that they mold and draw, of which there are shadows and images in water, they now use as images, seeking to see those things themselves, that one can see in no other way that with thought [19, Book VI, 510d, p. 191].

We discuss the consequences of Platonist and conceptualist approaches to mathematics that deny or dismiss the significance of the activity of doing mathematics and prize instead only the mental or cognitive reasoning faculty. We begin to read contemporary theories of embodied cognition that attack this approach philosophically ([13], [15], [16], [20], [21]). The students begin to imagine that diagramming (and other embodied activities) are not merely heuristic but rather necessary for thinking mathematically. We discuss what it might mean for thinking to occur in and through this activity rather than independent of it.

The readings in this section of the course range from physiological to more phenomenological approaches, and through these readings we begin to shape our ontological and epistemological discussions around the question of the role of the body in learning mathematics. We begin to consider how an approach to the body as part of the mathematical learning process impacts our philosophy of mathematics. As Núñez et al [18] state, situated learning and cognition radically shifts the terrain, and directs our attention to both social and physiological concerns:

> We argue that the nature of situated learning and cognition cannot be fully understood by focusing only on social, cultural and contextual factors. One must also take into account the non-arbitrary biological and experiential constraints that shape social activity and language, and through which cognition and learning are realized in a genuine embodied process. The bodily-grounded nature of cognition provides foundations for situatedness, entails a reconceptualization of cognition and mathematics itself, and has important consequences for mathematics education [18, p. 45].

More recently, and more specifically, Alibali and Nathan [1] direct our attention to how gesture and other micro-actions are taken up in mathematical behavior:

> We argue that mathematical cognition is embodied in two key senses: it is based in perception and action, and it is grounded in the physical environment. We present evidence for each of these claims drawn from the gestures that teachers and learners produce when they explain mathematical concepts and ideas. We argue that (a) pointing gestures reflect the grounding of cognition in the physical environment, (b) representational gestures manifest mental simulations of action and perception, and (c) some metaphoric gestures reflect body-based conceptual metaphors. Thus, gestures reveal that some aspects of mathematical thinking are embodied [1, p. 247].

Roth ([21] and later writings) draws on *material phenomenology* to make similar claims. He studies a young child's tactile handling of a cube in a classroom, where the teacher is teaching about shapes. Roth shows how the child's hands move spontaneously all over the cube, touching and stroking it, without conscious intention or conscious reflection.

According to Roth, the child comes to know the cube through his hands in such a way that his knowledge is pre-reflective. He also argues that it is in the hand—rather than or in addition to the brain—that the memory of the cube is immanent. In other words, knowing mathematics entails the material gestural encounter, prior to any synthesis or making linguistic or semiotic sense of it. Through touch and the pre-conscious coordination of the hand, eye, and other sensory modalities, the child comes to know what a cube is. This is a phenomenological study of learning, focusing on how concepts live in our embodied activity. Other scholars have argued this point with more advanced mathematics as well [7]. As Nemirovsky and Ferrara [15] suggest, "thinking is not a process that takes place 'behind' or 'underneath' bodily activity, but is the bodily activity itself." Roth [21] points out how the phenomenogical tradition helps us think differently about mathematics, moving away from the proposal that knowing rests heavily on mental representations, and towards the proposal that knowing is enacted or folded into activity of various kinds. In his words, this is a theory of learning mathematics that finally shifts away from the Kantian intellectualist mind model that dominates learning theories.

> In Kant's constructivist approach, the knowing subject and the object known are but two abstractions, and a real positive connection between the two does not exist (Maine de Biran, 1859a,b). The separation between inside and outside, the mind and the body, is inherent in the intellectualist approach whatever the particular brand [21, p. 9].

My aim in the course is to slowly help pre-service teachers begin to appreciate how this intellectualist and conceptualist model has dominated learning theory in mathematics education, and that contemporary philosophers of mathematics are offering alternative approaches to study the materiality of mathematical thinking. Our discussions of Roth, Nemirovsky and Ferrara, Alibali and Nathan, and Núñez invite the pre-service teachers to consider how philosophical assumptions are at work in the everyday mathematical activity of children as they encounter a physical cube or a symbolic equation. In the next section, I briefly discuss a key assignment of the course.

5 The Application Assignment

After the hard work of unpacking the core philosophical questions and gaining some basis in the literature in the philosophy of mathematics, the students end the course with an application assignment, in which they take these philosophical tools and questions, and use them to analyze observed mathematical behavior. I believe that this final assignment aligns with current teacher education concerns that pre-service teachers need to develop skills for analyzing and diagnosing student performance. Future teachers need to learn how to make sense of mathematical behavior for how it conveys student beliefs about particular mathematical concepts [23]. For the application assignment, pre-service teachers individually design their experiments, usually as an interview that includes a set of tasks. Interview questions have included simple opinion questions such as "How would you describe the difference between the infinity of 0,1,2,3,4, . . . and the infinity of the real numbers?" Tasks have included various mathematical problems such as "Which of the following arguments do you think best proves the following statement? Why?" "Generate as many examples as possible of a polygon" or simply "Talk me through how you would solve this puzzle." Pre-service teachers select the questions and tasks based on their interests, and hone the wording and the focus with my guidance. Part of my objective is to develop their skill at asking back-up questions that will help their own students become more willing to speculate or make conjectures about mathematics. Once we have a good working set of tasks, and a script for how they will interview their subjects, they proceed to collect the data. Some pre-service teachers are able to implement the experiment in classrooms, where the experiment becomes less an interview and more a survey, while others select a small group of friends or family or even strangers in a cafeteria. Because of our focus on diagrams, gestures, and embodiment, many of the students design experiments that shed light on how people do mathematics through visual and spatial methods.

In most cases, pre-service teachers learn just how hard it is to generate a task that actually produces ample data to speculate about a participant's thinking. But this is part of the point of the assignment—to force them to pay attention to what people did and said, and to begin to think about how mathematical thinking emerges through diagramming, gesturing, and talking. The analysis, I remind them, can only be tentative and linked to the participants under study. But the exercise of interviewing and analyzing, with the aim of studying everyday kinds of mathematical thinking, is an important moment in their training as future teachers. Teachers are more effective when they develop skills at noticing and decoding student gestures, diagrams, and verbal contributions in terms of beliefs about the nature of mathematics. Such skills allow teachers to understand why particular students make sense of mathematical tasks in particular ways. Moreover, the pre-service teachers find that people are interested in engaging in these tasks and in considering their philosophical questions, and thus they see first-hand how the philosophy of mathematics can be a powerful way of getting their future students to talk and think about mathematics.

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22

A Mathematics Inquiry Course: Teaching Mathematics in a Humanistic Way

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1 Introduction

This chapter discusses an inquiry course in mathematics for graduate mathematics education students. By *inquiry* we mean that students engage in a semester-long research and exploration in mathematics, posing their own questions and problems instead of being presented with teacher-prepared material to be learned or only solving problems posed originally by others. This use of the word *inquiry* is different from other uses in the community of college mathematics faculty, such as *inquiry based learning* [1]. The underlying philosophy of our course is to teach mathematics in a humanistic way. That is, the course stresses the view held by mathematical humanists and social constructivists that mathematics is human-made ([24], [17], [10]) by engaging students as active players in developing mathematical ideas and strategies as they pose and solve mathematical problems. They develop skills in problem solving, problem posing [4], and extending problems.

2 Context and Audience

The course we describe was co-taught in 2008 and 2012 by the first two authors at the University of Delaware, and it is mainly geared for doctoral students in mathematics education. The third and fourth authors took the course in 2008 and 2012 respectively and some of the comments they wrote as students are included in sections 4 and 5. The course is scheduled when a sufficient number of graduate students are available to take it, which means that several cohorts of students are together when the course is offered. Our course is an example of the work coordinated by the Center for Scholarship of School Mathematics, a project of the Education Development Center, which builds on previous work [5] and strives to institutionalize such experiences at several universities. The principles that guide the work of the Center also guide our efforts. They include that the mathematics that students explored felt personally relevant to them; participants would investigate mathematical ideas as mathematicians; and the focus of the work was on inquiry in mathematics rather than particular content.

The course is relevant to institutions that prepare mathematics teacher educators. If mathematics teacher educators have not engaged in posing and investigating their own mathematical problems or are uncomfortable doing so, it is unlikely that they will provide future and in-service mathematics teachers with these experiences. It is therefore important that mathematics teacher educators experience solving problems posed by others and have opportunities to pose and investigate their own mathematical problems. As Pólya points out, "a teacher who has had no personal experience of some sort of creative work can scarcely expect to be able to inspire, to lead, to help, or even to recognize the creative activity of his students." [26, p. 209]. Mathematics teacher educators and mathematics teachers need opportunities to find out mathematical ideas by themselves, if in the future they are to let students "find out by themselves as much as is feasible" [27, p. 116].

We did not make the philosophy behind the course explicit to our students. That is, we did not tell students that we were teaching mathematics in a humanistic way or that our philosophical stance was akin to social constructivism. As the Spanish proverb says, "la palabra mueve, el ejemplo arrastra" (words move; setting an example carries along). It was much more important for us (and for the students) to actually teach mathematics in a humanistic way than to label the course as such. We did, however, discuss that the course can be adapted for prospective or in-service teachers of mathematics, even if their students had less mathematical knowledge and sophistication.

3 The Humanistic Philosophy of the Inquiry Course

The course we describe focuses on teaching mathematics in a humanistic way, which is the first theme that White [33] identified when examining mathematics as a humanistic discipline. This approach seeks to "place the student more centrally in the position of inquirer than is generally the case, while at the same time acknowledging the emotional climate of the activity of learning mathematics." When teaching mathematics in a humanistic way we encourage "students to learn from each other and to better understand mathematics as meaningful, socially constructed knowledge" [32, p. vii]. According to Hersh [17, p. 48], "a humanist sees mathematics as a social-cultural-historic activity. In that case it's clear that one can actually look, go to mathematical life and see how proof and intuition and certainty are seen or not seen there."

Hersh [ibid, p. 60] says that "to the humanist, mathematics is *ours*—our tool, our plaything," and that is what we wanted our students to feel about mathematics. One of our goals was "to strengthen imagination and questioning" [31, p. 41]. Our course emphasized questions over answers [ibid, p. 40]. In our course, inquiry in mathematics was approached in a spirit of playfulness; there were many possibilities for invention [ibid, p. 43]. In addition to placing the students more centrally in the position of inquirers [32, p. vii], their voices were heard constantly, not just the voices of the teachers. Tymoczko [30, p. 12] states that "to introduce students to humanistic mathematics is to introduce them to a human adventure, an adventure of which humans have actually partaken in history." We wanted our students to participate in the adventure.

Two of the main features of the underlying philosophy of the course are, first, that the mathematical praxis has various facets, not just the finished product (the deductive rendering of theorems), and, second, that social interactions play a crucial role in mathematics, on the one hand, in reaching certainty (in the sense of absolute conviction) in mathematics, and, on the other hand, in developing the subjective knowledge of the individuals, and their enculturation into mathematics. It was therefore important to incorporate in the course aspects of mathematical practice beyond proof, especially to let students experience what mathematics in the making is like. Equally important was to reflect on the social nature of mathematical activity in the course.

3.1 Mathematics in the Making

The practice of mathematics comprises much more than proving theorems. For example, Riemann's contributions to mathematics go well beyond the theorems he proved. He contributed conjectures, the most famous being Riemann's hypothesis about the zeros of the zeta function. He gave definitions for important mathematical concepts, such as the Riemann integral, analytic functions, and the Riemann curvature tensor. He invented new concepts, such as Riemann surface, Riemann manifold, and the Riemann sphere (the extended complex plane). He also constructed important examples, such as what we now call hyperbolic space (Osserman, cited by Hersh [17, p. 50–51]). Posing and solving problems are also central to the practice of mathematics. According to Halmos [16], problems keep alive a research field and they are the heart of mathematics.

The creative phase in mathematics is also vital. Pólya points out that mathematics in the making resembles any other human knowledge in the making. "You have to guess a mathematical theorem before you prove it; you have to guess the idea of the proof before you carry out the details. You have to combine observations and follow analogies;

you have to try and try again. The result of the mathematician's creative work is demonstrative reasoning, a proof; but the proof is discovered by plausible reasoning, by guessing." [24, p. vi]. Laplace states that in mathematics the principal means to arrive at the truth (parvenir à la vérité) are induction and analogy [20, p. v]. Pólya [24, p. v] remarks that his book on induction and analogy is in a sense a philosophical essay, because he is addressing the philosophical problem of induction differently, "in closer touch with the practice of scientists" [ibid, p. vii]. Kazarinoff also highlights the importance of inductive thinking in mathematics. He states that "experimentation is a typical occupation of mathematicians":

> We conduct many, many experiments, only our experiments are made with numbers, geometrical figures, and various other abstract objects. Our experiments, like those of natural scientists, are mostly failures. Occasionally they are successful, and we discover a theorem. Then it often happens that more work is needed to give rigorous proof of the conjectured theorem, which we have come to believe is true by experiment [19, p. 14].

In their inquiries in our course students had to deal with tentative thinking, use many particular examples, deal with concepts that were not completely articulated or well formulated in the beginning, and try different approaches before finding one that was productive. This way to encounter mathematics stands in stark contrast with their previous experiences with the dominant style of exposition in mathematics, which "has been to insist on precise details of definitions and proofs, but to exclude or minimize discussion of why a problem is interesting, or why a particular method of proof is used." [18, p. 12].

3.2 The Social Practice of Mathematics

Another aspect of the mathematical practice illustrated in the course is that mathematics does not just happen in isolation. Wittgenstein remarks that mathematics is not only a branch of knowledge, but also an *activity* ([34, p. 227e], italics in original). Wittgenstein also highlights that the activity is done with other mathematicians. "*In* a demonstration we *get agreement* with someone." ([35, p. 62], italics in original.) Thurston also stresses the importance of the social setting and interactions. "We are inspired by other people, we seek appreciation by other people, and we like to help other people solve their mathematical problems. What we enjoy changes in response to other people. Social interaction occurs through face-to-face meetings. It also occurs through written and electronic correspondence, preprints, and journal articles." [29, p. 349].

3.3 Certainty in Mathematics

The interactions and communications among mathematicians follow conventional, but not completely arbitrary rules. "The rules of language and of mathematics are historically determined by the workings of society that evolve under pressure of the inner workings and interactions of social groups, and the social and biological environment of earth." [17, p. 8]. Hersh remarks that "when a mathematician submits work to the critical eye of her colleagues, it's being tested, or 'proved' in the old sense. The *proof* of the theorem (in the sense of being tested) is in the refereeing." [ibid, p. 52]. Hersh characterizes the proofs that mathematicians actually use in practice. "*Practical mathematical proof is what we do to make each other believe our theorems*. It's argument that convinces the qualified, skeptical expert." [ibid, p. 49, italics in original]. Thus certainty in mathematics, in the sense of absolute conviction, is reached in mathematics through social validation, by communicating and making the activity public, that is, by having other mathematicians test each other's statements. Thurston points out that mathematicians have a system that is "quite good at producing reliable theorems that can be solidly backed up." [29, p. 348]. The reliability "comes from mathematicians thinking carefully and critically about mathematical ideas." Hersh concurs that what is accepted as certain in mathematics is what has been verified by other mathematicians. "What mathematicians at large sanction and accept *is* correct mathematics." ([17, p. 50], italics in original.)

These descriptions of mathematical activity are consistent with the assumption of social constructivism that mathematical practices and institutions "are historically constituted and have a life of their own" [10, p. 148]. The complex relations between mathematicians and mathematical knowledge and truth described above are consistent with a theory of social practice of mathematics. According to Lave and Wenger,

> A theory of social practice emphasizes the relational interdependency of agent and world, activity, meaning, cognition, learning, and knowing. It emphasizes the inherently socially negotiated character of meaning and the interested, concerned character of the thought and action of persons-in-activity. This view also claims that learning, thinking, and knowing are relations among people in activity in, with, and arising from the socially and culturally structured world [21, p. 50–51].

In the course students worked in groups in class and shared their written reflections with the instructors. They worked individually on their inquiry projects, but shared their progress in their journals, and also in class, and received feedback from their peers as well as the instructors. They also wrote up their final findings and described their discovery process, and in turn received written feedback from several of their peers who acted as referees. Thus, we incorporated important components of mathematical activity into the course that are central to a theory of social mathematical practice.

3.4 Enculturation of the Next Generation into Mathematical Practice

In the same way that we consider the making of mathematics as a social process, we consider the learning of mathematics as a social process [3, p. 13]. Learned mathematical knowledge can be potentially unique and idiosyncratic, as Ernest [10, p. 221] points out, "because of the human creativity in sense making," which is one of the deep insights of the constructivist theory of learning. Children also bring a personal dimension to learning from their families, their home culture, their life experiences. As Bishop states,

> No two learners are alike and therefore even if the value messages being transmitted can be considered the "same", the message received will certainly be different because the receivers are different. The receiver contributes the conceptual context which gives meaning to the message, so that any communication is differentially affected by the personality of the individual [3, p. 15].

Continuous interactions and conversations allow subjective knowledge to be meshed with the knowledge of the mathematical community and to be considered as an interiorization of collective knowledge [10, p. 221]. In the classroom, within the constraints of the society and schools, "the teacher and the group mould, in interaction, the values which the individual child will receive concerning mathematics. Through activities, with reinforcement and negotiation, the child becomes enculturated into ways of thinking, behaving, feeling, and valuing." [3, p. 15].

Thurston also highlights the importance of communication to prepare the next generation of mathematicians, especially by putting "far greater effort into communicating mathematical *ideas*." ([29, p. 346], italics in original.) To accomplish this he says that we need to communicate not just definitions, theorems, and proofs, but pay much more attention to communicating mathematicians' ways of thinking. We also need "to appreciate the value of different ways of thinking about the same mathematical structure." [ibid].

In short, our philosophical stance is that to learn mathematics effectively, the enculturation needs to include explicit attention to elements of the creative aspect of the practice of mathematics such as active exploration that includes both trying to solve open-ended questions and posing new questions by varying aspects of a recently completed investigation or other people's results.

4 A Mathematics Inquiry Course at the University of Delaware

4.1 Implementation of the Course

A course designed to provide opportunities for independent research has to incorporate many elements of guidance and support so that students can deal with the intrinsic frustration and uncertainty of independent mathematical research. Thus our course is highly structured with specific activities.

The course includes short mathematics inquiries as a group and longer individual mathematics inquiry projects, and in each class we spend time on both group and individual inquiries. Students start thinking right away about a mathematical topic they would like to investigate for their mathematics inquiry project and writing down their current questions about it. Inquiry projects are discussed in class, and sample ideas for inquiry projects are introduced. Students have significant class time to develop and explore their ideas for their inquiry project alone and with the support of colleagues. For our group mathematical inquiries, in addition to documenting their thinking in their reflection journal, students write two "crystallizations"—explorations to extend their thinking about the group mathematical inquiry.

4.2 Reformulating Problems from a Given Problem

A first step in learning to pose problems that foster independent and potentially fruitful mathematical inquiry and research is to pose problems in a systematic way from a given problem. Students vary given problems by decomposing and recombining, introducing auxiliary elements, using generalization, specialization, and analogy [25]. Students can pose new problems by changing the conditions of the problem and the systematic use of the "What if not?" strategy [4]. For instance, one can change the perspective while looking at an equation like $x^2 + y^2 = z^2$. Usually students see the equation as a statement of the Pythagorean theorem, where x and y are the legs of a right triangle, and z is the hypotenuse, and where they interpret the equation as a question to find number triples that satisfy the equation, such as 3, 4, 5, or 5, 12, 13. Students can look at the equation $x^2 + y^2 = z^2$ in other ways; for example, it represents a circle with radius z on the coordinate plane, or the equation of a cone in three dimensions. Students can also change the statement of the problem, to deal with an inequality $x^2 + y^2 < z^2$, or change the exponents of the variables, and deal with equations such as $x^3 + y^3 = z^3$ or $x^n + y^n = z^n$ as in the case of Fermat's last theorem.

4.3 Class Inquiry Experiences

During class, we engaged in mathematical explorations as a group for the purpose of developing students' knowledge of processes for engaging in mathematical problem posing. The success of a course like this one rests on students' intellectual and verbal engagement during whole-class discussions and reflection during small group work. Table 1 lists some of the in-class explorations and the main mathematical connections they entail. Some of the explorations we derived from similar courses at other institutions [5], for instance, the determining the "squareness" of different rectangles, which can be also be found in a classic work of popularization of mathematics [9]. Other explorations, such as the geometry of numeric iterations, are based on previous mathematical explorations from one of the instructors [11]. We also vary class explorations from course to course.

Exploration	Mathematical connections
Squareness	Continued fractions; Euclidean algorithm.
Explorations with 142857	Modular arithmetic. Generators of cyclic groups. Isomorphic groups.
Geometry numeric iterations	Modular arithmetic. Isomorphic groups.
Generalizations of Pythagoras	Law of cosines. Pappus's theorem. Inner product of vectors. Trigonometric identities. Pythagoras three dimensions.
Explorations of the means	Average of speeds. Harmonic mean. Inequalities among means. Geometrical representation. Weighted averages.

Table 1. Explorations and mathematical connections.

We elaborate a little about the exploration of the means as an example where students took the exploration beyond what was proposed by the instructors, and how in turn their discovery enriched the course. The first time the course was offered, students were asked to explore representing the arithmetic, geometric, harmonic, and quadratic means of two numbers a and b on a trapezoid (see Figure 1) [2].

