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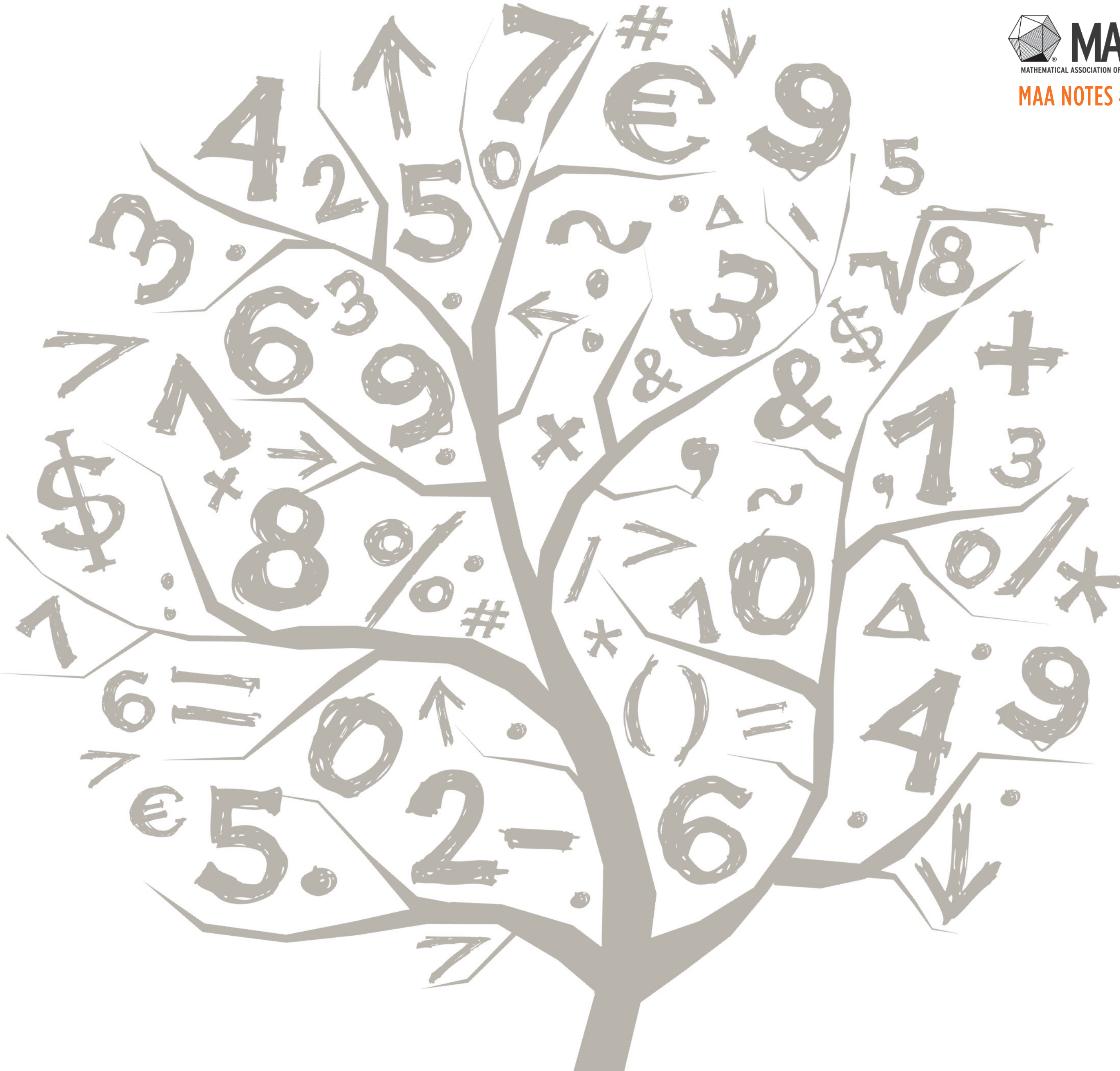

CHERYL BEAVER, LAURIE BURTON, MARIA FUNG, KLAY KRUCZEK, Editors

RESOURCES FOR PREPARING MIDDLE SCHOOL MATHEMATICS TEACHERS

Resources for Preparing Middle School Mathematics Teachers

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Resources for Preparing Middle School Mathematics Teachers

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Preface

Over the past decade there has been increased focus on the mathematical preparation of teachers at all levels as one of the main tools in improving mathematics education in this country. The mathematical knowledge of teachers arguably involves not only solid procedural and conceptual mastery of key mathematical processes and phenomena, but also understanding how mathematics content fits into and extends the school curriculum. In their coursework and professional development, teachers need to experience the process of learning and doing mathematics: experimenting, conjecturing, justifying, generalizing. and struggling. Teachers also need to experience consistently the beauty and power of mathematics.