Students remembered that the segment parallel to the bases and equidistant from them is the arithmetic mean of the bases. Prompted by the instructor, they discovered that the parallel segment that divides the trapezoid into two similar trapezoids represents the geometric mean; the segment dividing the trapezoid into two trapezoids of equal areas represents the quadratic mean; and the parallel segment through the intersection of the diagonals represents the harmonic mean. Two students took the exploration beyond and found a representation in the trapezoid for weighted averages of a and b, first for special cases like $\frac{3a+b}{3+1}$ and then in general $\frac{ma+nb}{m+n}$. Inspired by this discovery, one of the

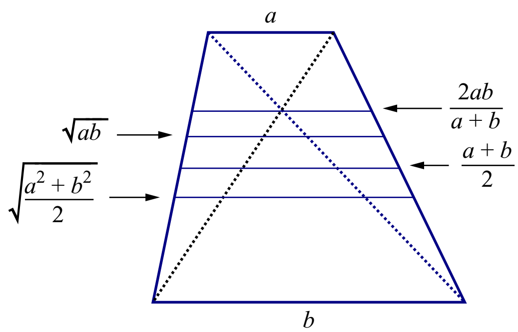

Figure 1. Different means of a and b.

instructors did later his own "crystallization" to explore whether the geometric mean, the harmonic mean, and the quadratic mean could also be represented as weighted averages [14]. The expanded class exploration was included the second time the course was offered.

4.4 Readings

We assigned readings about mathematics or processes of problem solving or posing. They were a basis for class or online discussions. We included several chapters of Brown and Walter [4], several sections of Pólya [25], the chapter on reinvention by Freudenthal [15], a paper on habits of mind by Cuoco, Goldenberg and Mark [8], and a paper by Sinclair [28] on the aesthetic in mathematical research.

4.5 Class Journals

Students kept a reflective journal documenting the development of their mathematical thinking about the mathematics explored in class. Each week during class twenty to thirty minutes were allotted for reflecting on their problem-solving experiences, including extensions to the mathematical insights the class had generated, as well as reflections upon new questions. The reflection was done in class, rather than after class, so that students would have the experience fresh in their minds. This increased the likelihood they actually would reflect. Students also reflected more generally on the course in these journals. Instructors read the journals to get a better idea of students' thinking, attitudes, and feelings, to provide comments, to support students' thinking, and to build future class activities on students' thinking.

4.6 Crystallizations and Online Discussions

Crystallizations gave students the opportunity to pose and investigate out of class their mathematical extension problems to explorations done in class and to engage in communicating mathematically about their investigations. For instance, during one of our in-class explorations, we examined ways of expressing which rectangles could be considered more "square." This led to developing an idea that we called "inner-square sequences" that had connections to continued fractions. For example, the 4×7 rectangle in Figure 2 leads to the inner-square sequence $(1, 1, 3)$ because it contains one 4×4 square, one 3×3 square and three 1×1 squares. This connects to the continued fraction $1 + \frac{1}{1+\frac{1}{3}}$.

Christine, one of the students in the course, wondered in her class reflection whether every number, including irrational numbers, could be translated into an inner-square sequence and, in turn, a continued fraction. Outside of class she wrote a crystallization [23] in which she used the number $\sqrt{2}$ to explore these ideas. By using rectangles and "squareness," she was able to think about the continued fraction for $\sqrt{2}$ by considering the rectangle $\sqrt{2} \times 1$. Her first step was to create a 1 by 1 square inside the rectangle. In her second step she had to consider squares whose sides are not integers, taking the exploration beyond what we did in class. Based on her exploration, she hypothesized that the continued fraction for $\sqrt{2}$ could be created, though she did not create the continued fraction in her crystallization. She ended up finding the continued fraction for $\sqrt{2}$ after she had submitted her crystallization. Students often worked on the mathematical ideas raised in class on their own time outside of class.

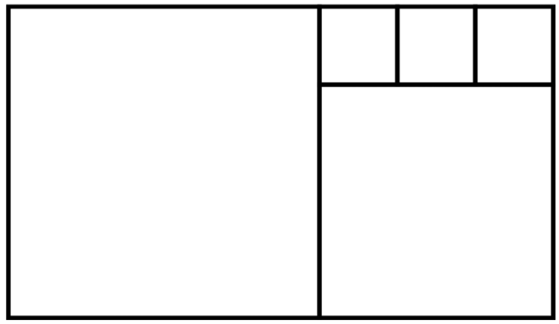

Figure 2. A 4×7 rectangle broken into squares.

In some weeks, we posted a question for discussion on a topic that needed more time to explore than we had in class. We expected each person to make a minimum of two substantive comments on the course web page.

4.7 Inquiry Project

Another key component of the course was that students were expected to conduct their own mathematical explorations by designing individual mathematics inquiry projects. They were opportunities for students to engage in mathematical research about problems that were relevant or interesting to them. Students' problems had to be new to them and without readily accessible solutions, but not necessarily open problems to mathematicians. Students worked on the projects throughout the semester. Inquiry projects were discussed in class, and students documented their progress and reflections on the project in their project journals. We collected students' project journals every other week and wrote comments and gave feedback to the students, supporting their progress on their inquiry. The final reports were submitted for "publication" to editorial boards formed by their peers.

5 Students' Reactions in the Inquiry Course

The reaction of our students to the course, experiencing that discovery entails both joy and frustration, is similar to the reaction of students in other courses where students have a central role in inquiry ([31], [22]). We share the experiences of two participants, Christine (mentioned before) and Laura, who were chosen for several reasons. They represent two extremes in terms of formal mathematical backgrounds: one had a master's degree in mathematics, the other was a former elementary school teacher. Their insights about their thinking and emotions were explicit and well-articulated. They also expressed feelings similar to those expressed by other participants. Both found the course to be valuable and worthwhile to them, personally and professionally.

Christine had previously earned a master's degree in mathematics and had teaching experience at the college level. Christine entered the course with a strong background in mathematics and an above average sense of comfort with problem solving prior to the course. Laura is a former elementary school teacher, and her formal mathematics training did not include courses that are common for mathematics majors. Some of the terms used by her classmates (such as isomorphism of groups) were new to her. She often felt that she needed more time in class to think things through, in part because some of the ideas that were common knowledge for her peers were new for her, but also because she is a deep thinker who likes to contemplate mathematical ideas from different perspectives.

Three salient points emerge from their experience in the course. First, in spite of the differences in their mathematical backgrounds, both Christine and Laura found that the course and their work with problem posing helped them develop a deeper and more connected understanding of mathematical concepts and processes. Second, the course helped them develop a deeper idea of the nature of problem posing and research in mathematics, including the inherent frustrations and tensions of the process. Third, problem posing was challenging and at times even frustrating, but also a joyful experience for them.

Although Christine had a strong content background in mathematics, she often found that the course pointed out areas of mathematics she didn't know as well as she thought, and allowed her to build on and extend concepts she had learned in previous courses. For example, in the class where the last part was exploring various means of two numbers described above, Christine wrote

> It felt like a nice thing to end the class with—exploring and learning a concept we all probably "knew" in some sense and yet we all could have known better (and probably still could—I bet we could do more explorations still). This feels like something we've done all semester—exploring and asking questions about math we may have been familiar with (like taking squares and talking about squareness) (5/21/08, class journal).

Both Laura and Christine found that the process of problem posing could be frustrating. For example, on week seven of her inquiry Laura wrote on her journal, "My brain hurts! Been thinking about this wrong—*again*—very frustrated!" ([6, p. 20], emphasis in original.) Christine wrote about her frustration after an exploration about the asymmetric propeller, in which the blades are three congruent equilateral triangles with a common vertex (Figure 3). Students moved the blades in a dynamic figure [13], and conjectured that the red triangle would always be equilateral, regardless of the relative positions of the other equilateral triangles. After using the "What if not?" process to develop questions related to this problem, Christine wrote:

> The question we chose to explore first (from the "What if not?" process) was what happens when you have four equilateral triangles. This didn't really get us very far. . . I know in math you often ask questions which don't have a clear answer or which lead nowhere after hours of work—but it is still frustrating (3/12/08, class journal).

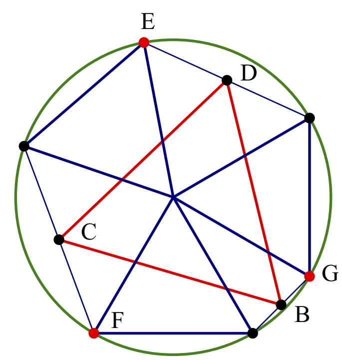

Figure 3. The asymmetric propeller.

As Christine's remark shows, the "What if not?" method does not provide specific guidance as to what to change in a problem. In the case of the asymmetric propeller, more interesting explorations would ensue from asking, "What if the equilateral triangles are not congruent?" or "What if there are only two equilateral triangles?" (the bowtie).

While Christine and Laura experienced some frustration as a result of problem posing, they also enjoyed it. In her crystallization about the squareness problem, Laura explored the ratio of the area of the first square to the area of the initial rectangle, and discovered it was equal to the ratio of the sides of the rectangle. She described this connection between ratios of areas, and ratios of lengths as "an exciting 'aha' moment" for her [7, p. 9]. Christine wrote, "One of the things I have learned in this course is that I really enjoy problem posing—it is fun to completely rethink problems" (04/05/08, online discussion). She often used the words "cool" and "interesting." At one point she refers to her work in the course as "playing" with mathematics (02/13/08, class journal). Laura wrote on her class journal (02/15/12), after the second session exploring the squareness problem

> Today was really interesting! Unfortunately, I feel like I miss stuff when I'm still processing something I just figured out or what we just talked about and other people are steps ahead already. But I think that's okay—because I'm having fun and learning still! I've thought a lot this past week about how we're all on our own mathematical journeys and what matters most is continuing to learn and make progress no matter where you're at.

6 Challenges and Changes

As we mentioned before, we have changed some of the class explorations from time to time. In some cases it was because an exploration proved to be too frustrating for some of the students. In some cases, students who considered themselves quite successful in mathematics found themselves not being able to explore mathematical ideas in a context in which they were not familiar, for example, using the kinematic method to derive geometrical properties of figures [12]. In other cases, the class explorations did not lead to additional discoveries or explorations by the students, so we replaced them with others. Of course, whether a class exploration is too frustrating or successful in generating further explorations depends on the activity and on the students in the class. We have expanded on class explorations that were fruitful.

Another issue that concerned us was the fact that there were wide differences in the mathematical background of the students. The instructors, especially Mandy, would on purpose slow down the presentations and explanations of students by asking lots of questions. Other participants were soon also asking questions during the presentations.

We found that co-teaching was an important aspect of the course. We were able to provide students with prompt and detailed mathematical feedback and timely emotional support that perhaps would be hard to provide were we to teach the course individually. Co-teaching is also deemed to be important for teams in similar inquiry courses at other institutions. This was shared by participants in the 2008 Summer Institute of the Center for Scholarship of School Mathematics.

We would like to see this type of course institutionalized at Delaware and other universities. For this to happen, the course would have to not depend only on two particular instructors. Also, we need to be able to run the course on a regular basis without requiring special releases or overloads. That would require finding a way to have the course taught effectively by just one instructor.

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X

Philosophical Issues That Can Be Related to Multiple Courses

Most of the ideas discussed in this book can be adapted to many mathematics courses, but here we include three that explicitly discuss several courses.

Chapter 23: Jason Belnap and Amy Parrott discuss the importance of creating opportunities for students to make mathematical conjectures in “Mathematical Enculturation through Conjecturing.” Their philosophical viewpoint is socioconstructivism (a term that arose from the mathematics education world at around the time that social constructivism was developing among mathematicians; they are closely related). Conjecturing is a mathematical practice that is accessible to students at many levels, gives students ownership over and commitment to mathematical concepts and ideas, and helps them learn to ask questions. It leads students to develop their own philosophical viewpoints concerning mathematics. They initially experimented with student conjecturing in courses for preservice elementary teachers who are mathematics minors (using in-house texts covering number systems, geometry, probability, infinite processes, and algebra). But now they incorporate student conjecturing in all levels of college classes. Examples are given from geometry, point-set topology, and linear algebra.

Although one does not have to subscribe to any particular philosophy of mathematics to teach conjecturing, it’s interesting that this chapter and the preceeding one both involve getting students to learn conjecturing, and the authors of both chapters subscribe to versions of social constructivism. The viewpoint that our knowledge is socially constructed naturally leads teachers to have students work with others to construct and develop their own understandings. In addition, social constructivists view bringing students into the community, teaching them the community’s norms and values along with the mathematical content, to be part of a teacher’s job.

Chapter 24: Linda Becerra and Ron Barnes focus on a specific topic in the philosophy of mathematics, that of the evolution of belief in the certainty of mathematics in “Consideration of Mathematical Certainty and Its Philosophical Foundations in Undergraduate Mathematics Courses.” They suggest considering different aspects of this topic depending on the course, by supplementing what is normally in a textbook with additional discussion, readings, and assignments. Topics considered include the relationship between belief, truth, and proof, and how our understanding of the certainty of mathematical statements and proofs has changed over time. They investigate the topic in a range of courses: Mathematics for Liberal Arts, Geometry for Teachers, Introduction to Modern Algebra, Differential Geometry, Set Theory, and History of Mathematics.

Chapter 25: Margaret Morrow, in “Proofs That Do More than Convince: College Geometry and Beyond,” discusses the role of proofs in mathematics: one role is to convince the listener or reader that the statement is true, but a more important role is to give the listener an understanding of why the result is true. (This argues against a purely formalist viewpoint.) A further role of proof is to display the strategies and concepts of mathematics. She gives a substantial example of the distinction from a college geometry course, but also illustrates the idea with examples from linear and abstract algebra. She notes that faculty need to be aware that what counts as explanatory depends on the background of the listener, and she gives an interesting example from linear algebra. An important lesson to take away from the chapter is that we should give more thought to the best proof to present, and perhaps to the value of presenting proofs from several perspectives.

You’ll find that this and several other chapters may seem to be more about pedagogy than philosophy. The balance does vary through the book, but please do keep in mind that justifying knowledge claims is an important part of philosophy, called “epistemology.” Since, in mathematics, our primary way to justify knowledge is via proofs, examining the nature of mathematical proof is both mathematics (when we are checking that the proof is correct) and philosophy (when we are thinking about how we determine whether a proof is correct, or why a correct proof lets us decide that a mathematical statement is true—or, as in this chapter, why one proof is superior to another). In addition, the process of gaining knowledge in mathematics is not limited to proof: we are working toward that goal when we make and test conjectures, for example.

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Mathematical Enculturation through Conjecturing

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1 The Need for Conjecturing in the Classroom

The work of mathematics is creative, exploratory, and generative; mathematics is an open discipline, growing at an astounding rate. It is full of exploration, conjecturing, example creation, argument, proof, and refutation. If we are not careful as instructors, however, we may lose our students in the mechanics of doing mathematics and miss the opportunity for them to really experience the rich and interesting aspects of the discipline—the creativity, exploration, and problem-solving nature of it.

Lockhart [11] pointed this out in his harsh critique of the mathematics commonly experienced during K-12 education. It tends to consist of teacher instructed content, skills, and procedures to be mastered and acquired by the students through rote practice. Successful students become adept at using formulas, solving canned sets of word problems, and memorizing and mimicking procedures. Throughout the process, many students fail to learn the value and meaning of the mathematics, misunderstand the concepts, or even lose interest in the discipline, finding it difficult, irrelevant, or boring, and thus many college students (including most future K-8 mathematics teachers) stop studying mathematics as soon as they graduate or their major requirements allow.

In undergraduate and graduate mathematics, we can face similar issues. If we focus too much on logical structure and proof writing, students can become good at making arguments regarding objects and relationships without understanding them. Proof writing can become a formalism, rather than a way of reasoning about the properties of interesting mathematical objects. Furthermore, the objects may become empty abstractions, with no conceptual foundation. The formalism can in turn cause difficulties for students as they progress on to graduate school. As Lockhart points out, "Many a graduate student has come to grief when they discover, after a decade of being told they were 'good at math,' that in fact they have no real mathematical talent and are just very good at following directions." [11, p. 6].

Much of what we love about mathematics comes from the culture of our profession and the opportunities we experience. Having been enculturated into the mathematical community, we have acquired and developed mathematical practices that enable us to independently and collaboratively access the community's knowledge, make it our own, and utilize it to solve new problems. As members of the mathematical community, we have and can create opportunities to explore mathematical objects, generate our own ideas, and then seek to understand, prove, and disprove them. As socioconstructivists, we believe that it is our responsibility as educators to bring both the overt and subtle practices of mathematical culture to light and infuse them into our classrooms.

Through our own studies (as researchers) and practice (as educators), we have found that conjecturing is a powerful mathematical practice that can enrich our students' mathematical experiences, complementing existing educational goals and practices. We have found conjecturing to be accessible to students at various levels and abilities and have found it to be an effective way of engaging students in meaningful, authentic mathematical activity. It shows students that mathematics is not something handed down from some remote mountaintop, but is a generative and sometimes uncertain process. It gives students personal ownership over and commitment to mathematical concepts and ideas. Finally, it provides a way to elicit questions from students.

2 Using Socioconstructivism to Revitalize Mathematics Instruction

The *socioconstructivist* perspective from which we write, also known as the social constructivist [1], [5], [9] or emergent perspective [6], is a merging of the *sociocultural* [5] and *constructivist* [22] positions. In this chapter, when we talk about constructivism, we are speaking about the cognitive theory of constructivism common to mathematics education [10, pp. 228–9], [9], not the foundational mathematical constructivism (intuitionism) common to the philosophy of mathematics that Hersh [10, pp. 21–2, 62–3, 139–40] describes but does not agree with.

The *constructivist* (cognitive) perspective is summed up well by the saying attributed to Anaïs Nin, "We don't see things as they are. We see things as we are." Knowledge is individual, subjective, and evolving. What we perceive in a situation is affected or colored by what we already know, understand, and value. By negotiating what we understood with what we perceive and observe, we continually add to and renegotiate our understanding of the world and everything we experience and encounter. In essence, learning involves the restructuring of our mental organization and is affected by our current understanding. If we encounter something that agrees with our current understanding, we easily assimilate it. Otherwise, we have to adapt our thinking, changing it to resolve the incompatibility. Because understanding is individual and subjective, the outside observer has no insight into how we think about, make sense of, or understand something without engaging us in discussion or activity that directly accesses it.

For example, in algebra students encounter the distributive property when working with algebraic formulas, such as multiplying 2 and $(x + y)$ to get $2x + 2y$. Students may understand this as two different ways to perform operations on unspecified numbers that result in the same value, or they may view it as a pattern of symbol usage that can be repeated whenever the symbols fit this form. One never actually knows what understanding they are building unless one tries to access it through discussion. When studying functions, students who do not understand the mathematics being represented are likely to generalize symbol rearrangements and write $f(a + h) = f(a) + f(h)$. The constructivist understands that to avoid such issues, one cannot simply focus on behaviors (e.g., how and what the student writes), because this leaves the student's developing understanding to chance. Instead, one must facilitate instructional activities that elicit and assess student thinking.

The *sociocultural* (social) perspective [5] views knowledge as communal, consisting of the norms and practices of a community of individuals as well as the artifacts created or goals accomplished by the community. Norms or rules of the community dictate what practices are acceptable and govern who has access to its resources and opportunities. Individuals progressively become a part of a community (enculturation) by engaging in the practices of that community. Through interactions and involvement in community practices, individuals who initially are on the periphery of the culture gain greater access, responsibilities, and opportunities to engage in authentic forms of participation.

For example, in many introduction to proof courses, students begin to acquire values and practices of the mathematics community regarding proof. Students are given opportunities to determine proof strategies and write formal mathematical arguments. They receive feedback, instruction, and correction on their work. Doing so, they acquire cultural aspects of the mathematical community, such as an understanding of what constitutes a valid proof, how proofs are written, how language is interpreted in mathematics (e.g., *or, and, necessary, sufficient, if . . . then*, and *if and only if*), and acceptable forms of proof.

In combining the cognitive and social, socioconstructivism recognizes both individual and cultural knowledge, but views neither as constant. Individuals develop understanding as they engage in social activities and social practices, including their understanding of the culture, along with its norms. At the same time, the culture (including its norms and practices) are developed, influenced, and changed by the decisions and actions of its members, especially at the local level. Student understanding is affected by much more than the text we use, the content we teach, or the problems that we give. It is affected by the experiences that students have and practices we establish in the classroom. For illustrations and discussion regarding the relationship between teachers' actions, classroom culture, and students' mathematical practices and understanding, see [3], [4], [7], [13].

From this perspective, effective instruction requires attention to both student understanding and the developing classroom culture. In fact, the mathematical community's values and practices empower people to make sense of and solve novel problems. Acquiring the community's values and practices enables individuals to independently access, understand, and even create new mathematical ideas, but lacking them can impede mathematical learning. Seaman and Szydlik [14] observed preservice teachers attempting to relearn and apply forgotten procedures and ideas from basic elementary school mathematics. Even with ample time, they were unable to relearn them because they, "displayed a set of values and avenues for doing mathematics so different from that of the mathematical community, and so impoverished, that they found it difficult to create fundamental mathematical understandings." [14, p. 167]. They lacked important values and practices of the mathematical community including

seeking to understand patterns

valuing and looking for structure

making and testing conjectures

creating mental and physical models (including examples and non-examples of mathematical objects)

valuing and using precise definitions

valuing an understanding of why relationships make sense

using logical arguments and counterexamples

using precise language

valuing symbolic representations and notation.

In mathematics we study ideal objects, abstract ideas, and powerful ways of thinking. The objects are communicated and negotiated through social interactions and practices. Therefore, in a real sense, mathematics is socially constructed and negotiated. What constitutes mathematical rigor and what are acceptable forms of argument are socially determined. The answers to the questions

How do we interpret language and definitions?

What questions are worth pursuing?

What are viable representations or models?

What axioms are acceptable? and

How much detail is required for a valid proof?

are all established by social convention. In fact, the review process preceding the acceptance of new mathematical results is itself a social process.

From this socioconstructivist perspective, learning mathematics involves acquiring the values, ideas, and practices of the mathematical community. The community's values and practices can only be developed through *enculturation*, that is, immersion in the culture by engaging in authentic mathematical activity. Without incorporating these practices into the mathematical work of the classroom, the "mathematics" of the classroom may scarcely resemble that of the discipline.

Establishing viable mathematical practices in classrooms is challenging, involving habits of mind that must be developed through experience and regular effort (i.e., enculturation). Teachers and instructors must accustom students to classroom and mathematical norms that may deviate from their previous classes, norms with which they themselves lack experience or are unaware of, due to their subtlety. The Conference Board of the Mathematical Sciences, in the *Mathematical Education of Teachers II* [8], observed

> A primary goal of a mathematics major program is the development of mathematical reasoning skills. This may seem like a truism to higher education mathematics faculty, to whom reasoning is second nature. But

> precisely because it is second nature, it is often not made explicit in undergraduate mathematics courses. A mathematician may use reasoning by continuity to come to a conjecture, or delay the numerical evaluation of a calculation in order to see its structure and create a general formula, but what college students see is often the end result of this thinking, with no idea about how it was conceived. (p. 55–56)

Educators and researchers need to seek out, understand, and bring to light these mathematical values and practices. We need to find ways to integrate them into instructional activities.

We argue in this chapter that engaging students in conjecturing is an effective approach to enculturation, and that it presents the opportunity to make students aware of both the overt and subtle practices of mathematical culture. As we look at examples of conjecturing tasks, we will indicate some of the aspects of mathematical culture that these tasks bring out.

3 Engaging Students in Conjecturing

As an entry point, we have focused our recent research and teaching efforts on understanding conjecturing and integrating it into our classroom practices. Our work has taken place at the University of Wisconsin at Oshkosh, a comprehensive, masters-granting, public university. We teach a variety of mathematics courses there, including content courses for preservice teachers, traditional courses for mathematics and secondary education majors, and mathematics courses for graduate students.

While we teach a variety of courses at assorted levels, most of our teaching opportunities are with preservice teachers—elementary education majors and mathematics education minors. These classes aim at providing students with non-traditional mathematical learning experiences to develop an understanding of fundamental concepts in elementary and middle school mathematics topics and an understanding of the culture and values of the mathematical community. Our faculty have developed an in-house series of non-traditional texts to support this type of instruction, the *Big Ideas in Mathematics* texts, dealing with number systems [19], geometry [17], [15], probability [20], [21], infinite processes [16], and algebra [18]. Each section of the texts begins with a class activity, which students work on in groups. Students collaborate to solve the problems and determine why their solutions work. Afterward, students present, discuss, and debate their solutions and reasoning through whole-class discussion, mainly consisting of students listening to and responding to one another. The instructor fosters and moderates student debate (rather than sanctioning answers) and supports the development of classroom norms previously discussed.

In order to understand and implement conjecturing, we began with an investigative qualitative case study (for details, see [2]). It involved creating novel conjecturing tasks in Euclidean geometry and studying how participants with varying levels of mathematical background and maturity approached conjecturing. While they varied in their conjecturing approaches and skills, participants at all levels were able to engage successfully in this authentic mathematical activity.

Since observing our novice participants' success, we have found ways to incorporate conjecturing into all levels of our college classes. We now discuss several such ways that we have done this: formal tasks (embedded-conjecturing tasks and overt-conjecturing tasks) and teaching strategies (informal teaching episodes).

3.1 Embedded-Conjecturing Tasks

One way we engage students is through embedded-conjecturing tasks. They are formal tasks or activities that (from the students' perspective) have some specific goal, objective, or solution (other than conjecturing) as the target of the mathematical work. Conjecturing is a means to an end, which naturally takes place as the students seek to solve the problem, producing some conjecture (i.e., formula, solution strategy, etc.).

For example, on the very first day of Number Systems, the first of our mathematics courses for elementary education majors, students do the following activity from the *Big Ideas in Mathematics* series:

> *Class Activity 1: The Penny of Death*
> The game is this: place n pennies on the table between you and one opponent. Now you and your opponent must take turns removing pennies from the table. Either one or two pennies may be removed in each person's turn. The object is to force your opponent to take the last penny.

> Your group's job is to write careful instructions (that some innocent person walking by in the hall could understand) for how to win at this game for every number of pennies [19, p. 2].

As in this activity, embedded-conjecturing tasks provide students with a natural context and problem to solve. Their problem-based nature naturally situates and motivates students to draw upon (and consequently develop) their conjecturing skills and abilities. The problem's structure provides a clear direction or goal for the activity, motivating the investigation and enabling students to assess their progress, recognize task completion, and determine the reasonableness of their solutions—making them particularly accessible to novices.

The biggest challenge posed by embedded-conjecturing tasks is finding or creating a suitable task, because suitability is not a simple matter of content and problem structure, but depends on student background, skills, knowledge, and understanding. Suitable problems must be challenging and novel to the students. Otherwise students will not actually engage in inductive work (i.e., creating examples, looking for patterns, experimenting with formulas, etc.); they will instead simply be practicing what they already know. They should be sufficiently unstructured, having solution pathways that are not obvious nor specified nor unique (e.g., not just applications of the current section, method, or approach). Otherwise students will not have the freedom to explore the problem, render judgments on the suitability of different mathematical ideas, and determine their own courses of action. They also need to have a solution general enough to transcend a handful of examples, so that there is a need for both inductive work and a conjectured solution. Tasks with ample solution pathways that are sufficiently obscure can be well suited for in-class group work because they foster discussion, decision, and debate within the group.

At the same time, suitable problems need to be accessible to the students. Students need to understand (or have access to) relevant definitions and have developed sufficient problem solving strategies to make sense of the problem and find entry points into it. The problem needs to be susceptible to instantiation and modeling. Students need to have the knowledge, understanding, and skills to create examples, model the object or situation, and collect and organize data. Thus problem suitability depends not just on the nature of the problem, but on its relation to the students' knowledge, skills, and understanding.

The structure that embedded-conjecturing tasks provide makes them an ideal task for early enculturation. They commonly engage students in various problem solving activities, including modeling, translating between different mathematical representations, persisting, and making and testing conjectures. These are important mathematical practices for students to acquire. Whole-class discussions taking place during or after the tasks are also important opportunities for enculturation. As a representative of the mathematical community, the instructor can initiate discussions that reflect on cultural aspects of the problems, such as discussing the value and appropriateness of various strategies or representations, limitations of graphical and numerical representations, the need for justification and proof, the meaning of discipline-specific language, the role and nature of mathematical definitions, and the meaning of conditional statements. We assert that having these discussions is critical to mathematical enculturation.

3.2 Overt-Conjecturing Tasks

Another way of engaging students in conjecturing is through overt-conjecturing tasks, formal tasks or activities in which making mathematical conjectures is the explicit activity and goal, unshrouded by other questions. These tasks describe a mathematical context, situation, or definition, and students explore, observe, or experiment with the goal of generating as many mathematical conjectures as they can. For example, in our research study participants were given the task:

> Consider a generic quadrilateral $ABCD$ and the three types of derived quadrilaterals:[1] the angle bisector quadrilateral,[2] the midpoint quadrilateral,[3] and the perpendicular bisector quadrilateral.[4] Your task is to

[1] The quadrilaterals were defined to create novel objects for study participants. Of the three, only the midpoint quadrilateral is recognized and studied by the geometric community (and always exists); thus the other two were novel, even for the professional mathematicians in our study. Each defines a quadrilateral $A'B'C'D'$ based on a quadrilateral $ABCD$.

[2] In the angle bisector quadrilateral, A', B', C', and D' are the intersections of the angle bisectors of the corresponding and subsequent vertices (e.g., A' is the intersection of the bisectors of $\angle DAB$ and $\angle ABC$).

[3] In the midpoint quadrilateral, A', B', C', and D' are the midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ (respectively).

[4] In the perpendicular bisector quadrilateral, A', B', C', and D' are the intersection points of the perpendicular bisectors of the two adjacent sides (e.g., A' is the intersection of the perpendicular bisectors of $\overline{DA}$ and $\overline{AB}$).

> explore the relationships between quadrilateral $ABCD$ and the derived quadrilaterals; make conjectures based upon your observations [2].

The task then included formal mathematical definitions for the objects specified. Participants were given ample time to study, explore, and experiment with the situation in order to generate conjectures.

Overt-conjecturing tasks (like this example) directly engage students in generating new mathematical ideas. The lack of a problem structure affords students freedom of observation and expression; however, it makes the tasks difficult for novices because there is no clear exit strategy or objective (i.e., there is nothing to solve). The novices in our research study were uncomfortable with this type of task because (at least initially) they were not certain what to do, where to go; there was no clear problem to solve. They were unclear of how to approach such a task and unsure as to whether their conjectures were legitimate or correct.

Narrowing the scope of the investigation can ease the difficulty by providing some structure. For example, the second author implemented a matrix multiplication commutativity task in her introductory linear algebra course. She gave students the procedure for matrix multiplication and then gave them fifteen minutes to work in groups, conjecturing conditions for when, for matrices A and B, $AB = BA$. The commutativity condition provides extra structure, namely a way for testing conjectures.

As with embedded-conjecturing tasks, task selection is critical to overt-conjecturing tasks and many of the same considerations apply. Based on our experiences, there are some additional considerations that are critical for the creation and generation of overt-conjecturing tasks.

One is the context or topic selection. Suitable contexts must have a large pool of potential conjectures, facilitated by having ample dimensions or properties. Situations posed must be accessible to inductive work and example generation; students need to have the knowledge, tools, and strategies needed to recreate, manipulate, or vary what is being studied. Instructors should not give the tasks expecting students to produce a specific result.

Related to this is the openness of the task, largely determined by the amount of structure and information provided. Too much structure closes an investigative task, funneling student thinking toward specific results instead of encouraging inductive thinking processes; it also reinforces the false notion that all mathematical work aims at seeking a predetermined answer. On the other hand, too little structure may broaden the scope of an investigation to the point that productive debriefing or follow-up activities become difficult; furthermore, novice students may not be prepared for such tasks.

Finally, time is a consideration. True mathematical investigations take time, and one cannot (on an individual basis) predict how long it will take for someone to make conjectures. Therefore, while we have been successful using these activities in class (individually or in small groups), we have found them particularly useful when given as individual out-of-class assignments. We follow up by selecting conjectures that we redirect to the students to be tested, validated, and proven or disproven.

In our experience, overt-conjecturing tasks are rarely used in mathematics courses, but could provide opportunities for mathematical enculturation that go beyond those provided by embedded-conjecturing tasks. Because of the lack of structure, students can experience unguided mathematical inquiry affording them opportunities to develop the ability to generate new mathematical ideas, ask mathematical questions, and guide their mathematical investigations. In addition to the cultural topics brought out by embedded-conjecturing tasks, instructors can use overt-conjecturing tasks to discuss the articulation of mathematical ideas, and strategies for generating examples and making sense of definitions. They can also help students understand what ideas or questions are non-trivial, interesting, and important to the mathematical community.

3.3 Informal Conjecturing Episodes

Conjecturing can also be incorporated subtly or informally during whole-class discussion via instructor-student interactions. For example, when the first author was teaching point-set topology, the class was discussing how topologies on a space (under the relation of subset) form a lattice. While discussing upper and lower bounds for a pair of topologies, one student asked if the intersection of the topologies would be their greatest lower bound; another student followed by asking if their union would be their least upper bound. The instructor recognized this as a question regarding the content and as a way to reinforce the idea of making and testing conjectures. Instead of answering the questions, he wrote two conjectures on the board: “The greatest lower bound for two topologies is their intersection” and “The least upper bound of two topologies is their union." He then commented that they were interesting conjectures and asked the class to determine if they were true, giving them time in their groups to create and play with some examples. After a few minutes, he orchestrated whole-class discussion, in which a student posed a counterexample showing that the union did not work as a topology. The instructor then extended the investigation, initiating further conjecturing by asking, "What is missing? What would we need to make sure that it did work?" Since class time was not sufficient to bring the activity to closure, students were encouraged to continue their investigations outside of class. Even without closure, this pedagogical episode engaged the students in making and testing conjectures.

Informal conjecturing episodes require a departure from traditional teaching practices and (if used regularly) can lead to an important shift in classroom norms. It is common practice for instructors to ask a question, wait for a response, and then evaluate it, asserting one's authority as instructor to sanction, correct, or refute the student's contribution [12]. This establishes the view that mathematical correctness is dictated, rather than determined by logic, reasoning, and argumentation. As in the above example, instead of simply answering the student's question, the instructor can reformulate and redirect it to the class as a student-generated conjecture, asking them to determine (alone or in groups) whether it is true, or including it in a homework assignment. Notice the change in the instructor's and students' roles. The instructor's role changes from mathematical performer to mentor, and the students' role changes from observer to apprentice. By doing so, the instructor is also establishing two important norms of the mathematical community: questions are mathematical ideas worthy of study, and the truthfulness of conjectures is established through investigation and reasoning, rather than by asking someone with more authority and experience.

4 Conjecturing as the Means to an End

Conjecturing naturally affords individuals the opportunity to practice inductive reasoning and develop relevant mathematical skills and practices, including example generation, pattern recognition, formula generation, and observational skills. It can also be utilized for other pedagogical purposes, strategically accomplishing various ends.

4.1 Building Mathematical Practices and Habits of Mind

One of our goals is helping students develop the practices and habits of mind of the mathematical community. Students need to develop and diversify their problem-solving strategies, increase their mathematical persistence, and find ways of accessing novel problems and mathematical knowledge.

Embedded-conjecturing tasks, like the Penny of Death problem, are well suited to the goals because they provide a problem-solving context. The problem is accessible to preservice teachers because they can easily understand the game and find ways to model and experiment with it. At the same time, its solution is complex enough that it is not obvious, warranting collaborative work. The goal of conjecturing a solution facilitates and encourages mathematical simulation, pattern recognition, organization of data, and persistence in problem solving. In groups, students experiment, play the game with counters, and work together to conjecture a winning strategy. To facilitate students' development of mathematical practices, in particular their reliance on mathematical argumentation (rather than authority) to determine mathematical validity, instructors must not give out or even sanction correct solutions, but rather challenge student solutions, raise important questions, and encourage discussion, debate, and further inquiry (as needed) to enable the students to resolve the solution.

Resolution of the problem eventually comes through whole-class discussion in which students present, explain, and justify their solutions; however, the answer is not the goal or end of the activity. Important debriefing discussion includes having students share their problem-solving approaches, the things they tried, how they organized their data, and how they found their solutions. The overt discussion about mathematical practices helps students focus on the more important aspects of the experience (i.e., the mathematical practices), rather than the temporary goal posed by the problem (i.e., how to win a game).

4.2 Establishing the Value of Proofs

Conjecturing activities can also provide opportunities to establish or reinforce the mathematical community's social values. For example, one value we struggle to establish in our preservice teachers is the need for a deductive argument, because many preservice teachers are willing to accept (as fact) mathematical statements generalized from at most a few examples, with no justification. The following embedded-conjecturing problem addresses this value.

> *Class Activity 3: Pizza Problems*
> 1) Suppose that you take a round pizza and make n straight cuts across it. What is the maximum number of regions into which you could divide the pizza? Argue that you are correct.
>
> 2) Now place n points on the edge of the pizza (on the rim of the crust). Now connect those points with all possible chords. What is the maximum number of regions you can divide the pizza into using n points? Argue that you are correct [19, p. 15].

Working in groups, students begin drawing pictures, seeking to model the problem. They begin collecting data, organizing tables, and finding a formula based on n. On the second part of the problem, many students quickly create a clear, simple, and consistent conjecture that works for $n = 1, 2, 3, 4$, and 5; however, they have no idea why it works and are often content with showing their table of data as justification. Challenging them to show the diagram for $n = 6$ creates a direct conflict, because the obvious conjecture fails, starting at this value. Consequently, the task provides an excellent context for overtly discussing the limitations of inductive work, the nature of conjectures, and the need for a deductive argument (proof) for establishing mathematical truths. It affords the instructor the opportunity to emphasize the need for understanding why, a critical value of the mathematical community.

4.3 Generating Classroom Content

Contrary to common practice, new mathematical ideas and topics do not have to originate with a lecture or textbook. Suitable conjecturing tasks can lead into or even generate new mathematical classroom content.

In teaching topology, the first author used Willard's text [23], a thorough, rigorous, and traditional text. It provides definitions, examples, theorems, and proofs, but the only deductive work it leaves to the students is proving statements furnished by the author. It begins with metric spaces along with epsilon disks and continuity; it then uses this to motivate the definition of a topology (and its reasonableness).

The instructor decided to use student conjecturing to motivate the transition to and general definition of topologies, rather than having the book simply give the definition to the students. After studying several different metrics on $\mathbb{R}^2$ and examining different examples of open and closed sets, he gave them the following overt-conjecturing task as homework:

> *Conjecturing exercise*: Without reading ahead in the book or looking in other resources, look for extreme cases of what can happen when you take the union or intersection of open or of closed sets. Write down as many conjectures as you can about this situation and bring them to class.

In all, students returned with and shared twelve different conjectures, including

A set is open if it contains none of its boundary points.

If an open set is unioned with a closed set, the result is an open set.

The intersection of infinitely many closed sets is closed.

Every open set can be written as the union of open disks.

As follow-up, for as many as class time allowed, he had the students discuss and determine whether each was true or false. The students produced counterexamples to several and editorial revisions to some statements. Then he gave students the assignment to prove or disprove the remaining statements. Students presented and discussed their arguments during the following class. Using what they had proven, he presented the formal definition of topology. The students' own proven conjectures had raised the required properties of open and closed sets, providing the motivation for and transition to the new material, the definition of a topology.

4.4 Developing Mathematical Understanding of Objects and Properties

The generative and inductive aspects of conjecturing make it well suited to helping build understanding. Formal mathematical definitions are critical to mathematics because they are precise, providing specific, testable criteria for something to qualify as that object. As anyone experienced in teaching and doing formal mathematics knows, a deep understanding comes from the experience of instantiating the definition, creating examples and non-examples, and comparing and contrasting their properties. Teachers and instructors typically provide these for the students. However, students benefit more by generating examples and discussing them for themselves—at least to the extent that they examine diverse and extreme examples. Conjecturing can afford students this opportunity.

For example, when the second author gave the aforementioned overt-conjecturing task on matrix multiplication commutativity, the students produced and shared conjectures including

A and B must be square.

If they are both diagonal, then they commute.

If A and B are upper triangular, then they commute.

A and B commute if B is a power of A.

The class deepened their understanding of matrix multiplication as they proved or found counterexamples to their conjectures and discussed which conjectures were equivalent (e.g., several groups had different forms of the first conjecture). Students became more familiar with the operation and how its properties differ from usual real number multiplication.

5 Summary

Mathematics is a rich and beautiful discipline, full of puzzles, problems, and challenges, the solving of which is amazingly fulfilling. The ability to make sense of, apply, and understand mathematical ideas lies in large part in the mathematical culture, in our mathematical values and practices—practices that will empower students to understand and access mathematical knowledge. The practices are primarily developed through enculturation, and at present students rarely experience them until they are well into a mathematics major.

We need to extend the periphery of the discipline by including the practices in all undergraduate mathematics courses, especially those taken by preservice teachers—thus empowering them to understand what it means to really do mathematics. If we do this and discuss mathematical culture with preservice teachers, we encourage the incorporation of mathematical practices in K-12 teaching, narrowing the K-12 to college gap. As we provide the experiences to undergraduate majors and potential majors, we will attract and better prepare students for the work of the discipline.

Conjecturing is a form of authentic mathematical activity that can enrich our students' mathematical experiences while meeting our pedagogical goals. It is accessible to students at all levels. It is easily included in any mathematical course through formal and informal conjecturing activities. It can be used to introduce new content, deepen student understanding, and support mathematical values. It increases student ownership of and investment in mathematical ideas. It affords students the opportunity to experience firsthand the often hidden inductive side of mathematics, which can shape students' views about mathematics. Finally, as a culturally based activity, it can help students gain the values of the discipline, such as understanding the need for deductive arguments.

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Consideration of Mathematical Certainty and Its Philosophical Foundations in Undergraduate Mathematics Courses

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1 Introduction

In this chapter, we discuss integrating topics related to the evolution of mathematical certainty into various mathematics courses to increase student understanding of the significance of mathematics and its contributions to culture. Students are introduced to the philosophical foundations of mathematics and its responses to some philosophical questions including how, or even if, one can arrive at mathematical certainty via axiomatic foundations. Students also learn about the inherent limitations of axiomatic formulations of mathematical systems. We illustrate ways we have involved students in these investigations in a variety of courses.

2 University of Houston-Downtown

The University of Houston-Downtown (UHD) is a publicly supported comprehensive urban university that offers baccalaureate degrees in 44 areas and seven master's degree programs. Enrollment is over 14,000 students. The courses to which this chapter is relevant are Mathematics for Liberal Arts, Geometry for Teachers, Introduction to Modern Algebra (which includes some set theory), Differential Geometry, Set Theory, and History of Mathematics.