Middle school mathematics teachers occupy a unique place in the mathematical development of students. These teachers need to be proficient in all elementary mathematics topics, together with some secondary mathematics topics. They demand a special kind of preparation that differs from both that of their elementary and secondary colleagues. *The Mathematics Education of Teachers* (MET) document published by the Conference Board of the Mathematical Sciences set forth criteria for the preparation of mathematics middle school teachers which made it critical that special programs and courses for this group emerge. All middle school teachers need to know the mathematics content for elementary teachers. In addition, in the strand of number sense and operation, middle school teachers need to be fluent with the number line model for the real numbers, proportional reasoning, and elementary number theory. They need to be able to distinguish among rational and irrational numbers and their representations. In the algebra strand, middle school teachers need to see algebra not simply as a generalization tool but also as a study of linear, polynomial, and exponential functions and as a body of techniques for equation solving. In the geometry strand, middle school teachers need to be proficient with transformational plane geometry, congruence and similarity, and connections of geometry to art and nature. They also need to be able to use a variety of strategies for developing and using formulas for areas of plane figures and volumes of polyhedra. In the strand of probability and statistics, middle school teachers need to experience the design and execution of a small statistics project, and get familiar with normal curves and sampling distributions, and the basis of inferential statistics. Some of the pedagogical techniques that facilitate the learning for middle school teachers are hands-on approaches, cooperative learning, effective classroom discourse, writing, and technology.

This collection of articles is in response to the MET document and the result of several gatherings of mathematics educators and mathematicians training middle school teachers. We, the editors of this volume, under the sponsorship of the Committee on the Mathematical Education of Teachers, organized two contributed paper sessions at the Joint Mathematics Meetings entitled “Content Courses for the Mathematical Education of Middle School Teachers” in 2007 and “Curriculum Materials for Pre-service Middle School Mathematics Teachers” in 2008. We invited participants from these two sessions as well as colleagues heavily involved in the mathematics education of middle school teachers to write articles on both programs and courses.

We have chosen the articles that appear in this volume for several purposes: to disseminate various middle school programs structures, to detail methods of teaching specific middle school teachers content courses, and to share materials and resources. While each article describes the unique program or course of its respective institution, each also includes a common core of information to provide some consistency to the volume. In particular, all articles describing middle school programs contain information about the host institution, a history of the program, degree and testing requirements for the program and for state licensure, learning goals and objectives for the program and courses, and any available assessment data. When applicable information is included about particular courses, for example, some articles provide sample activities or syllabi and some have a description of courses in the appendix. Most articles have links to websites containing further information about the program, courses, state requirements, or resources that can be downloaded and used directly.

Each article in the second half of the volume describing courses provides information on the history, purpose and learning objectives of the course. It also gives the structure and format of the course as well as a summary of the content and essential course elements. In addition, many contain or link to sample class activities, assignments, teaching notes, class projects, syllabi, assessment materials or dedicated course websites. The articles provide a rich set of readily available, classroom-tested resources.

We expect that the reader of this volume will be either a faculty member who is new to the teaching of courses for middle school mathematics teachers or a seasoned teacher of pre-service teachers who is interested in trying some new approaches and perhaps starting a middle school program at his or her institution. In either case, we hope that the reader find these expositions beneficial and stimulating.

Programs for Middle School Teachers

We begin this volume by showcasing several pre-service training programs. We asked each contributor in this section to incorporate as many components of the following outline into the description of their program as possible.

Core Program Paper Outline

- I. Instructional Information
  - A. Size and location of school
  - B. Student participation in the program
- II. Program History
  - A. Intended audience
  - B. Program beginning
  - C. Background and philosophy of the program
- III. Degree Requirements
  - A. State requirements (in appendices)
  - B. Learning goals and objectives
    1. For the program
    2. For the courses
  - C. Certification tests
  - D. Courses requirements
    1. Mathematics courses
    2. Mathematics education courses
  - E. Mathematics education courses
    1. Instructor information
    2. Course list specifically for middle school mathematics teachers
    3. Enrollment for these courses
    4. Syllabi links
    5. Activities
       - a. Resources
       - b. Connection to classroom
       - c. Connection to program and course learning goals and objectives
- IV. Assessment
  - A. Student reaction
  - B. Student employment perspective
  - C. Middle school administrators’ reactions
  - D. Student performance on required state examinations
  - E. Retention in education
- V. References and links