3 Consideration of Mathematical Certainty in a Range of Courses

The discussion of mathematical certainty includes a number of viewpoints or topics. The topics to be explored here are taken from our article "The Evolution of Mathematical Certainty" [1], and from a considerably enlarged annotated version that is available from us. Most of the topics as they relate to the courses are also dealt with in the texts required for them. A brief discussion of the individual topics, and how we have used them, follows.

3.1 History and Development of Deductive Reasoning and its Applications

This topic is integrated into Mathematics for Liberal Arts, a general education course, in the discussion of logic, a chapter in the textbook. Here, students are introduced to propositional logic and Aristotle's syllogisms. They see how to determine the validity of arguments by using truth tables. In typical textbooks, the logic chapter makes no mention of pre-Socratic philosophers. In addition, the relationship between Aristotle's treatment of the principles of correct reasoning (with his method for determining when an argument had been proven in an axiomatic way), and the use of the axiomatic method in Euclid's *Elements*, is also not usually mentioned. Our lectures introduce Thales and Parmenides, the pre-Socratics, who made the first recorded attempts to explain nature and the universe by using reason. We discuss Aristotle's development of the ideas of inductive and deductive reasoning. We briefly describe Aristotle's development of correct reasoning (logic) and discuss his method for determining when an argument had been proven in an axiomatic way. We mention that Euclid incorporated Aristotle's ideas into his mathematical work, the *Elements*. It has been recognized ever since as the paradigm for how mathematics should be written: well thought out axioms, precise definitions, carefully stated theorems, and logically coherent proofs. Euclid's geometry appeared to be a perfectly solid foundation for mathematics, and its methodology of rational investigations permeated mathematical, philosophical, and scientific thought. For over 2000 years, Euclid's geometry was held to be the strongest, most reliable branch of knowledge. A helpful resource is Katz's book, *The History of Mathematics* [8], which contains short discussions of most if not all of the topics. Another useful resource is the online *Stanford Encyclopedia of Philosophy* [11].

During these lectures, we challenge students with questions including

When would you consider an argument to be convincing?

When would you consider an argument to be valid?

Is there a difference in your mind between the concepts? If you answer yes, please give examples.

Is it possible to arrive at truth by a method other than reasoning?

Is there more than one kind of reasoning?

Can you describe

something that was believed to be true (possibly for a long time) but which was later proved to be not true?
something that was believed to be true (possibly for a long time) and which was later proved to be true?
something that was believed to be true (possibly for a long time) but to this day it has not been determined whether it is true or false?

Describe what you perceive to be the differences between truth, belief, and theory, and give examples of each.

Is it possible for a mathematical statement to be true but not capable of being proven to be true?

Student replies to the questions are invariably incomplete and most students are not aware of distinctions among the various terms. To help them obtain background knowledge to suitably respond to them, they are provided with references including *Proof and Other Dilemmas* [6], which includes articles on the nature and significance of proof, the roles of proof in mathematics, when a problem is solved, and other papers related to these issues. Other helpful resources provided are [2], [5], and [9]. Web sites, including Wikipedia [12], the MacTutor History of Mathematics [10] and the Stanford Encyclopedia of Philosophy can also be used by students with the caveat to employ them as first steps to initiate their investigations. Since web sites can vary in their reliability, students are advised to use more than one reference for their investigations. Using these references, students are now able to clarify relevant concepts including true statements, provable statements, valid arguments, convincing arguments, logical reasoning, scientific theory, belief, proof, Goldbach's conjecture, and Fermat's last theorem. Students are required to present brief oral reports (five minutes) on their research. They then discuss their reports and compare their findings in class discussion. As a result, students acquire a better understanding of the terminology and the development of the rational framework of reasoning, and they are able to formulate more appropriate responses to our questions and related ideas emanating from them.

We also integrate the topic of the history and development of mathematical reasoning into our History of Mathematics course which has a heterogeneous mix of students, including mathematics education, statistics, mathematics, and computer science majors. In the course, we integrate the topic into the material on Greek mathematics and the material on the development of mathematical logic initiated by the discovery of non-Euclidean geometry. Depending on the students, the degree to which we integrate the material varies. All students are challenged with questions similar to those posed to the Mathematics for Liberal Arts students. However, we go into them in more depth and add additional queries. For example, we mention the four-color problem and note it was relatively recently solved by using the computer. We ask them if they think proof by computer is a valid proof. As in Mathematics for Liberal Arts, student responses to the questions we pose are invariably incomplete and here too students are often not aware of distinctions in the various terms. As before, we direct students to various resources, including those described earlier ([2], [5], [8], [9], [10], [11] and [12]) and require them to report back orally on the results of their investigations. As a result of their investigations the students achieve a more complete understanding of concepts including logical reasoning, true statements, provable statements, and what constitutes a valid argument. The students then demonstrate their understanding of the concepts in their presentations about the four-color problem, Goldbach's conjecture, the twin prime conjecture, or similar topics. The *Universal Book of Mathematics* [3] briefly describes a number of such outstanding problems and conjectures. *Philosophy of Mathematics: Selected Readings* [2] contains articles by noted mathematicians and philosophers dealing with a number of these issues, including "On the Nature of Mathematical Reasoning" by Poincaré and "Russell's Mathematical Logic" by Gödel. With sharpened definitions and concepts, the students are equipped to better understand the ideas motivating the non-Euclidean geometry investigations and Gödel's incompleteness theorems that follow. In addition, all students are required to complete a significant writing project and oral presentation. Some of the projects have reflected student responses to the challenge questions. Others have included further investigations into pre-Socratic contributions (including Parmenides and his Eleatic school) and those of Pythagoras, Aristotle, and Euclid. Many textbooks for the course do not stress Aristotle's connection to both inductive and deductive methods nor do they emphasize Aristotle's and Euclid's development of the ideas that culminated in the axiomatic paradigm of how mathematics should be done. Some student projects have dealt with Aristotle's inductive and deductive methods, or Aristotle's and Euclid's developments leading to the modern axiomatic paradigm for mathematics. Students then discuss their investigations and projects and describe their findings in class presentations. Through written and oral presentations, students achieve a better and more complete understanding of the issues.

3.2 Euclid's Geometry as a Perfectly Solid Foundation for Mathematics—Its Methodology begins to be called into Question

We integrate the topic into our two geometry courses, Geometry for Teachers and Differential Geometry (for general mathematics majors). Both courses briefly introduce students to non-Euclidean geometries. However, the textbooks used do not emphasize the philosophical implications of the existence of non-Euclidean geometries and their exposure of the vulnerability of geometric intuition and its implications. We explain the widely held opinion in the eighteenth century, championed by the eminent philosopher Immanuel Kant, that (Aristotelian) logic was "a closed and completed body of doctrine." Kant believed that Euclidean geometry was the only possible geometry because it was considered the model of the known universe and space. In Kant's time, Euclid's geometry appeared to be a perfectly solid foundation for mathematics. Its methodology of rational investigations permeated mathematical, philosophical, and scientific thought well into the next century.

We introduce non-Euclidean geometries and point out that since their development, the mathematical community has accepted the possibility of more than one valid geometry. This led to the realization that mathematics could deal with completely abstract systems of axioms that no longer had to correspond to beliefs based on real world experiences. This indicates the vulnerability of using geometric intuition alone as a basis of mathematics. The loss of certainty in geometry—that Euclidean geometry was no longer the only possible geometry—might suggest the loss of certainty in mathematics and hence in human knowledge. We explain that to address the problem, mathematicians turned from geometry to arithmetic as the foundation for mathematics.

Students are asked to comment on Kant's belief about geometry. In particular, do they think that the different possible geometries are just as successful models? Students are asked to investigate (individually or in groups) which properties of Euclidean geometry no longer hold in non-Euclidean settings and to prepare brief oral reports (five minutes) on their findings for class discussion. Resources for the student investigations include Greenberg's *Euclidean and Non-Euclidean Geometries: Development and History* [7], the text chapter on non-Euclidean geometries, and the previously mentioned references.

Student responses have included

> While the sum of the angles in a triangle is 180 degrees in Euclidean geometry, in non-Euclidean versions, the sum may be less than or greater than 180 degrees.
>
> While the shortest distance between two points in the plane is a straight line in Euclidean geometry, on the sphere the shortest distance is the arc of the great circle connecting the points. (In this case, on the sphere, arcs of great circles replace the straight lines in Euclidean geometry in the plane.)
>
> From a point not on a line, zero, one, or an infinite number of lines may be drawn parallel to the given line (depending on which geometry one is considering).

Such examples suggest the vulnerability of the idea of geometric intuition. The student investigations and class discussions have led students to a better and more complete understanding of the questions and to appropriate responses to them.

3.3 The Search for a Solid Foundation for Mathematics

The topic is integrated into our upper-level Set Theory course in the chapter on the axioms of set theory. The relationship between mathematics, logic, and set theory is not considered in most texts for the course, yet it led to the axiomatic foundations for mathematics that are generally accepted in mathematics today. Most texts at this level do not even mention David Hilbert's program or the ideas of consistency and completeness. We introduce Hilbert's program for the development of the foundations of mathematics. He required that all mathematics should be put on a sound basis using the axiomatic method. To Hilbert, this meant that in each field of mathematics, a finite set of axioms must be formulated from which the facts of the field could be proved. He required any such system to be complete (any statement is either provable or refutable, i.e., its negation is provable) and consistent (it is impossible to prove contradictory statements). Hilbert's attempt to prove that all mathematics can be described by a finite complete system of axioms is sometimes referred to as *the axiomatic approach*. In addition, Hilbert's philosophy of mathematics, called formalism, asserts that the essence of mathematics is symbols and the rules for manipulating them. To formalists, mathematics is not and need not be about anything else. The distinction between the axiomatic approach and formalism is worth making: virtually all modern mathematicians believe that (at least in some fields) the axiomatic approach is extremely important. However, many mathematicians would categorize themselves not as formalists but as platonists, who consider mathematical entities to be abstract, eternal, and unchanging.

Students are asked what axiomatic systems, if any, they have already encountered in their studies in mathematics. Do they believe it is possible to construct a finite set of axioms to describe, for example, algebra or the real number system? In our sections of this course, a short written report is required. Students can elect to focus their reports on the topic. A brief discussion of Hilbert's ideas enables students to see the big picture of how axiomatic systems came to the forefront in mathematics.

We then describe a second approach to the development of the foundations of mathematics, called logicism. Its thesis, briefly stated, is that all mathematics is derivable from logic. The laws of logic were accepted by almost all mathematicians as true. Hence, logicists contended that mathematics must also consist of truths. And since truth is perforce consistent, then mathematics must be consistent. Russell and Whitehead, building on the work of Frege, attempted to achieve these goals with their *Principia Mathematica*.

At this time, set theory appeared to be the same as logic. So it seemed possible that set theory and logic could serve as the foundation for all mathematics. We explain that if all mathematics can be reduced to logic, and if logic is the same as set theory, then in order to understand the foundations of mathematics, one would only need to consider the foundations of set theory. The communication of Russell's famous paradox to Frege indicated difficulties this approach could and would encounter.

A third foundational school, intuitionism, is not considered in our discussions.

Excellent resources considering the three foundational schools can be found in *Proof and Other Dilemmas* [6], which presents relatively brief but succinct discussions, and in *Mathematics: The Loss of Certainty* [9], which devotes chapters to these topics.

We lead discussions of the topics and ask students to consider the questions

Which, if either, of the programs (formalism-axiomatics or logicism) do you think has the possibility of being successfully carried out? Why or why not?

Do you believe that Euclidean geometry is consistent? complete? Why or why not?

Do you believe that arithmetic is consistent? complete? Why or why not?

Is it even possible to determine if any of these systems is consistent or complete?

Student replies to the questions are often naive and in many cases students are unaware of the difficulties involved in these topics. Web sites, especially Wikipedia and the MacTutor History of Mathematics, offer a good starting point for investigations—in this case, discussions of consistency and completeness. Kline [9] provides considerably more detailed discussions. Students are asked to investigate and contribute brief responses to one or more of the questions in the next class discussion. Through the investigations and class discussions, students arrive at a better and more complete understanding of the questions and what constitutes appropriate responses to them.

3.4 The Demise of the Russell-Whitehead and Hilbert Programs

We incorporate this topic in our Set Theory course. We explain why Russell and Whitehead were ultimately unsuccessful in achieving their goal of deriving all mathematics from the laws of logic by utilizing set theory. In particular, it is necessary to add non-logical axioms such as one stating that there is an infinite set. Most texts at this level consider slightly, if at all, Gödel's incompleteness theorems. Informally, students are asked whether or not they believe it is possible to achieve Hilbert's goal of a finite set of axioms to describe arithmetic that is both complete and consistent. Also, do they think it is possible that inconsistencies exist in mathematics? If so, how could they possibly know? Student responses are invariably that mathematics is consistent and complete.

We next describe how Gödel's first and second incompleteness theorems ended Hilbert's dream of putting all mathematics on a sound basis using a finite axiomatic method that was both consistent and complete. Gödel's results come as a surprise to most students based on their naive answers to our earlier queries. We posed informal queries including the question "Is it possible that inconsistencies exist in mathematics?" to entice the students to conjecture that mathematics is indeed consistent and that the students expect that eventually one would be able to prove this. This latter error Hilbert and Russell-Whitehead fell into. Students also naively conjecture that mathematics is also complete—again an error shared by Hilbert and Russell-Whitehead. Gödel's theorems showed the conjectures to be false, contrary to the students' naive intuitions. These events illustrate the limits of intuition and limits on finite axiomatic systems.

In the short paper that students are assigned, they investigate in more detail connections among logic, set theory, and mathematics. This enables them to better understand the interactions among the topics in the development of mathematics. As noted earlier, Wikipedia and the MacTutor History of Mathematics offer a good starting point. Kline [9] provides considerably more detailed discussions.

The discussion can also be integrated into upper level mathematics courses that use set theory. In particular, in our Modern Algebra course students are assigned to read our paper "The Evolution of Mathematical Certainty" [1] and write an essay on their interpretation of its implications to modern algebra in regard to its set-theoretic foundations.

In the History of Mathematics course, we present the material on Gödel's theorems and their implications. It is also a possible topic for the required student writing project in this course, as are the Hilbert and Russell-Whitehead programs.

In the Mathematics for Liberal Arts course, we mention that for any finitely describable system as rich as arithmetic, if it is consistent then it is incomplete. This discussion allows students to see one of the philosophical implications of mathematics—in certain mathematical systems that are consistent, not everything that is true can be ultimately proven, within the system, to be true.

4 Some Issues to Consider

Instructors should take into consideration the level of the course (lower or upper division) and the student preparation when determining the extent to which they can effectively incorporate the development of the mathematical and philosophical foundations of mathematics into their course. There is a temptation to go into greater detail than the student audience may be able to absorb and understand. Student feedback is essential in the successful implementation of the ideas. If the students don't readily volunteer questions and comments, the instructor should draw them out by posing questions or situations relating to the topics.

For example, in Section 3.1, students are challenged with the question "Is it possible for some mathematical statement to be true but not capable of being proven to be true?" If student response is minimal, we draw them out by an informal discussion of the problems of squaring the circle, Fermat's last theorem, and the four-color problem. We note that the ultimate resolution of the problems evolved over a considerable amount of time, and ask "Because of the difficulty in resolving the problems, do you (the students) think it is possible that among the many still unsolved problems that at least one cannot be resolved using all the tools of mathematics?"

Another possible way of drawing students into a discussion is to request that they explain, in their own words, some of the ideas under consideration. For example, ask your students to explain in their own words the difference between consistent and inconsistent, and between complete and incomplete systems.

While Wikipedia and the MacTutor History of Mathematics are reasonably reliable resources, we caution students that some internet resources are of questionable provenance. So students are advised to consult two or more references for their research.

In conclusion, our implementation of these ideas has been very well received as reflected in student comments and course evaluations. In addition, many lively discussions with considerable student interaction have taken place, indicating the students to be genuinely interested in the development of the mathematical and philosophical foundations of mathematics. Overall, the student presentations, both written and oral, have reflected a serious interest in the topics as well as an appropriate understanding of some of the underlying issues. We believe that our efforts have been successful and rewarding. We encourage you to consider including some of the ideas in your courses to enlighten your students as to the mathematical and philosophical foundations of mathematics. We believe your efforts can be equally rewarding.

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Proofs That Do More than Convince: College Geometry and Beyond

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1 Introduction

Most mathematicians would agree that proof is central to mathematics, at least in the current view of mathematics. But when one tries to pin down the exact nature and purpose of mathematical proof, it turns out to be surprisingly problematic. To the extent to which one's conceptions of the nature and role of proof are unclear, one will likely make misguided decisions about proof in undergraduate courses and the curriculum.

In this chapter I will consider the explanatory power of proof. I will also reflect on proofs as a vehicle for communicating strategies and concepts of mathematics. I will present an extended example from college geometry, and a few shorter examples from other courses to illustrate how consideration of these features of a proof might be used as a touchstone in making pedagogical decisions.

1.1 Context: the Students

The courses discussed in this chapter are all housed in the mathematics department in the Arts and Science division of SUNY Plattsburgh, a four-year public college. A significant percentage of the mathematics majors at the college plan to become teachers at the secondary level. Indeed many of them are in a five-year combined BA and MS program through the teacher education unit in the division of professional studies. Elementary education majors who have chosen mathematics as their concentration also take these courses. The elementary education majors are found in lower division mathematics major courses and in courses dedicated to them. One of the dedicated courses is a 100-level college geometry course. (Strictly speaking, the course is available to non-education majors as well. However, in practice very few non-elementary-education majors enroll in it.) The college geometry course is the context of the extended example presented in this chapter. Hence the course is discussed a little more fully in what follows. However, the issues raised inform my teaching in virtually every undergraduate mathematics course, including linear algebra, abstract algebra, and upper division geometry. Toward the end of the chapter a few simple examples from the other courses are presented to illustrate this.

Let us go back then to the students in the courses. It is a rare individual among them who would agree with Edna St. Vincent Millay's view that "Euclid alone has looked on beauty bare." They are seldom seduced by the sheer logical perfection of axiomatic geometry—be it Euclid's slightly flawed attempt, Hilbert's approach, or the breed of axiomatic geometry that has evolved specifically for high school. In fact, the students in the college geometry class have often had very little experience working with geometric figures in any way. Many of them are arguably at the lower levels of the van Hiele categorization.[1] So we begin with informal explorations, and take a gentle approach to proof. When the students do start writing proofs, several of them cling to the formalities of two column proof that they have been taught in high school. When using the format, some students seem to write down as many facts as they can, based on the diagram for the problem, without regard to whether they are relevant to what must be proved, and with little or no attention to the logical connections between them. (It may be that they were taught this strategy to optimize the number of points they could earn on state exams.)

There is ongoing debate on the characterization of proof in the philosophy of mathematics, and concomitantly there is ongoing debate on the appropriate view of proof in mathematics education. From the perspective of the philosophy of mathematics, while Euclidean geometry, and its ultimate expression in Hilbert's version, remains an astounding feat of reason, this model of proof is generally considered inadequate for a complete account of mathematical proof in a broader context. The focus has shifted to characterizing mathematical proofs as being either explanatory or non-explanatory. But even this is inadequate, in that the notion of "explanatory" remains problematic [8]. An even more recent idea is that proof is quintessentially the communication of the tools, strategies, and concepts of mathematics [11]. These perspectives are discussed more fully in later sections of this chapter.

What then might be the appropriate role and purpose of proof in college mathematics courses of the type described? All too often students in the classes lack a sense of what it means to deeply understand a piece of mathematics (beyond procedural fluency). Given the context, my goal for students in the college geometry class (and in all my classes) is that they come to see the essence of proof not as formality, but rather as deep and convincing logical necessity. Further, while acknowledging that the notion of "explanatory" is problematic, as far as possible I want the proofs that the students encounter to help them to gain insight into why the assertion is true. I want proof to operate, at whatever level is appropriate for the students, as a vehicle to help all students (and in particular education majors) to gain insight into what it is to understand mathematics deeply.

Thus I am confronted in my teaching, course by course and day by day, with decisions about which proofs to consider, about the level of formality of classroom discourse related to proof, and about who (I or the students) will develop and present the proofs. In facing these choices, I am inevitably faced with questions such as those above about the nature and purpose of proof, and about how this should influence pedagogy. My philosophical framework draws on the literature, and is influenced by observations of how mathematicians and students actually operate with proof. This is discussed more fully in what follows.

2 Proofs That Explain, versus Proofs That Simply Verify

An early influence on my approach to teaching proof was provided by David Henderson. (For example, see his insistence that proof is "a convincing argument that answers—Why?" [7, p. xvi].) Henderson talks about the way in which the formality of his first exposure to ideas in high school geometry initially led him to become disaffected with mathematics. This struck a chord with me, and seemed especially pertinent to the outlook and experiences of many of my students. I was also intrigued by the extreme difficulty that some students experience in using definitions and logic to create what seem to me to be the simplest of proofs. The research literature in undergraduate mathematics education, and in mathematics education in general, provides insights into how students view proof. For example Harel and Sowder provide an interesting categorization of the proof schemes of students in college mathematics courses. (They define a proof scheme as "what constitutes ascertaining and persuading" for the student [6, p. 244].) As an example, under the umbrella of external conviction proof schemes they include subcategories such as proof as symbolic manipulation (possibly devoid of meaning), and proof as ritual (in which the form is all-important, rather than careful

[1] Based on their research, the van Hieles posited five levels of conceptual development in geometric thinking, each dependent on the previous stages. The first stage is characterized by only very simple perceptual recognition of shapes. At the second stage there is a focus on analyzing the characteristics of shapes (such as opposite sides congruent). At the third stage the individual is capable of simple abstraction and simple geometric reasoning (for example, being able to recognize that all squares are rectangles). In the fourth stage deductive reasoning can be appreciated. Only in the much later fifth stage is the individual able to make sense of a rigorous axiomatic approach to geometry [2]. Most of the students in the college geometry class need to start with additional work on abstraction and simple geometric arguments, and some even need more experience at the level of the second stage.

scrutiny of the correctness of the reasoning) [ibid, p. 246]. This sheds light on some of the behavior I observe in students in relation to proof. Another insight is provided by Raman's concept of the "key idea" in a mathematical proof, which she characterizes as a heuristic idea (a sense of why the theorem should be true) that can be turned into a rigorous proof [10].

Gila Hanna [4] discusses the varied factors that lead to acceptance of new mathematics by mathematicians. They include consistency and relevance in the context of related mathematics, the reputation of the author, and "a convincing argument for it (rigorous or otherwise) of a type that they have encountered before." [ibid., p. 58]. Hanna goes on to point out that, despite the varied factors that lead to acceptance of new mathematics, mathematical results are ultimately published in the form of theorems and proofs, so that "competence in mathematics might readily be misperceived as [only] the ability to create the form, a rigorous proof." [ibid, p. 60]. She asserts that from this some conclude that "To teach a beginning student is assumed to involve teaching the formalities of proof." But, as she goes on to point out, by placing too much emphasis on the formalities, the part most important to mathematicians, namely the ideas and the relation between the ideas, is likely to get lost. Indeed it is quite probably the too-early emphasis on the formalities of proof, without the substance, that leads to some of the inadequate proof schemes noted by Harel and Sowder mentioned above.

Hanna [3] refers to proofs that explain, versus proofs that prove. The distinction goes back to Mark Steiner, who remarks that "mathematicians routinely distinguish proofs that merely demonstrate from proofs which explain." [12, p. 135]. Steiner goes on to posit that what characterizes an explanatory proof is that it "makes reference to a characterizing property of an entity or structure mentioned in the theorem, such that from the proof it is evident that the result depends on the property." [ibid, p. 143]. This view influences my approach to teaching proof in all my courses.

2.1 Putting This into Practice

Consider the previous reflections about proof in the context of the 100-level college geometry course taken by our elementary education mathematics concentration students. When I teach the course, I no longer use a standard text, since I have been unable to find one that in my opinion strikes the right balance between providing students with opportunity for informal exploration of geometry and including sufficient work with formal proof. The available texts seem all to be either too formal or too informal. So instead of requiring a text, I provide the students with extensive course notes.

The following example comes from near the end of the course, at which stage we synthesize much of what has been learned by applying it to the geometry of circles. The students have already had a little experience working with dynamic geometry software. The theorem in question is that which asserts that the angle subtended by two points on a circle at the center of the circle is twice the angle subtended at the circle in the major arc. The traditional approach in texts is to prove the theorem first, and to then prove, as a corollary, that angles subtended by two points on a circle on the major arc of the circle are congruent [1, p. 254].

The proof traditionally proceeds by cases. Case 1 is illustrated in Figure 1. (The point O is the center of the circle.) The proof here is fairly straightforward. Using the exterior angle theorem and the angle equality in the isosceles triangles formed in the circle, we obtain $a + b = y + y' + z + z' = 2(y + z)$, as required.

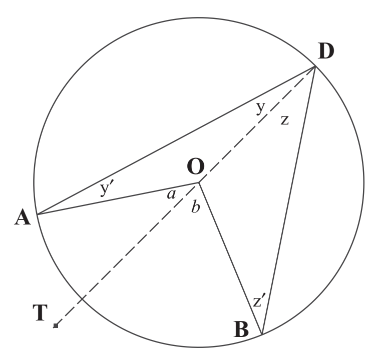

Figure 1

Case 2 is illustrated in Figure 2. The proof of this case is considerably less transparent, and does not follow simply from the proof of case 1. Worse, while the exterior angle theorem is again useful, its role is not a simple generalization of its role in case 1. Yet the traditional approach in texts at this level is to provide the proof of case 1, and leave the other cases to the exercises. (See for example the treatment in Alexander and Koeberlein [1, p. 254].)

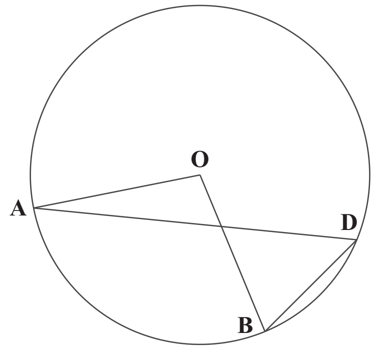

Figure 2

When exploring ways to help the students to develop intuitions to assist in proving this second case, and in particular when using dynamic geometry software to try to find a key idea that could lead to structuring a proof, it was striking that a direct perception that one angle has twice the size of another is elusive. (Of course dynamic geometry software allows one to verify this by measurement, but this does not serve the purpose of developing a proof.) On the other hand, when moving point D in Figure 1 around the circle, it certainly appeared that the size of the angle subtended at the circle remained constant. Even more, it appeared that the relative sizes of the angles y and z changed, but that triangle AOB did not change, and in particular the angles opposite the congruent sides in triangle AOB did not change. Surely there was a key idea here that could lead to an explanatory proof for this theorem?

So, consider going against the traditional order here, and first proving that the angles subtended at a circle in a major arc of the circle are all congruent. Initially it seems we need to consider separate cases. Case 1 is shown in Figure 3.

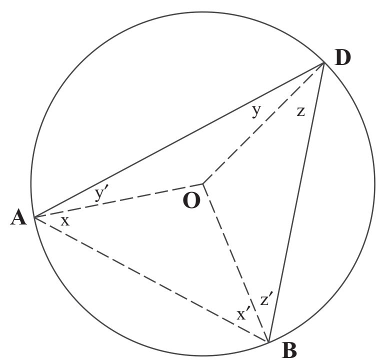

Figure 3

The proof proceeds by using the sum of the angles of triangle ABD, combined with the congruence of the angles opposite the congruent sides in the three isosceles triangles, to obtain $2x + 2y + 2z = 180°$, so that $x + y + z = 90°$. Noting that x is constant, we obtain $y + z = 90° - x$ is constant. This completes the proof of the first case.

Now consider case 2, illustrated in Figure 4.

Using the dynamic geometry software again, it is clear that as D is moved to the relevant position for this case, the angles in triangle OAB are, as before, constant. The main difference from case 1 is that the angles marked y and y' lie inside triangles OBD and OAB respectively, instead of outside, as happens in case 1 (see Figure 3). This suggests a simple modification of the proof of case 1 to provide a proof for case 2, as follows. Referring to the angles in triangle ABD, and using the isosceles triangles formed in the circle, we obtain $2x + 2z - 2y = 180°$, so that $x + z - y = 90°$. Noting that x is constant, we obtain $z - y = 90° - x$, which is constant, and more particularly is actually equal to the angle size obtained for case 1.

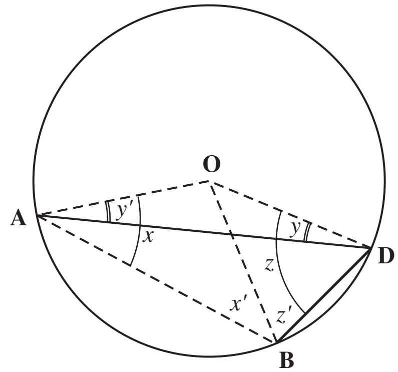

Figure 4

Case 3, illustrated in Figure 5, presents no difficulty (indeed one can simply set $y = 0^\circ$ in the proof of either of the previous cases to obtain what is required).

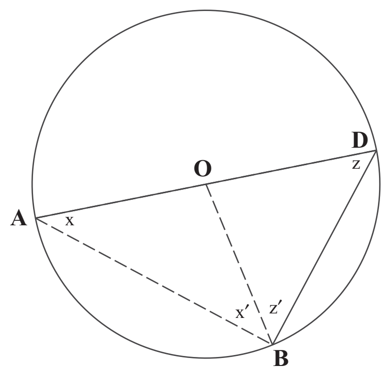

Figure 5

The theorem that the angle subtended by two points on a circle at the center of a circle is twice the angle subtended in the major arc is an easy corollary of the theorem (now proved) that angles subtended in the same major arc are congruent. (One can use the sketch of case 1 for the proof of the corollary.)

So far this example illustrates how dynamic geometry software can be used to provide a key idea that translates into a valid proof. In this example, a single key idea elucidates why the theorem is true in all necessary cases, and the key idea is visually apparent. The example also illustrates a proof with greater explanatory power at a level appropriate for the students in a 100-level college geometry class.

Further, in contrast to the traditional approach, the approach outlined above demonstrates a strategy that is often useful in mathematics: that one can sometimes adapt an approach that works in one case to prove another case. And there is even more. If one allows negative angles (not usually contemplated in geometry, but arguably a nice strategy for students at this level to consider) then one can regard the angles y and y' as negative in case 2, and as zero in case 3; with this convention, the proof provided for case 1 actually covers all three cases at once. In the following section we discuss a philosophic framework within which the additional features of the proof may be situated.

3 Beyond Explaining: Proofs as Bearers of Mathematical Knowledge

Rav argues that proofs themselves are the bearers of mathematical knowledge. He argues that mathematicians develop new strategies and concepts in their pursuit of solutions to open problems, and in particular in the search for proofs of open conjectures. He argues that proofs are the means for communicating the new strategies and concepts. “The whole arsenal of mathematical methodologies, concepts, strategies and techniques for solving problems, . . . the entire mathematical know-how is embedded in proofs.” [11, p. 20].

Keith Weber provides a somewhat related view, claiming that proofs serve purposes beyond simply convincing and explaining (such as illustrating techniques), and that taking this perspective provides a potential route for helping students to more fully appreciate the relevance of proof [13]. Hanna remarks "Educators have overlooked . . . the role of proof as a bearer of mathematical knowledge in the form of methods, tools, strategies and concepts that are new to the student and add to the approaches the student can bring to bear in other mathematical contexts." [5, p. 352]. In the example from the college geometry class, searching for a key idea by using dynamic geometry software, modifying the proof of one case to prove another, and generalizing an idea by allowing negative values for a variable are all strategies that can be applied in other mathematical contexts, and all the strategies are unfamiliar to most students in this course.

4 More on Explanatory Power of Proofs

4.1 Explanatory Power is Relative

As a caveat, it should be noted that there are grey areas in considering the explanatory power of a proof; in particular the degree of explanatory power depends on the audience. To illustrate this, consider an example drawn from our 200-level linear algebra course.

After considering properties of the dot product, students were assigned the following as a homework problem: Prove that for vectors $\mathbf{u}$ and $\mathbf{v}$, $(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) = ||\mathbf{u}||^2 - ||\mathbf{v}||^2$.

I was anticipating application of the distributive property of the dot product to prove this (and I should add that I am pleasantly surprised when students at this level attempt any proof). I was surprised when a good freshman volunteered the following proof in class (the student clearly assumes that $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$):

$$\begin{aligned}(u_1+v_1, u_2+v_2)\cdot(u_1-v_1, u_2-v_2) &= \left(\sqrt{u_1^2+u_2^2}\right)^2 - \left(\sqrt{v_1^2+v_2^2}\right)^2 \\ (u_1+v_1)(u_1-v_1)+(u_2+v_2)(u_2-v_2) &= u_1^2+u_2^2-v_1^2-v_2^2 \\ u_1^2-v_1^2+u_2^2-v_2^2 &= u_1^2-v_1^2+u_2^2-v_2^2\end{aligned}$$

Putting aside that the proof is restricted to two dimensional vectors, and putting aside as well the common student error of working backwards from the identity they must prove, what is striking is that this proof seems to arise from a paradigm of proof as simply a variant of algebraic manipulation. In my experience many beginning students operate within such a paradigm. In terms of Harel's categorization, the dominant proof scheme here seems to be symbolic manipulation. (Recall that the proof scheme constitutes the mode for ascertaining and persuading.) Assuming that the student operates within a paradigm of proof as simply a variant of algebraic manipulation, it is quite likely that the proof provided by the student has more explanatory power for the student than the proof based on the distributive property. In fact it is possible that the student's proof indicates significant insight that vector operations are not simply operations on real numbers. It is even quite likely that the proof based on the distributive property of the dot product would for many students reinforce an inadequate view of proof as simply symbolic manipulation, and mask just how different the operations on vectors are from the operations on real numbers.

In class I praised the student for a good start, and moved perhaps too quickly to pointing out how the distributive property could be used as an alternate proof strategy. It was only on further reflection that I gained the insights in the previous paragraph. However it is by repeated reflection on incidents like this that one hones one's skills in translating philosophical ideas into classroom practice.

4.2 One More Example; Different Proof Routes in Abstract Algebra

A final illustrative example can be drawn from our junior-level introduction to abstract algebra class. I use an inquiry based learning (IBL) approach; students develop sequences of proofs for themselves, based on problems provided to them. (Of course the students find this very challenging.)

In one part of the course, students work through a sequence of problems about cosets, which culminates in a proof of Lagrange's theorem (that the order of a subgroup of a finite group divides the order of the group). The elegant approach to linking the idea of cosets to Lagrange's theorem is via the following two problems:

1. Suppose G is a group and H a subgroup of G. Define a relation $\sim$ on G as follows: for all $a, b \in G$, $a \sim b$ if and only if $a \in bH$. Prove that $\sim$ is an equivalence relation.
2. Suppose G is a group and H a subgroup of G. Prove that the collection of distinct left cosets of H in G form a partition of G.

Of course the second proposition is an easy consequence of the first if one understands the connection between equivalence relations and partitions. The students in my abstract algebra class have encountered equivalence relations and partitions in the introduction to proofs course that is a prerequisite for abstract algebra. That said, this is a difficult concept for the majority of the students. There are those who would argue that one should back up and redevelop the theory of equivalence relations and partitions if students are not able to easily see the connection between the problems. But from repeated experience, I find that a more effective approach for my audience is to simply insert the following two problems in the sequence, between the two problems shown above.

(A) Suppose G is a group and H a subgroup of G. Prove that for all $a, b \in G$, either $aH = bH$ or $aH \cap bH$ is empty.

(B) Suppose G is a group and H a subgroup of G. Prove that $\bigcup_{(a \in G)} aH = G$. (Recall: $\bigcup_{(a \in G)} aH$ denotes the union of the sets aH taken over all $a \in G$; that is, $\bigcup_{(a \in G)} aH = \{g \in G | g \in aH \text{ for some } a \in G\}$.)

A few of the stronger students sometimes notice that problem 1 implies that we have a partition of G, which then by definition of a partition provides the required results in problems A and B. But for those who do not have this facility with the relation between equivalence classes and partitions, problems A and B provide an approach that in my view has enormous advantages. First, the approach provides an accessible explanation based on basic principles for why the relation gives rise to a partition; this is in contrast to simply applying a result (an equivalence relation gives rise to a partition) that is not well understood by the student in the first place. Secondly, the approach exemplifies proof as displaying strategies and concepts that have application in other contexts. Strategies called for in proving A and B from first principles include working with set operations, working with cosets, and working with the defining properties of an equivalence relation. In terms of concepts, problems A and B in a sense allow the student to reinvent the idea that an equivalence relation gives rise to a partition, and to revisit why this is true; this reinforces precisely the ideas that the students found difficult when first they encountered equivalence relations and partitions in the introduction to proofs class.

When we go through these problems in class, I call on students who have done problems A and B from first principles to present their solutions, and only after that do I invite students who have taken the simpler route to present their ideas. This example illustrates making a deliberate choice of a proof approach based on philosophical views about the explanatory power at the level of the students, and about proofs as the vehicle for developing mathematical concepts and strategies. Problems 1 and 2 are from published notes in the *Journal of Inquiry Based Learning in Mathematics* (*JIBLM*) [9, p. 24], while A and B, as well as some of the considerations discussed above appear in the notes in the instructor's version of the same course notes [ibid, p. 31]. As an aside, it is interesting to note that Rav uses Lagrange's theorem as one of his examples in arguing that proofs are the bearer of mathematical knowledge [11, p. 22], although I was unaware of this when writing the course notes published in *JIBLM*. Rav remarks "there is more technical information, more mathematical knowledge embodied in the whole proof with all its methodological links and inter-theoretical connections than in the statement of the theorem." [11, p. 22].

In the abstract algebra course, students are usually required to write up and turn in the proofs after they have been discussed in class. With the very rare exception of an occasional student who sees the connection between the equivalence relation in problem 1 and the propositions in problems A and B almost immediately, students write up the proofs for problems A and B based on first principles, despite some in-class discussion of the shorter route. This suggests that students find some appeal in the explanatory power of the approach based on first principles.

5 Student Reactions to my Approach to Proof

It is a difficult to tease out cause and effect in a classroom context even in a carefully designed research setting, much less in the normal flow of teaching. I have persisted in emphasizing the explanatory power of proof in my classes over the years, precisely because I see in those who take my classes modest but persistent shifts in the way they interact with proof. Most students in my college geometry course make good progress toward writing simple but lucid proofs. Most students in my abstract algebra course leave the course with far more confidence in their ability to develop and write proofs than when they started. To what extent the increase in confidence is due to the IBL approach, and to what extent it has to do with choices about sequencing of problems based on ideas about the explanatory power of proofs, and proofs as bearers of mathematical knowledge, I cannot say. Most likely both aspects are important. Here are a few comments (from journals) from students who have taken this course: "The biggest achievement from this class is about the way to present a proof. I have written my proofs more clearly so people can understand." "As long as I complete the proofs on my own I feel I can only improve, I do not like hearing other people's proofs because I do not think the same way they do. When I develop the ideas on my own I can remember them better and understand them because, in a way I created them." "At first, I was very unsure of how to start proofs, and this caused me to never be able to really complete one completely on my own. I now really understand how a proof is supposed to start, which makes it a little bit easier to get it complete. There are still proofs where I get stuck and cannot complete them without further help but I do feel that I have come a long way from when I began."

6 Conclusions

Two pedagogically valuable features that a proof may have are

1. explanatory power at a level appropriate for the students
2. a display of strategies and concepts that are applicable in other mathematical contexts.

These features should be taken into consideration when making decisions about which proofs to bring before students, and about the classroom strategies to be used.

In my experience, emphasizing the explanatory power of proofs, and highlighting key ideas via appropriate exploration, leads my students to become more adept at creating and communicating mathematical proofs. Considering proofs as bearers of mathematical knowledge is an ongoing exploration in my teaching, and is beginning to impact my pedagogical choices vis-à-vis proof.

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XI

Philosophy of Mathematics and Teaching

The chapters in this section are really more about philosophical issues related to mathematics education, rather than specific applications of philosophy to mathematics courses, but involve approaches that we feel are worth sharing with the community. Although the focus is on a philosophical issue, both suggest ways of directly applying their conclusions to teaching mathematics.

Chapter 26: Sheila Miller proposes, in "On the Value of Doubt and Discomfort in Education," that there are two tools faculty can use to help students develop the habits of inquiry and persistence that are critically important characteristics of educated adults: doubt and discomfort. Students need to learn to insist that they understand how things work, rather than simply accept conclusions from authorities. She uses the topic of mathematical infinity in a range of classes to move students beyond their comfort level. Students first learn that one property infinite sets have is that they can be put in one-to-one correspondence with proper subsets of themselves. Then, via Cantor's proof, they discover that there are nonetheless infinite sets that are of different sizes. She spends varying amounts of time on this topic, depending on the level of the course and the time available, from fifteen minutes to six class meetings. In classes from developmental mathematics and college algebra, precalculus and calculus, to teacher preparation courses and courses for mathematics majors, she uses this topic effectively to sow doubt and discomfort, and, gradually, deeper thought.

Along the way, a range of philosophical issues arise, challenging students' implicit philosophical beliefs about the nature, meaning, and practices of mathematics. Students' discomfort with actual infinities echo those that arose in the mathematical community throughout history. Reflecting on the struggles the mathematical community had with the issue gives students confidence to pursue their own intellectual curiosity.

Chapter 27: Alejandro Cuneo and Ruggero Ferro, in "From the Classroom: Towards A New Philosophy of Mathematics," introduce a general theory, coming from the experience of teaching mathematics, of the acquisition of (mathematical) knowledge from our experiences and direct perceptions and from mental operations. The chapter is, in effect, a reverse engineering of the usual approach in this book, which takes a philosophical stance and applies it to teaching. Instead, here the authors look at how questions in the teaching of mathematics lead to a philosophical position about mathematics that is consistent with it.

Their approach is motivated, in part, by results from logic: that it is not possible, using language, to uniquely determine an infinite set such as the natural numbers. In fact, there's not even a way to say, in a finite language, what's meant by a set being infinite. So how can we, as teachers, be sure that when we speak of the natural numbers, our students have the same set in mind? The authors attempt to show how we can refer students to experiences all people have, and develop mathematical concepts from them. They then bring their conclusions back to the teaching of mathematics with an example from teaching calculus.

The authors are not claiming that the paths they propose to the development of notions of time, element and set, and natural numbers are in fact the paths that everyone naturally follows to acquire them. People normally do not care, nor pay attention to, how they acquire primitive mathematical concepts, just as children learning to walk don't need to know the laws of physics nor the anatomy and physiology of their legs. But a physiotherapist has to know all that to correctly help a person having problems with walking. Teachers have a similar need, although currently a complete knowledge of what it is meant by learning and knowing is not available. What Cuneo and Ferro hope to achieve in this chapter, beyond introducing their new philosophical viewpoint, is a contribution toward an account of learning and knowing.

While little in this chapter is immediately applicable to our teaching (which is why it is the last chapter in the book), we believe that the ideas presented are worth serious consideration when we have time to think back on why our students are having difficulty learning what we are trying to teach.

Because the motivation for the chapter comes, in part, from results in logic, it is probably helpful to know a bit about logic and foundations (Gödel's completeness theorem and its corollaries, issues with impredicative definitions and the contradictions in naive set theory, for example) when reading it.

26

On the Value of Doubt and Discomfort in Education

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1 The Role of Doubt and Discomfort in Education

The evaluation of ideas requires the challenging and uncomfortable task of revealing—and often changing—underlying assumptions. This chapter presents a case for the deliberate and constructive cultivation of doubt and discomfort in the mathematics classroom as tools to facilitate learning. We claim that all people have philosophies of mathematics, and that teaching with those philosophies is part of teaching mathematics to our students. In particular we suggest that students' doubts about and struggles with mathematical ideas frequently have philosophical roots. Revealing and exploring these deep and often unconscious philosophical ideas can help students to identify the origins of their discomfort and, ultimately, to resolve it through engagement with philosophy and mathematics. We include an example of how one might foster (and resolve) doubt and discomfort using Cantor's celebrated result that there is more than one size of infinity.

The cultivation of wisdom—the conscious and fruitful direction of knowledge—is a fundamental expression of our humanity. As educators we aspire to advance each student's ability and inclination to apply intellectual tools to his or her life, namely to contribute to the expansion of each student's wisdom. Two fundamental tools that can be applied regardless of discipline, course, and teacher are doubt and discomfort.

As educators our ultimate goal is to help our students develop the methods and confidence required to understand not only what is taught to them, but anything whatsoever in which they wish to engage their minds. Our task as teachers of mathematics is not to persuade our students to take mathematical truths on faith; it is to help them understand why they are true, to facilitate and extend understanding, which is not the same as mere acceptance. The ability to critically analyze ideas, claims, and arguments is crucial for informed participation in civic life.

To be effective in our lives and responsible citizens of the world we must be able to determine which of the things we take to be true we understand and which we believe on some other evidence, such as testimony. The study of mathematics is an excellent and relatively emotionally safe environment for learning the crucial skill of separating what we accept from what we understand. Understanding is the conclusion of reason, the natural consequence of seeing the premises and inferences that guide one to a position. Beliefs, on the other hand, can be based on the province of something external to ourselves or outside of the domain of reason: the knowledge of our teacher, scripture, or direct experience. The dominance of understanding over belief in mathematics is not a subjugation of the value of direct experience; indeed, mathematical inspiration often comes in the form of spontaneous insight without a logically traceable origin. Yet no matter how powerful the experience of the experiencer, the insight is not accepted as mathematical truth until it has been justified by reason—until it has been understood and, furthermore, communicated.

By focusing on issues such as student underpreparation and the material we must cover in a course, it is possible to rationalize behaviors that promote ourselves as authorities and foster in our students a dependence on us. As teachers we are all subject to the attractive and comforting qualities of being needed, of being, for example, good at explaining things. Nevertheless, part of our job is to teach students the difference between acceptance and understanding. Of course this kind of deep learning is by its nature difficult and challenging to facilitate. When we must ask them to believe something without understanding it—to accept some black box—this should be a conscious decision with valid reasons.

Unsurprisingly, many students resist self-reliance, especially intially. Coming to our own conclusions is much more disruptive to our belief systems and our emotional states than comparmentalizing knowledge and holding mathematics as separate from those things for which we can be held accountable. Understanding mathematics rather than memorizing it imposes a period of uncertainty during which we feel ill at ease because we are outside of what is usual and accepted. Part of our role as teachers is to train and embolden our students to embrace discomfort rather than consider it a signal to shut down inquiry, to recognize it is often an indicator of a conflict between something they believe (such as their mathematical philosophy) and a (possibly incompatible) mathematical claim.

It's easy to teach students that they don't know; it's hard to teach students what they don't know, especially as that changes from student to student. Indeed, one of the hardest things about learning a new subject is figuring out what it is that we don't understand. Ultimately we hope students will be able to identify and diagnose their misunderstandings and partial understandings on their own. Our role is to coach them in how to ask helpful questions of themselves, and to do this we must consider comprehension to be more valuable than confirmation, more important than the right answer.

Giving our students the freedom to doubt is among the greatest gifts we can give them. As teachers we are perfectly positioned to illustrate that doubt is not an indication of stupidity or inferiority but is instead an opportunity to transform the way we see mathematics, learning, and indeed much of our lives. If we allow ourselves to feel doubt and explore that doubt, not rushing, not trying to get the answer, but instead working to resolve the doubt, then we possess the most important tool there is for understanding in any area of our lives.

To affirm the value of doubt is to offer the treasure of our discipline. If one avoids doubt and the discomfort that accompanies it, it is not possible to experience fully the joy of intellectual work, and the joy of mathematics in particular—the onset of clarity as the resolution of doubt.

2 Setting: Where I Teach and have Taught

Currently I am an assistant professor at the City University of New York's New York City College of Technology (City Tech), a 16,000 student public college with one of the most racially and ethnically diverse student populations in the country and with high variance in student preparation. Before my position at City Tech, I held a postdoctoral fellowship at the United States Military Academy, West Point, a public liberal arts college with approximately 4,000 exceptionally well-prepared students. Prior to that I taught throughout graduate school at the University of Colorado, Boulder, a 26,000 student public university.

At each institution I have employed the methods here described in courses ranging from college algebra, precalculus, and calculus, to teacher preparation courses and various courses for majors in mathematics, applied mathematics, and engineering. In each case I have been pleased and honored by the resulting growth in my students. For more on student responses, see sections 6 and 7.

3 How Mathematics Captured Me

Cantor's argument that there are, in a precise sense, more real numbers than natural numbers (i.e., $|\mathbb{R}| > |\mathbb{N}|$), called Cantor's diagonalization argument, created in me a tension that could only be resolved by studying more mathematics. The proof was clearly correct, and yet I didn't believe the result. There couldn't be more than one size of infinity. Such a possibility defied every idea I'd had about what it meant to be infinite, a concept that to me was then the same as to be uniquely transcendent. When I made tentative inquiries to my professors into how to resolve this impossibility, I was urged not to pollute my budding mathematical mind with questions of philosophy. Disturbed and confused (why was the meaning of infinity not relevant to mathematics?), I learned to keep such thoughts to myself and eventually to suppress them altogether. Mathematics became for me a beautiful, poetic, technical exercise, separated from all other disciplines and experiences, haunting only for the way in which it reveals the vastness of the potential of the human mind.

This lofty untouchability was comforting in the beginning. To some extent it was why I continued to study mathematics. It gave me something legitimate to do while hiding from the horrible, overwhelming mess out there: environmental destruction, human suffering, vanishing species. After a time, though, I began to feel stale, divorced from everything vital. Recovering my interest in philosophy and the philosophy of mathematics reanimated my intellectual life and helped to reconnect my mathematical interests with my human ones.

The jarring discord I experienced when exposed to Cantor's set theory is not unique to me or to set theory. At times learning requires the updating, alteration, or abandonment of previously held beliefs, a process that can be uncomfortable, even painful. In the remainder of this chapter I explore one way the infinite can be used to uncover philosophy in the classroom and to constructively foster doubt and discomfort.

4 What Philosophy of Mathematics?

Even if they aren't aware of it, all people, students and working mathematicians alike, have philosophies of mathematics, namely (possibly coherent) collections of beliefs about the nature and meaning of mathematics and the practices of mathematics. Furthermore, those philosophies influence how and even if one learns mathematics. Conflicts between them and new material are often the origin of their discomfort. Coaching students to explore their philosophies of mathematics empowers them to trust their own minds. In the course of our classroom discussions, some students have told me that if the existence of an actual infinite set means that there exists another, larger[1] infinite set, then they reject the existence of the actual infinite. Some students accept the existence of each natural number but reject the existence of the natural numbers together as a set. Many students have said that while they can see how an argument would work if we could iterate infinitely, they reject the idea that we could repeat any process infinitely as we are, after all, only finite beings. Some students are also disinclined to consider non-constructive existence proofs to be meaningful mathematics. I've heard students argue that numbers and functions, their properties, and the proof of the properties are as real as the material universe of atoms. Other students have countered that what we are doing is utterly devoid of any inherent meaning, that we are merely symbol-pushing.

These examples of objections raised by students might sound familiar; they are the same objections expressed by well-known mathematicians and philosophers of mathematics. Intuitionists such as Brouwer tend to agree with the rejection of infinite constructions and the rejection of the existence of the actual infinite (denying the existence of the set of natural numbers while affirming the reality of each natural number). Some constructivists reject existence proofs[2] and in particular the law of the excluded middle.[3] There is satisfaction for students in knowing that they are in great company, whether they are inclined toward platonism or formalism (toward viewing the abstract entities of mathematics as external to humans in the same way that the laws of chemistry are, or toward considering higher mathematics as a game of manipulating strings of symbols according to set rules). It serves them well to know about those points on which they agree with Gödel or Hilbert, and to know who those people are[4].

The primary counterargument given by some instructors to students' objections to induction and infinite constructions amounts to: "what you don't understand about induction (or infinite constructions) is that it works." This is a curious response considering that often students do see that it works. Their disagreement is with the underlying assumptions that enable induction to fall within the mathematical rules of engagement. Saying "these are the rules because we said so" is a response that, while time-saving, stifles the mathematical minds we are meant to foster. There was a time before induction, before set theory, and we are even now living in the time before the next mathematical revolution. If we deny students their objections, especially to those most fundamental concepts and constructions, we cut down

[1] In the sense of cardinality.
[2] By "existence proof" I mean a proof that asserts the existence of an object without producing such an object.
[3] Given a statement ϕ, either ϕ or the negation of ϕ must be true.
[4] The people mentioned here are not the end of the story. For those interested in the contemporary philosophy of mathematics, see [2] or [4].

their creative capacity and inclination. We thus squelch their natural curiosity, the very trait which may be the most effective motivator of learning.

To provide our students the opportunity to explore these ideas is to support their natural right to see themselves as mathematicians, to see their minds as capable of deep and compelling arguments. Wouldn't it be wondrous to learn at eighteen that Cantor himself, the creator of the marvelous machinery of set theory, struggled with the irreconcilable conflict between the belief that the infinite is identified with a unique and transcendent God and the undeniable plurality of the infinite asserted by his own theory?

5 How the Infinite can be Used to Encourage Doubt and Discomfort

Most students come to our mathematics classes with the belief that there is one size of infinity, and when asked to give a definition of "infinite", they usually give one of these responses:

to go on forever;

to not be finite;

to be bigger than anything else.

Starting from these beliefs about what it means to be infinite, which are shared by most students, it is possible to have an engaging discussion about infinity. For this exploration we need some tools; in particular we need to argue that bijections are the right tool for demonstrating that two sets have the same size. The concept of a bijection is a straightforward one and does not require a foundations course. Most students already understand what a bijection is—it is the terminology and application that are new. For example, if I bring to class a plate of brownies, my students will quickly determine whether or not there is a bijection between the set of brownies on the plate and the set of students in the class. They don't need to be taught the concept of bijections; they need to see how it is linked with ideas they already have.

It doesn't take long before students agree that counting is itself a special form of bijection that facilitates comparison between groups of objects. In the case of finite collections, we use counting to compare the sizes of sets. Given that counting is a special form of bijection, it is reasonable that, in order to do something like count—namely to compare the sizes of—infinite sets, bijections are the natural tool for doing so.

At this point I have a class full of people who understand bijections between finite sets and who also agree that the concept of pairing could also apply to infinite sets. From here I like to introduce one way in which infinite sets are unlike finite sets. From a lifetime of dealing in a finite world, most people have developed a strong intuition for the behavior of subtraction. It leads them to believe that a proper subset always contains fewer elements than the original set. They believe this even without knowing what sets and elements are, and of course it is true and correct for finite sets. It is possible to contradict their well-established belief that the result of subtraction is always a smaller entity than the original one while simultaneously reinforcing their belief that there is only one size of infinity.

To start students thinking, I have them try to find good—and bad—bijections between $\mathbb{N}$ and $\mathbb{N}\backslash\{0\}$ and between $\mathbb{N}$ and $\mathbb{N}\backslash\{0, 1, 2, \ldots, m\}$. It is important to emphasize that in order for sets to have the same size, we do not require that every one-to-one function is a bijection, but rather that some bijection exists. This is an important departure from the finite case. Hilbert's Hotel can be a useful device for exploring these ideas. Hilbert's Hotel has infinitely many rooms. The doors to the rooms are labeled with the natural numbers. The vacancy sign at Hilbert's Hotel is illuminated at all times, even when every room is full. How can they get away with this? It is always possible to fit one more guest by having the existing guests move down one room. Indeed, once we see that one more guest can fit, it is almost no more work to see that any finite number of guests can be accommodated.

The new properties of set subtraction are an excellent opportunity to investigate a curious phenomenon—something was true, and now it's not true. In particular, subtraction of things with positive size always gave a result smaller than our initial value. Now we see that this is not always so for set subtraction.

Indeed, for subtraction, matters get worse. The set of natural numbers without the infinitely many odd numbers is the set of even numbers. The correspondence $n \mapsto 2n$ shows there is a bijection between $\mathbb{N}$ and $\mathbb{E} = \{0, 2, 4, 6, \ldots\}$. This demonstrates that sometimes, even if we take away as many elements as we started with ($k \mapsto 2k + 1$ gives a bijection between the naturals and the odds), we have nonetheless not changed the size of our original set.

Given that the sets we are dealing with are not finite, there is no reason our intuition, which is based on finite sets, should apply. In the absence of any substitute, of course, it is almost inevitable that we will try to rely on it. Embedded here are human tendencies that extend far beyond mathematics, and it is worth investing time exploring the resulting observations. Releasing a previously useful intuition is difficult. Yet, in defiance of their lifetime of experience with subtraction, students see that it is possible to take something away from an infinite set and still be left with an infinite set of the same size. This is a tremendous departure from the known, and as such it can be exciting, troubling, liberating, and alarming. By holding our own ideas loosely, by scrutinizing and doubting them, we improve our chances of noticing indications that what we once knew is no longer true.

Students are rightly reluctant to abandon their intuition for finite sets, as it applies to everything else in their experience. The violation of that intuition by infinite sets is an indication that there is something we do not yet understand, a way in which subtraction, at least for infinite sets, might not be as we believed. Indeed it leads to the question of whether we have found a better characterization of the infinite than those proposed at the beginning of this section. Namely, perhaps this strange property of violating one of the fundamental properties of being finite better characterizes what it means to be infinite than our first attempts at a definition. In particular, after showing that there is a bijection between the natural numbers and the natural numbers without zero; between the natural numbers and the natural numbers without the set $\{0, 1, 2, 3, 4, 5\}$, or $\mathbb{N}\setminus\{0, 1, 2, \ldots, m\}$; between the natural numbers and the even natural numbers; it seems that a better definition of infinite might be that the set A is infinite if and only if there exists a proper subset B of A such that there is a bijection between A and B.

Even using this new definition of infinite, students can find evidence in support of their prior (though not correct) belief that there is only one size of infinity. We show that the natural numbers, the integers, and the rational numbers all have the same cardinality (which we can call simply "size")[5]. The lovely proofs support their feeling that there is only one size of infinity, and now we have them right where we want them. If there is only one size of infinity then the natural numbers and the real numbers have the same cardinality. It is worth asking, of course, what it would mean for two infinite sets to have different cardinalities. The answer is that there would be, in the entire universe, no bijection between them.

Alternately exploring the claims that there is a bijection between the set of natural numbers and the set of real numbers (i.e., $|\mathbb{N}| = |\mathbb{R}|$), and that there is not, we can introduce Cantor's diagonalization argument and prove the result that there is more than one size of infinity. And here we should not rush.

Students have all manners of objections to these ideas, proofs, and conclusions, and they should! Exploring the burrs on which their acceptance gets snagged is what allows mathematics into their minds, and what gives them the chance to see that they and their classmates, even while disagreeing with each other or with us, their teachers, have ideas that other great scholars have had. There is precious little affirmation of one's potential to think creatively as a mathematician, especially early in one's training, and this is in part because we, as educators, present sanitized theorems. The story is sucked dry, and all that is left at the end is the machine. It is beautiful, even awe-inspiring, and it is austere and unapproachable. The knowledge that they often take the same missteps while approaching a result as those who created it grants students a basis on which to have faith in their own minds, to trust their experience of doubt, and accept where it leads them.

Engaging critically with the philosophy of mathematics, their own and that of others, enables deep learning and, perhaps more importantly, facilitates their development as mathematical thinkers who know themselves to be capable of engaging with profound mathematical ideas and having such ideas themselves as fresh and personal ones.

6 Has This Approach Helped Students to Learn?

It has been my experience that encouraging philosophical explorations is especially helpful for students who enjoy contemplation, regardless of whether they consider themselves mathematically inclined. There are certainly students who prefer not to integrate what they learn in their mathematics classrooms with the world outside. However, I don't find their ability to learn mathematics hampered by philosophical considerations, whereas for many students the space and encouragement to explore their philosophies of mathematics vitalizes, energizes, and directs their studies. In short,

[5] For a nice, elementary exposition of this topic, up to and including a version of Cantor's diagonalization proof and other results on cardinality mentioned in this section, see [1].

students are more engaged, put forth more effort, and ultimately learn more than they did in the segment of the course before we began discussing the philosophy of mathematics.

7 Student Responses

There are always a few students, often among those who are struggling with the course material, who are frustrated by a discussion of content not in the syllabus (in the cases when cardinality is not). There are generally at least as many struggling students who are relived to discover that there is more to mathematics than that with which they are struggling.

Aside from the few complaints from frantic students concerned about diversion of class time from the syllabus, almost all the student responses I've received have been positive. Here are some of my students' comments from a range of courses in which I presented Cantor's proof.

A student in a precalculus class wrote me a letter at the end of the semester saying she learned more about life in our class than in the rest of her university education combined. She is now a high school mathematics teacher at an urban high school. One student who saw Cantor's proof in a calculus class went on to do an informal independent study with me in set theory, then take set theory and logic courses, ultimately earning a Ph.D. in set theory. He now has an excellent faculty position. Another student, already a mathematics major, who was in an introduction to proofs course wrote me a letter at the end of the semester saying that she had learned more about what it means to persist in the face of doubt and fear from our mathematics class than she had in her military training.

8 Difficulties and Modifications

One perpetual difficulty is finding time to teach material that is not in the syllabus of most courses. To help create available classroom time, it is often possible to construct scaffolded out-of-class assignments for one or two of the more approachable topics in a syllabus. In addition to creating the opportunity to talk in the classroom about Cantor's proof and about philosophy, it also reinfoces to students the importance of learning independently.

I have sometimes had to give a significantly condensed version of the presentation described in section 5. The most time I have ever had for the presentation of the material was six one-hour lectures in a teacher preparation course. The least was fifteen minutes in a linear algebra course (in which the students were already familiar with negation and proof by contradiction). I skipped the presentation entirely in a remedial (pre-college credit) course, as the students' algebra skills were not yet developed enough to follow the proof. We did, however, play the diagonalization game [6] and talk in more general and admittedly less fruitful terms about the fact that there is more than one size of infinity.

Having a class full of argumentative philosophers might initially seem like a difficulty. With a little bit of playfulness and a willingness to say "I don't know—look into it and let us know what you find out," it is often delightful. Reading a bit of the philosophy of mathematics is good preparation. One need not become an expert, but having some background can provide ideas for where to direct interested students and help arbitrate and encourage classroom discussions (not conclusions). Bonnie Gold's article [3] (reprinted in this volume) is a good introduction. To become aquainted with both the classical philosophy of mathematics (such as most of the ideas mentioned in section 4) and the contemporary philosophy of mathematics, I suggest the excellent books of Shapiro [4] and Giaquinto [2]. The Stanford Encyclopedia of Philosophy [5], available online, is a free and reliable source for articles containing details and subtleties on many philosophical matters, including mathematical ones.

9 Final Thoughts

Several years ago a mathematics major told me that he chose mathematics because it didn't challenge him as a person. He just learned the rules and the facts, returned them when asked, and collected praise. I found his attitude chilling.

[6] In this game, "Dodgeball", Player 1 fills a six-by-six matrix with Xs and Os, one row per turn. Player 2 builds a single row, one entry per turn. Player 2's objective is to ensure that the single row she creates (analogous to the real number constructed in Cantor's diagonal argument that is not in the image of the bijection) does not match any of the six rows made by the first player. Player 2 places one mark in the nth column after Player 1 creates his nth row. The winning strategy for Player 2 is to change the diagonal entries of Player 1's rows. More can be found on this game in [1, pp. 8–9].

And infuriating. What were we really teaching him? To be a liability in society, to accept the authority of others as a substitute for personal responsibility, and to ignore his natural ability to ask and answer hard questions.

Some college educators view discussing that which forms our students' identities to be outside the scope of their professional purview. Many feel that it is not or should not be relevant to mathematics classrooms. Such discourse belongs on some other part of campus, they say. That is not my view.

How our students see themselves is central to the question of how to teach them. For many students, their ideas are their intellectual identity, and to discover that an idea or belief is incorrect is menacing to their concept of self. Surrendering our attachment to our ideas empowers us to subject our ideas and beliefs to deeper, more rigorous examination, thereby increasing the coherence of our collections of ideas and beliefs (and perhaps even their correctness). Adjusting our beliefs is not a threat to our deeper selves.

We can encourage students' development by liberating them from the conditioning that they have been subjected to, that when they are confused, when they have doubt, it means they are lacking in talent or ability. To be good doesn't mean to not struggle. The discomfort of not understanding is not a sign of deficiency. Discomfort is a signal to send an invitation to curiosity, which protects us when we venture far from what is familiar and safe.

The road to knowledge, wisdom, and correct perception is not defined by exam grades. It is a path of doubt, of fortitude, and of the inestimable magnificence of the human mind courageously applied. There is joy without measure in the mind-shift from incapable to capable, when the veil lifts, and the object of study becomes clear. It is this joy I hope to share with my students by helping them see the value in their doubts and discomfort.

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From the Classroom: Towards A New Philosophy of Mathematics

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1 Introduction

In this chapter we would like to emphasize the importance of the mutual interplay between the philosophy of mathematics and the teaching of mathematics. We will not present a lesson plan that includes philosophy in the regular mathematics curriculum. Rather, we propose a philosophical standpoint, suggested by observations coming from routine teaching of mathematics, which will permit teachers to acquire an open-minded vision of mathematics and to discuss on solid ground the meaning and the worth of their pedagogical actions.

We will start from questions that teachers of mathematics must consider, suggesting that a key question lies behind them, "What is mathematics?" We will try to present a viable philosophical position about mathematics (different from the traditional ones) stemming from our view of the teaching and learning experience. To make the presentation more clear and concrete we will analyze specific topics such as time, element and set, and natural number. We will conclude by applying that position to the teaching of a specific topic, thus showing the unexpected depth and richness, often hidden in familiar topics, that is brought to the surface by the proposed considerations.

2 What is Mathematics?

A serious teacher concerned with teaching mathematics meaningfully will consider two questions:

"Why am I teaching this topic?" and

"Why do I teach it in the manner that I do?"

Any answer could raise further questions of why it is correct and convincing. But deeper and deeper questions along this line all lead back, eventually, to the basic question "What is mathematics?" The pedagogical viewpoint helps pinpoint inadequacies in the answers the standard philosophies of mathematics give to the questions.

Logicism attempted to reduce mathematics to logic and set theory in order to ground its acceptability and meaning. But further results in logic demonstrated the limits of languages to describe mathematical notions involving infinitely many elements. For example, such theories are not categorical (they always have non-isomorphic models), key notions (such as finiteness: there is no sentence that will be interpreted as true if and only if a structure is finite) are not definable, and basic constructions are not predicative (often definitions of sets invoke the set being defined). Further, it is not easy to base all mathematical concepts on the notion of set, even though it may seem intuitively easy and immediate. First, it assumes the notion of element, which again appears intuitively immediate, but requires some effort to describe properly, unless we consider it an inborn notion. It also requires the notion of membership and of infinite set. Furthermore, many mathematical concepts (order, for instance) are not present in set theory, and can be represented in it in several ways, one of which may then be chosen by convention to treat that concept within set theory.

The formalist approach wanted to free mathematics of any reference to meanings: mathematical entities were to be purely abstract elements obeying the syntactical rules of a meaningless game. The unique constraint for such a game would be its consistency. Gödel's second incompleteness theorem, however, proved that there cannot be any proof of its consistency within the game itself, the essential aim of the approach. The theory thus obtained could be a nice and elegant game, but without any reason why it should be played, why impose it on everybody through the curricula of the school systems?

In the twentieth century, both logicism and formalism converged on an axiomatic approach to mathematics: mathematics is that which can be deduced from sets of axioms in a formal language. Axioms are sentences of the language that are assumed to form a consistent set, as otherwise the resulting theory would be trivial. From this viewpoint, talking of their truth is nonsense. Actually, if formulas are not to be interpreted, in order to avoid linking them to specific meanings, then they are meaningless sequences of mere scrawls. Often we think of truth as a property of sentences, but in fact truth is a binary relation between sentences and the structures in which they are interpreted.

Even though sentences without an interpretation are just scrawls, they are often thought to be adequate to determine almost uniquely (i.e., up to isomorphism) a structure, namely the structure in which they can be interpreted as true. Unfortunately, unless the structure is a completely described finite one with a well-determined fixed number of elements, there are non-isomorphic structures in which all the given sentences are true. Thus no set of sentences, no matter how rich it is, can determine (even up to isomorphism) a single structure with infinitely many elements. As a consequence, we should be aware that even the very familiar notion of natural numbers cannot ever be unequivocally characterized and thus it cannot be unequivocally[1] presented through the language.

Brilliant scholars may immediately grasp some single structure falling under an axiomatically presented notion, and in this way they give meaning to what they are doing, but many others may start working with the axiomatically presented notion without ever considering any instance of it, just by utilizing its stated properties. In so doing they have very little idea of the meaning and relevance of the notion considered. This underlines a danger of the axiomatic approach, the shortcomings of considering axiomatically presented notions per se rather than viewing them as abstractions from already obtained mathematical entities.

An empirical approach to mathematics that denies the existence of inborn ideas and that accepts only notions that refer to concrete objects has to face some other immediate difficulties. We have never met a number or a geometric figure or a function on the street. So we would need to show how to construct a path from immediate sense perceptions to mathematical entities.

This kind of constructed path is not found in classical empiricism. We don't have to stick to classical empiricism, and we may consider a more modern form of it going under the name of constructivism. The main claim here can be summarized as follows: knowing subjects construct their own knowledge, rather than passively receiving it from others. But one would like to understand how the construction is done. Social constructivism suggests that the culture in which the subject lives determines the accepted knowledge: the reliability of mathematics is based on linguistic rules that are socially accepted. Thus ultimately $2 + 2 = 4$ is correct because most people accept that it is.

But mathematical statements should be accepted because they enable us to adequately grasp and represent situations, not due to the opinion of a large majority. If we are not to eliminate mathematics from the general education curriculum as being meaningless, we should search for points of view about mathematics from which each person can find some meaning and interest in the subject. Thus we need a philosophical stance based on a theory of knowledge that assigns meaning to mathematical expressions, even when the meaning cannot be defined through the language. The only way out is to study what the experience of knowing tells us about knowledge.

[1] As a consequence of the compactness theorem in logic, a theory with an infinite model has non-isomorphic models. For more details on this and other remarks here concerning logic, see [1].

Let us look at the above review of the philosophical positions from a teacher's point of view. If the primitive notions of mathematics are innate (or synthetic *a priori*), the role of the teacher with respect to such fundamental notions is limited to ascertaining whether the student was born with them, and hence whether the student is mathematically talented, since there is no possibility of altering or improving the situation. On the other hand, if all notions are constructed from experience, the teacher may help the student to construct a more convenient notion for the task at hand. To accept the latter case, and undertake the pedagogical task of leading the students to construct their own notions, one has to show how the notions could be constructed. In what follows one of our goals will be precisely to show that the central notions of mathematics can be constructed.

3 Proposals from the Teaching of Mathematics

From a pedagogical point of view an essential component of a theory of knowledge is how a concept is acquired. In particular the manner of acquiring mathematical notions, especially primitive ones, is essential to understanding them. Thus we should investigate what problem a mathematical concept is supposed to address, what direct personal experiences lead to the elaboration of the concept, and through which personal mental operations the construction of the concept can be developed. The knowledge of these steps should permit the instructor to communicate how to construct an intended concept beyond both definitions that are impossible and characterizations that cannot distinguish the intended entity from others. Clearly the construction will be internal to the individual since the steps suggested to reach it, described below, are. Nevertheless, given the basic similarities supposed among humans, the concepts, and their consequences, arrived at by different individuals should be very similar, despite perhaps being described differently, and an interactive discussion can ascertain the degree of similarity. This should permit anyone considering a concept to recognize that certain basic characteristics (i.e., a set of axioms) intended to describe it are pertinent to it. The same path of acquisition will give meaning and hence interest to what has been constructed. Sharing a set of axioms permits cooperation in achieving the knowledge of further aspects[2] of the concept. Furthermore, if the axioms are presented in an adequate formal language, a calculus can be developed that permits a check of the elaborations in the most effective possible manner.

4 A Sketch of a Theory of Knowledge

So let us investigate the path from sensory perceptions to mathematical concepts. In doing so, we will not hesitate to use introspection. The word "introspection" should not be read as a deep analysis of the most hidden aspects of my[3] most obscure personality characteristics, but simply as a direct observation of perceptions from my internal senses happening within me.

First of all, we shall list the internal individual features of a human being that will operate along the path. By calling them human features, I am claiming to see them in myself and I am supposing that they are also in other humans because of the similarities among us, and because my daily dealings with others don't falsify the hypothesis.

The direct perceptions from the classical five senses—not the causes of a perception, but just the perception *per se*—are obviously individual features. But besides these five senses there are others that produce direct perceptions of exactly the same type as those prompted by the five classical ones. For instance I do not use the usual five senses to

[2] Here, and in the following occurrences of the word "aspect" we are stretching its usual meaning. Usually "an aspect" refers to one of the ways something may appear to an observer. This interpretation will certainly be one part of what we mean. But a concept doesn't include only ways of appearing to an observer. It also has its constituents, components and parts; there are properties that it satisfies, and through the word "aspect" we want to refer to all of the items as perceived by an observer. For instance, consider the concept of community: the community is composed of the several individuals that belong to it, it has a story that brought it to its present state, it has leaders, but also it may contain groups of individuals with particular links among themselves. It also has properties (large, small, rich, active, secluded, organized, and so on), relations both within itself and with the outside, and standards of evaluation of what it is happening. Furthermore the concept of community should account for why a group of persons is indicated as a community, on which basis and according to which criteria this is done. Our use of the word "aspect" includes each one of these types.

[3] Here and in the following, the switch in the presentation from the plural (indicating a claim by the authors) to the singular is intended to indicate that what is asserted in the singular form can be experienced and seen only by the single individual asserting it, even though both authors share the claim. The generalization of the claim to other individuals requires assuming that the others can experience the same things that the single individual is aware of doing.

perceive my body's position and my movements: I just have a direct perception of it through internal senses different from the five usual ones.[4] But there are other internal senses that are non-physical. One informs me whether an action of mine is done according to my will to do it or not. For instance it is I who is now willingly hitting the keys on the keyboard. By contrast, my feeling of being cold while sitting here does not depend on my will. We would like to point out a further internal non-physical sense: the one that makes me perceive that I am myself. Let it be clear that here we do not mean the more or less vast knowledge of myself, but just the perception of being myself. I was a child, I grew up, now I am very different from what I was: perhaps not one of the atoms that were forming my body then is still in my body, but I perceive that I am still myself. I fell asleep without being aware of myself. I even fell into a coma for a couple of days, but when I woke up I was still the same person. Even after a long time during which I may also have forgotten several things, I am still myself. Thus the perception of being myself is something beyond just having a memory. This is as genuine a perception as any from my senses.

Only individuals can be aware of their own perceptions, in particular the internal ones, that are not prompted by something external. Introspection is thus needed and will play a central role in our discussion of how a concept is acquired.

In addition to senses, humans are endowed with a memory that records both the perceptions (impulses) from the senses and also the impulses coming from the mental elaborations of which humans are capable. Furthermore, what is recorded in the memory can be recovered through impulses (perceptions) that encode the entire state of the memory as well as parts of it.[5] Thus we posit an internal sense, the perceptions of which encode the content of the memory, and in decrypting the code one has the entire content of the memory available. Usually we do not compare perceptions directly, but via their recording. Also the recollected states of memory give rise to a perception, on which mental operations can be applied. In turn this very perception from memory, and those produced by the mental operation, can also be stored in the memory. We call all that is encoded in a single perception coming from the memory *a state of memory*.

We also have to consider mental activities that people perform, of which we are aware by introspection. In order to deal with mental activities, we hypothesize a few simple and powerful mental operations, intending that they will be sufficient to describe a possible path of acquisition of mathematical concepts. The mental operations should conveniently organize and give account of the different mental activities in a manageable way.

A rather simple and common mental operation that humans perform is *abstraction*. By this we mean that among all the perceptions that we have at our disposal we decide to consider only some of them, the ones that we deem most relevant to the task at hand. The mental operation of abstraction is very important because we otherwise could not cope with the complication (an enormous disordered set of pieces of information)[6] of the perceptions that we perceive. Through abstraction we consider just a few selected perceptions, simplifying the situation to be handled to the point that we have a chance of managing it. However, by abstracting we depart from a complete vision of reality (here by "reality" we mean what doesn't depend on the individual's will), hoping that this will not cause problems since the aspects omitted are assumed to be irrelevant to managing the situation.

Another easily recognizable mental operation is *generalization*. By this we mean that, if something happened in every situation of a certain type experienced to date, we assume that it is going to happen also in any situation that could be recognized of that type, even if we don't experience it.

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[4] Our learning from experience to coordinate the perceptions from the posture sense with those from the classical five doesn't mean that this sense can be reduced to the others.

[5] Clearly this is a simplified notion of recollecting from memory, in which we are not considering all the problems coming from the possibility of forgetting. It may very well be that there are impulses that do not even reach the memory, and memorized impulses that are not recollected. Nevertheless, we deem that, for a theoretical approach to this phenomenon, it is convenient to assume, at the beginning, that every impulse from the senses is recorded and can be recollected. The idea for these assumptions may be grasped from an analogy with elementary physics: nothing moves at a constant velocity forever; nevertheless we accept the principle of inertia to study motion. Forgetting, and how it operates, can be considered at a second stage building on the present proposal, without altering substantially what we are going to discuss here. Thus we feel justified in accepting the stated assumptions.

[6] We distinguish between complication and complexity. "Complication" indicates a situation with far too many disorganized pieces of information, making it impossible to have an idea of what is going on, to understand how things are correlated and could evolve. On the other hand, "complexity" indicates a state of affairs difficult to understand and follow, due to a rich organization of a large amount of information, possibly with several connections of different types and at different levels.

4.1 Abstraction and Generalization: the Acquisition of the Concept of Time

Already at this early stage of our proposed theory of knowledge concerning how concepts are acquired, we can recognize the path of acquisition of the concept of time, which is relevant in part because it is needed for developing many other concepts, including counting. It is natural to call some perceptions "past" if they are recorded in our memory. Also, we can say that a perception A is further in the past (we will call it "trans-past") than another perception B if a state of memory in which B is recorded also has a recording of a recalling perception (recollection) that encodes that the first perception A is recorded. No state of memory includes the recording of the recollection of itself, and if a state of memory includes the recording of the recollection of another, the second does not include the recording of the recollection of the first. Furthermore if a state of memory includes the recording of the recollection R_1 of a second state of memory and this includes a recording of the recollection R_2 of a third one, this third is already recorded in R_1, in the form of an encoding of an encoding (because R_2 is present in the state that is encoded by the recollection R_1). Hence the third state of memory is also recorded in the recollection R_1 within the first state of memory, and the relation introduced is transitive. (See footnote 5.) Generalizing on the three types of contingent facts, we can safely say that a state of memory C follows another, D, if the recollection of C includes a recording of a recollection of D. Thus the states of memory are presented in a totally ordered[7] fashion that we can call the "past time." Notice, further, that the past state of memory is not mechanically determined by a trans-past state of memory, as there is no complete information in a trans-past state of memory T to determine the past states of memory that follow it (i.e., for which T is trans-past). Moreover, we may distinguish among states of memory using the recollections that are present in them, and thus they are all different. We say that two perceptions are simultaneous if neither is trans-past with respect to the other.

It is part of our experience that every experienced past perception becomes trans-past; i.e., every experienced past perception is trans-past with respect to another past experience. Therefore any experienced (i.e., present in the memory) state of memory is trans-past with respect to (i.e., is followed by) other experienced states of memory. Generalizing, we claim that every state of memory (whether experienced or not) will be followed by other states of memory. Before this generalization, with the past state of memory we had built an ordered notion of the past referring to the trans-past. With the new generalization, we are introducing the future: we don't know what will be perceived after the experienced states of memory, but there will be further states of memory. Now, with past and future, we have a notion of time.

At this point a couple of remarks are in order. First, we didn't define time, nor did we characterize it. Rather we pointed out some steps that we used to acquire that notion. If other humans operate as I do, following the indications on how to acquire the notion of time, most probably they will arrive at a notion very similar to mine. Second, when we generalize, we depart from experience: we assume that something will happen without any guarantee that it will, and we accept that risk.

4.2 Idealization by Transfer: the Acquisition of the Notions of Element and Set

In trying to handle the multiplicities of various kinds that we see around us, we need to build concepts that require further mental operations for their construction: for instance, the one that we are going to call *idealization*.

To explain this mental operation and to see why it is needed, let us consider an example. We often say "consider this element." Instead of "element" we could use the words "thing" or "object" but here we use them synonymously. What do the words "this specific element" indicate? What is this specific element? How do we acquire the notion of a specific element? I receive many impulses (perceptions) from my senses: looking at a certain area I see some colors from different points; I might feel a temperature, a consistency, a weight, and have many other perceptions. But from them, how do I arrive at the claim that I am considering a specific element? What appears to be missing is the assertion that all these perceptions should always be held together. But why? My senses do not inform me that the perceptions that I am considering should be kept together. On the other hand, perceptions concerning *me* are kept together just because I have the further perception of "being myself." This last perception gives unity to the others because they are all referred to me.

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[7] Here we are not introducing a notion of order, but simply a word that summarizes the properties that were noticed about the recollections present in a state of memory.

To assert that certain experienced perceptions are to be kept together, we add to them the perception of "being myself," even though to be myself can never be experienced together with perceptions not referring to me. This perception of "being myself" inserted in a different context will be called "being itself." This is exactly what the mental operation of idealization by transfer permits us to do. The added perception is unrelated to the others being considered. Nevertheless we decide to add it to obtain, as a result, a new set of perceptions that should be kept together, since it contains the added perception introduced exactly for that purpose. We have now completed the path of acquisition of the concept "being exactly the specific element" that we want to consider.

The described path of acquisition of a concept shows the power, the convenience, and the relevance of the mental operation of idealization by transfer: I can add to the other perceived perceptions one that is not perceived together with them, but is perceived in a totally different setting (the perception that I have of being myself in the earlier example).

Now, to obtain the notion of "an element," without specifying *which* one, we should abstract, from each distinct notion of specific element, the aspects of the specific element that are peculiar to each one, leaving the only aspect of being itself and different from other elements. Being itself is just the perception of being myself, except that, each time, the subject asserting this is the single element. The perception left in this concept is a perception that I cannot have, since it is internal but not to me. Nevertheless I can accept the entire path of acquisition of the concept, thanks to the mental operation of idealization. Other people could perform their own mental steps in an analogous manner and reach their notion of an element. Thus the notion of element is not an immediate one, but is rather worked out very carefully. This notion is acquired, however, at a very young age, and used extensively without wondering how it was obtained.

To develop mathematics, the idealization operation cannot be ignored because it is indispensable in developing the concept of element. This is indeed at the root of many mathematical notions, such as the notion of collection (obtained by abstracting which elements to consider in a specific collection)—because a collection has, as its members, just elements—and also the notion of set, which is a collection that is itself an element.

4.3 Idealization by Negation: the Acquisition of the Concept of Natural Number

Hopefully the application described above made the operation of idealization by transfer clearer. Indeed, the process of idealization in general is the one that allows adding to a set of aspects a further one that is either experienced in a different context (transfer) or the negation of something experienced. In the previous example we added a perception that was perceived and well known in a different setting. In the next example we would like to add the negation of an aspect. Here by "negation" we do not mean the difference, i.e., everything in a certain environment not having that aspect. (In this case an adequate mental operation of abstraction would suffice.) Rather we refer to the "total negation," which considers absolutely everything not sharing the aspect, and, since we cannot know what absolutely everything consists of, we cannot have a precise idea of the negation of an aspect.

The operations of idealization by transfer, and, even more, idealization by negation, can lead to concepts rather different from the intended ones, departing from reality and hence possibly causing contradictions. Nevertheless we choose to use it because, on the one hand, reality is so complicated that we cannot know it completely no matter what we do, and, on the other hand, through the operation of idealization we want to construct a simplified model of reality that should be manageable thanks precisely to the features idealization introduces.

To illustrate how the operation of idealization by negation can profitably use features that go beyond our control, we will analyze how to acquire the notion of the system of natural numbers.

The natural numbers are an instrument constructed to solve the problem of comparing, distinguishing, and appreciating various quantities of elements. We say that two sets have the same quantity of elements if there is a bijection between the two, and that one has a smaller or equal quantity of elements than another if there is a bijection of the first onto a subset of the second. The mathematical terminology used here doesn't need an already acquired notion of natural number, just a few initial steps of set theory (such as the notions of "set of ordered pairs," "function," "bijection," etc.) that immediately follow from the concepts already introduced.

Finding such bijections is not always easy because in most cases the elements of two sets cannot be physically arranged in pairs. The strategy to solve this problem is called counting; that is to say, starting from having considered nothing, iterate the step of "considering a further element" (different from the ones already considered) of the set, remembering each iteration of this process, eventually with the aid of symbols, until all the elements of the set are considered. The recourse to symbols is due only to the limitations of human memory.

With this strategy the bijection is between the set that is being counted and the set of iterations of the step of considering a further element of that set, starting from not having considered any and continuing until all the elements of the set are considered. Clearly counting is based on the ordering of the elements of a set by considering them one at a time. (Notice that the notion of counting requires the notions of time and element.) The bijection between a given set to be counted and the set of iterations of considering a further element, obtained by counting the set, permits us to use the second set instead of the first one in order to compare its quantity of elements to that of other sets. The sets for which we can be aware that the counting process ends are called finite; we are only interested here in these sets, even though it is possible to consider also sets for which the counting process doesn't come to an end.

We can call natural numbers these series of iterations of the step of considering a further element. The natural numbers become a standard by which we grasp the size and relevance of quantities of elements of finite sets. Instead of comparing finite sets directly via bijections that could be difficult to construct, they can be compared by referring to the natural numbers having the same quantity of elements as the sets under consideration and comparing these natural numbers.

We didn't define natural numbers, nor did we characterize them using some of their properties, but once again we indicated a path to acquire the concept of natural number via concepts already acquired (element, set, bijection) and the activity of counting (that is, of considering a further element, iterating an action).

If we add one element to a set to be counted, the number associated with the new set can be obtained from the number associated with the previous set by adding to it an additional iteration of considering a further element. We call the *immediate successor of a natural number* the natural number obtained from the original by adding one iteration of considering a further element. We call zero the natural number counting the empty set. It can be seen that every natural number is obtained by starting from zero and iterating the application of the passage to the immediate successor.

We could ask how many of these entities are necessary. Maybe the class of them could be finite? In practice the answer could conceivably be yes. Not only must every single count end in order to get to know the quantity of elements of a set with this method, but also, after counting for a while, we would use up all the resources for counting: once we've used all the combinations of elementary particles of the universe we would no longer be able to keep counting, having used up all the resources of our memory and their aids. This argument could be used to assert that from a practical point of view the existence of a biggest entity is possible. Also the choice of considering a biggest entity would agree with experience. In fact, human beings have only experienced processes that end; consequently we don't have a precise notion of what a non-ending process might be. This example shows how, even if a notion is well known, its negation is not, unless in the setting of a known domain. This is because we can't know every one of the possibilities we are excluding. For all these reasons, it might be reasonable to think that the construction of these entities would admit a biggest one.

However, if we were to do that, before being able to represent with a natural number the passage to a further element, we would have to check first if the biggest number has been reached, in which case the operation would be impossible. It could be argued that such checking is unnecesary, because in practice the biggest number can never be reached because it would be chosen so big as to avoid any risk of reaching it. In this way we would avoid the need to perform a check in every practical situation, even though, in theory, we would still need to do so. Indeed, the decision not to check in any theoretical situation is in direct contrast with the choice of the existence of the biggest number, because, by choosing not to do such checking, we can consider the successor of any number, denying the existence of a biggest one.

On the other hand, is it possible to deny the existence of a biggest number? The position that any natural number has an immediate successor, and that hence there is no biggest natural number, leads to the endless construction of these entities, without any idea about how long we should continue, but certainly it would not be a finite number of steps. We call "infinite" a not-finite number of steps. So, in building the concept of the natural number system, we add via idealization by negation to the previously considered features of this concept, the aspect that the system is not finite. In fact, historically, this was the preferred choice in the construction of the entities that represent counting, even though it raises new difficulties, because the elimination of the need to check whether the maximum number has been reached leads to substantial simplification.

There is a possible vagueness introduced by the mental operation of idealization by negation in this case. We understand that there should not be a maximal number because we can keep passing to the successor, but how far should we go, how long should we proceed, how big is infinity? These questions are not at all trivial, since we can envision substantially different unending processes, posing the problem of choosing the one we want to consider as obtained through this idealization.

To respond to this, we will use the notion of an inductive set. By *inductive set* we mean a set of elements with one special element and an injective function whose domain is the set and whose range is the same set minus the special element. The set can be viewed as the set of elements introduced to count, where the special element is the starting point for counting (zero), and the injective function is the function that maps an element to its immediate successor: thus the system of natural numbers is an inductive set. But from an inductive set (say a set A with special element a and function f) it is easy to construct new inductive sets.

For instance, let b be an element not belonging to A, B a set obtained from A by adding the element b to it ($B = A \cup \{b\}$), and f' the function obtained by extending the function f to the set B by sending b to itself ($f' = f \cup \{(b, b)\}$). B with the special element a and the function f' is also an inductive set. This example may be not very useful since in general we don't want elements to be images of themselves. But it is not difficult to build other non-isomorphic examples without this drawback by adding infinitely many elements.

Thus there are several ways of lacking a maximal element, of being without an end, of being infinite. The aspect of being infinite, introduced by idealization, needs to be made more precise in order to capture the proposed concept, the natural number system. It is reasonable to decide that we will be happy with the smallest (that is, included in any other, up to isomorphism) system that allows us to count. Hence we will specify that the idealization to be performed requires that the system that we want to obtain is infinite, and is contained in any inductive set. Finally, the *natural number system* is the concept thus acquired.

5 The Role of Formal Languages

Even though there are many other mathematical concepts whose manner of acquisition we could indicate, continuing in this direction is not the goal of this chapter; doing this is left to the reader or to future papers. Instead we would like to link what has been discussed to the daily practice of teaching mathematics. Before we can do this, however, we need to consider briefly the role of formal and semi-formal languages.

So far we have insisted on the importance of meanings in mathematics. They are central to the pedagogy of mathematics that prompted our sketch of a theory of knowledge, in which we criticized the formal approach as unable to specify the meanings of what is considered. On the other hand, there are plenty of formal expressions in mathematics: should we do without them completely? No! Even in our framework it is an essential part of mathematics once we specify its role.

In order to cooperate as efficiently as possible, we need means of communication. Exhibiting or indicating something is an adequate one as long as we are concerned with concrete objects or behaviors. But in mathematics we would also like to communicate meanings obtained via mental constructions of the type considered above. To achieve this, humans developed more and more articulated languages to be linked to the meanings to be communicated. The Greek philosophers thought they already had a language so perfectly adapted to express everything that, to analyze humans' mental activity, it would suffice to analyze the linguistic expressions representing it. Even though that was not the case, already they could check the correctness of some arguments by examining their linguistic manifestations: that's what their syllogisms were doing.

Roughly a thousand years later, another big step in this direction was taken by the Indo-Arabic culture with the introduction of finite sequences of digits to represent natural numbers in performing arithmetic operations. Indeed, if you ask most people to describe addition of natural numbers, they will rarely respond with the meaning of this operation (determining the quantity of elements of the union of two disjoint sets that have as many elements as is specified by the numbers to be added) but rather with the description of the algorithm used to determine the total.

Let us examine the algorithm. It operates on the finite sequences of symbols, called digits, representing the numbers, without having to understand what a natural number is. Using the table of addition of digits, it produces a sequence of digits, putting in each place of the sequence the digit determined by the application of the table to the digits of the given numbers in the same column, or possibly the successor digit (where the successor digit of 9 is 0 plus overflow) in case of an overflow from the previous step. Thus the algorithm not only does not consider natural numbers (which are the meaning of sequences of digits), but also ignores what an addition is. Similarly, we have algorithms for each arithmetical operation, and considering them all we have the arithmetical calculus (more commonly simply called arithmetic)! We should be surprised that by playing with sequences of digits in a funny way we get the name of the correct answer (but we were told that it is correct, and almost none of us ever questioned why). Notice the importance of an adequate language and notation: there is no similar algorithm for multiplication using the Roman notation of natural numbers.

Some centuries later, independently Newton and Leibniz built adequate notations (languages) to represent functions and their derivatives and rules to transform expressions, to pass from the notation of a function to the notation of its derivative. Whereas, from the point of view of the meaning, to obtain a derivative requires reaching the end of an infinite procedure, the algorithms based on the rules permit one easily to reach the correct name of the desired derivative. What a success: the differential calculus!

We have used the word calculus in two very different contexts, but there are many others in which the word calculus is sometimes used: algebraic calculus, vector calculus, predicate calculus, etc. Why are these called calculi? What do they have in common?

In any calculus, relative to a certain environment, with its elements, relations and operations on which we plan to work, we have an adequate language (not a phonetic one but rather an ideographic one) to represent what is in the environment, and algorithms, corresponding to the operations of the environment, that transform given expressions of the language into a final expression. This expression should be the name of the result that would have been obtained applying the corresponding operation to the elements named by the initial expressions. Thus, instead of operating on the meaning to obtain a result from some initial data, we can use the algorithm on the linguistic representations of the initial data and obtain the name of the same result. Any such algorithm involves only linguistic expressions (we say that it is *purely syntactic*) without requiring any reference to the corresponding meanings (that is, to the *semantic*).

We say that a calculus is correct if it indeed leads to the correct name of the result of the corresponding operation on the meanings.

Thus if we succeed in the demanding task of building a calculus for a subject (introducing an adequate language, rules, and algorithms, over expressions of the language, which use the rules) and prove that it is correct (even harder), we have at our disposal two ways to tackle problems: one operating on the meanings, and a second one applying the corresponding algorithms to the notations representing the meanings. Wouldn't one way be enough, without having to go through the whole trouble of building a calculus and proving its correctness?

No! Inventing and developing calculi is among the great achievements of mathematics. The formal (syntactic) procedures of a calculus can be easily performed (even by adequately built machines) and checked, whereas operating with the meanings is usually rather difficult, involving consideration of unmanageable quantities of aspects, even infinitely many. Just think of adding two large numbers according to the definition instead of via the algorithm, or computing a non-trivial derivative according to the definition rather than obtaining it using the rules of the differential calculus. On the other hand, to know which steps of a calculus to perform, the knowledge of the corresponding meaning and its relationship to the solution of a problem is needed.

Calculi are extremely important nowadays: they allow a stupid machine that has no idea of what it is doing to appear intelligent, getting the correct names of the sought-after results if it is operated by someone understanding the task. This assumes that person is able to identify the algorithms in a calculus corresponding to the task and to read the meaning of the output, thanks to a knowledge of the subject matter.

A calculus is useful only if it is correct; otherwise it is pure nonsense. Furthermore, a clear understanding of the meanings involved is needed to design a calculus and to obtain a proof of its correctness. To devise a correctness proof, one is forced to understand the subject, and the correspondence between the meanings and the tools of the calculus, deeply. Programming a computer is devising an algorithm to perform a task. A common experience of programmers is producing an algorithm that obeys all the formal conditions to be executed by a machine but that in fact performs a different task than the one intended: such an algorithm is not correct.

Thus, to build and use a calculus one first has to know the meanings of what it is being considered. Then one must devise an adequate language to represent them, as well as rules of transformation on the representations that are faithful to the meanings. Therefore one has to set up a formal (syntactic) tool (the calculus), correct in the sense of yielding the representation of the sought-for results, useful to manage the data and to check the steps performed. As already pointed out, the formal language is unable to tell what it is talking about, but, once the subject is known, if the calculus is correct, it is a wonderful tool to automate processing of information.

Along this line, using the logical calculus, given some mathematical notions, acquired as we pointed out previously, it is convenient to state formally some of their basic properties, select some of them as axioms for the notions, and formally deduce from them other properties using correct deduction rules operating on the linguistic expressions. This way of proceeding is simple to perform and easy to check, once it is guided by the understanding of the meaning and of what it is being sought.

To exemplify the last point, consider the specification, made describing the process of acquisition of the natural number system, that its infinitude should be the least. It would be useful to know how to argue this precisely using the minimality of the infinitude of the system. Thus one attempts to formally state an axiom that captures and is justified by this minimality. That same axiom can profitably be used to deduce further features of the natural number system. Such an axiom was devised by Peano, the *axiom* (or principle) *of induction*. It states that for any property P, if the property holds of 0 ($P(0)$, the hypothesis called the *base* of the induction) and if for any natural number n, if the property holds for n then it also holds for the successor of n ($\forall n(P(n) \to P(\text{succ}(n)))$, which is called the *induction step* of the induction), then, just due to these assumptions, the property P holds for all natural numbers ($\forall n(P(n))$). In the statement of the axiom, "succ" is the name of an injective function whose domain is the universe under discussion, and whose range is the same universe deprived of the special element 0. To see the adequacy of the induction axiom to the goal, let X_P be the set of the elements satisfying P ($X_P = \{x : P(x)\}$), and notice that X_P with the special element 0 and the injective function succ is an inductive set. Thus the hypotheses of the induction axiom require that X_P be the name of an inductive set. On the other hand the conclusion of the axiom ($\forall n(P(n))$) states that all elements of the universe, the natural numbers, have the property P. Thus the axiom is saying that any natural number is an element of the inductive set X_P; i.e., the set of natural numbers is contained in the set X_P. Hence, in view of the universal quantification of P, the set of the natural numbers is contained in any inductive set that can be named, making the former the minimal inductive set (among those that can be named). Therefore the induction axiom expresses (alas not completely) what we meant, in a manner that is useful (and often essential) to derive many properties of the natural number system.

Thus formal languages are a great achievement if taken in their proper role as we just specified. However, to benefit from it we must go through the huge task of proving that the formal system is correct for the intended purposes.

6 The Pedagogical Problem of a Rule of Differentiation

We will now try to show how what we have done so far could help a teacher face the questions initially posed. We will do that with respect to the teaching of a simple and common topic: the rule for the derivative of the power functions, i.e., the functions of the type $f(x) = x^n$, where n is a positive natural number and x a variable ranging over the real numbers. (We will not present recipes for how to run a class on this topic, but rather we investigate its meaning, what underlies it, the notions that should already be known, and the motivations for taking certain steps.)

Let us begin by considering the elements. The meaning of x^n consists in the repeated multiplication of x by itself n times. Thus, first of all we should know what multiplication of real numbers is: it is a concept very different from that of multiplication of natural numbers (repetition of additions of a number to itself a certain number of times) even though it is an extension of the latter and it is obtained after an adequate development that we will assume to have been acquired without specifying it here since it is beyond the scope of this contribution.

Now we can discuss how to teach the power rule, that the derivative of the function x^n is the function nx^{n-1}. We are considering a family of real functions, one for each positive natural number n, and we are claiming that the result holds for all such numbers n. Clearly we would first require knowing what the derivative of a function is, not only from the definition (limit of the incremental ratio as the increment of the independent variable tends to zero), but also being aware of the problems that prompted it, understanding why the definition is presented in such an awkward way, and why we need to consider a limit (and what it is). Again let us assume that all these were adequately presented and are understood by the students.

At this point we could ask ourselves why we should present this rule, and the answer is: because it is an essential rule of the differential calculus. This answer brings to the fore the deeper question: why should we teach differential calculus? Here we could respond with many answers: because it is on the syllabus for this course, because it is needed in many applications, because a large part of the understanding of the physical and technological world is based on it, etc. But we would prefer a deeper answer justifying all the previous ones. In our simplified vision of reality we consider many phenomena to be continuous and smooth. To see the effect of instantaneous rates of change (e.g., speeds, slopes, rates of development, and so on), or even to determine them, we would need either negligible quantities or finer and finer approximations, and it is very hard if not impossible to work with them. Fortunately a calculus adequate to this environment was developed, and we want to take advantage of it, since the abovementioned problems surround us in our daily life. Thus introducing the differential calculus is worthwhile .

Thus finally we decide that it is worthwhile teaching the rules of differential calculus and in particular the power rule. But how should we to teach it? Of course we must show that it is correct, as otherwise we cannot accept it as a rule of calculus. This implies that we cannot simply tell students to just memorize the rule. Students who do this may make disastrous mistakes that could not be detected by the maker of it due to a lack of knowledge of why the rule should be as it is. Furthermore, just memorizing the rule without understanding it gives few hints on when to use it. Therefore we should answer the question of how to present the topic by insisting that we have to prove the correctness of this rule.

And the proof is simple: it is a straightforward application of the induction principle once the rule of the derivative of the product of differentiable functions is known. The sketch of the proof is little more than a line, but a line that is full of deep notions that deserve consideration.

Besides the notions of a real unary function of one variable and of multiplication of functions defined pointwise, notice that the rule for the derivative of a product of two differentiable functions is itself very important in the framework of showing the correctness of the differential calculus. It is probably even more important than the power rule since it is needed in more instances, including the use for the proof that we are analyzing. Thus remarks of the sort that we are presenting now should have already been considered in dealing with that rule, and again we assume that they were already well developed. The proof of the rule for differentiating the product of differentiable functions is almost as straightforward as that of the power rule, but it needs the small trick of adding and subtracting an appropriate quantity. Here again, this proof should be presented to students because it justifies the peculiar details of the rule, helping them avoid memorizing it incorrectly; we are thus taking a position on how this subtopic should be taught.

Continuing, let us consider the other tool on which the simple proof depends: the induction principle. The rule applies to a denumerable family of functions indexed by the exponent, which is a positive natural number, and we want to prove its correctness for all positive natural numbers. One could start with the case $n = 1$, recalling the trivial proof that the derivative of the identity function is 1, a result agreeing with the rule in this case. Next one can consider the case when $n = 2$, which is a direct application of what is known about the derivative of a product, still agreeing with what we want to show. One could even go to the case $n = 3$, looking at the function to be differentiated as the product of the functions x^2 and x (both with known derivatives), justifying our claim one more time. From the consideration of the last case, one can get the idea that, in general, to obtain the derivative of x^{n+1} one could view this function as the product of the functions x^n and x and again apply the product rule. Of course, in this case, one has to assume that the derivative of the function x^n is nx^{n-1}. In effect what we have done was to prove that the claim holds for $n = 1$ and that if it holds for any positive natural number n, then it holds also for the immediate successor of n, i.e., it holds for an inductive set of indexes.

At this point one is tempted to affirm that the initial claim is proved: for any fixed positive natural number n that can be reached and considered in our humanly finite procedure of considering one natural number after the other, we have proved that the derivative of the function $f(x) = x^n$ is the function $f'(x) = nx^{n-1}$. But we must wonder: are all positive natural numbers thus reachable? Yes, because of the choice, stated in section 4.3 and made formally explicit in section 5, about what the natural numbers are: the smallest inductive set! This choice about our concept of natural numbers was captured by the induction axiom, and indeed it is what is needed for the proof of the correctness of the power rule. The proof that we gave proved the base step and also the induction step, and thus, exactly by the induction principle, we can conclude that our claim is proved.

Along the path developed, we justified the importance of the proof of the correctness of the power rule, and during the proof we had to recall concepts together with their ways of acquisition, which were needed to justify the formal statements used in the proof. Thus we showed how our approach is essential for the acquisition of concepts in order to justify and develop important results in mathematics.

7 Conclusion

Good pedagogy of mathematics should take care of and pay attention to the students, particularly to their interests and curiosity, as well as to requirements of meaning and relevance. Thus it urges us to look at how mathematical concepts can be acquired. These aims led us to propose a new position in the philosophy of mathematics.

So far we tried to sketch feasible paths of acquisition of a few initial mathematical concepts. Of course, the task would become much wider and beyond the scope of this chapter if we were to consider many more mathematical concepts. But the wider task can be performed using the methodology presented here.

The constructive processes we are proposing of acquisition of mathematical concepts, as outlined, start with the perceptions from our senses (including the physical and non-physical internal ones). It has the memory playing an essential role, and it relies very much on the mental operations. (The three presented, abstraction, generalization, and idealization, seem sufficient to reach the mathematical concepts.)

All the steps of such a path of acquisition of a mathematical concept are internal to the individual person and seen by introspection. Indeed the act of knowing is individual, and to understand it we cannot avoid looking into ourselves: all attempts to explain it from outside lead to incomplete accounts of the phenomenon and hence to failure. Furthermore, we believe that being aware of how a concept has been acquired can be an effective way of communicating it, simply because we have plenty of indications that other people can operate in the same manner as we do. Thus they can easily receive the communication indicating which kind of perceptions they have to consider and which mental operations to perform along the path followed to acquire a mathematical concept. Performing the steps described, they will obtain results (the mathematical concepts) that should be very similar to the intended ones, indeed so similar as to be indistinguishable through the language. Furthermore, using this approach, the transmitted concepts keep being meaningful and relevant to the listener, exactly because they are based on contingent experiences that are of common concern in daily lives, thus meeting one of the needs of the teaching of mathematics.

These ways of acquiring knowledge could, of course, be applied to any kind of concepts, not only to mathematical ones, possibly after enlarging the set of mental operations. What, then, is typical of mathematics? Given our results we can go back to the main question of what is mathematics. Now we know what we are talking about. There is vagueness and approximation in the meanings, but they are limited to what the language cannot express. In mathematics we attempt to approximate reality with models of reality (although as observed, mental operations may yield concepts departing from reality). We do this to obtain a manageable model that overcomes the extreme complication of reality, which if not so approximated and simplified would prevent any handling and understanding of it. We avoid solipsism because, even though there can be many models, they are not at the discretion of the individual constructing one of them, since they are designed to best capture reality. Still, we do not know whether the claims of mathematics are true or not (which is good, since there are no mathematical truths, simply sentences true in a specific interpretation). They are just correct deductions from assumptions that we feel are adequate to model reality well enough for our purposes. Our purpose in doing mathematics is being able to handle and manage multiplicities of any kind. This being the goal of mathematics, it is evident that it finds lots of applications because multiplicity is present in any discipline, and the capacity to handle it is essential independent of the specific features of the discipline.

Finally, this path of acquisition of a vision of mathematics should allow each teacher to build a personal view of mathematics in which to ground the answer to the questions proposed at the beginning.

For a further discussion of some issues related to our proposal we suggest the following readings:

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