Author responses

Angel Abney, Nancy Mizelle, and Janet Shiver discuss the collaborative effort of mathematics and education faculty at Georgia College and State University in delivering a successful program. Cheryl Beaver, Rachel Harrington, and Klay Kruczek show an example of a program built around the quarter system that is both rigorous and nurturing. Ira J. Papick focuses on the program at the University of Missouri; this program motivated the development of a series of middle school teacher curriculum materials centered on making connections between concepts in the middle school and the college classrooms. Finally, Jennifer Szydlik, John Beam, Eric Kuennen, and Carol Seaman describe the structure of the program for middle school teachers at the University of Wisconsin Oshkosh.

We then showcase several in-service teachers' programs. These articles illustrate how institutions have created effective, rigorous, in-service programs for middle school teachers even when faced with the many challenges resulting from the busy schedules and varying mathematical backgrounds of in-service teachers. Julie Belock shares experiences establishing the only middle school teachers' preparation program in Massachusetts. Ruth M. Heaton, W. James Lewis, and Wendy M. Smith discuss a mathematics institute for middle school teachers. M. Elizabeth Mayfield and Christy Danko Graybeal give an example of an effective professional development program for in-service middle school teachers in Maryland.

Courses for Middle School Teachers

The second part of this volume focuses on specific course descriptions and resources. We asked each contributor in this section to incorporate as many components of the following outline into the descriptions of their courses as possible

Core Course Paper Outline

I. Introduction and Background
   A. Course history, development of course
   B. Course structure
   C. Prerequisite and enrollment information
II. Course materials and resources
   A. Syllabi
   B. Teaching Notes
   C. Classroom materials
III. Course structure and format
IV. Course content
V. Essential course elements (possible)
   A. Assignments
   B. Student Experiences
   C. Class Projects
   D. Assessment
VI. References and links
VII. Appendices (possible)
   A. Table of contents for author written materials
   B. Assessment materials

Author responses

There are two approaches to discrete mathematics offered, one by Tanya Cofer, Valerie A. DeBellis, Cathy Liebars, Joseph G. Rosenstein, Bonnie Saunders and Margaret Wirth and one by Mary Flahive and Reva Kasman. Both articles present a wealth of ideas of what such a course could involve. Bethany Noblitt describes her geometry course for middle school teachers with a variety of exciting activities and ideas. Michael Mays and David Miller write about their number theory course which is largely taught online, while Theresa Jorgensen focuses on pre-calculus and calculus ideas. Laurie Burton and Klay Kruczek present an intriguing way to teaching algebra with visual techniques. The University of Wisconsin Oshkosh group discusses a probability and statistics course. Finally, there are two articles on integrated courses: one by George Ashline and Marny Frantz about the Vermont Middle Level Mathematics Initiative, and one about a series of courses at the University of Nebraska Lincoln by Ruth M. Heaton, W. James Lewis, Michelle R. Homp, Steven R. Dunbar, and Wendy M. Smith.

It has not been our intention to write the definitive volume on how to teach mathematics to middle school teachers. Indeed, students can have successful learning experiences in many different types of programs. Included here are only a portion of the programs and courses that are now being developed. We hope, however, that the ideas contained in this volume will stimulate readers to think about programs at their own institutions and courses they could develop specifically for middle school teachers.

The editors would first of all like to thank the authors of the articles for their interest in the project and their patience throughout the editing process. Special thanks go to Professor Stephen B Maurer who has guided us through this entire process and to the entire MAA Notes Editorial Board, especially the unnamed members who were on our review panel, for their careful reading and valuable suggestions. We are grateful for the insightful evaluations done by the Notes Editorial Board as we worked to prepare a final document. We thank Beverly Ruedi and the rest of the production staff at MAA headquarters

We appreciate the continuing support of our institutions, Southern Connecticut State University, Western Oregon University and Worcester University. This project would not have been possible without the hard work of the folks at the Mathematical Association of America. With all this support and encouragement, we have been able to create a volume that will, we think, have an important impact on the mathematical education of middle school teachers.

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I

Programs for Middle School Mathematics Teachers

A. Pre-service Training Programs

1

Preparing Middle Grades Mathematics Teachers at Georgia College & State University

Angel R. Abney, Nancy B. Mizelle, and Janet Shiver
Georgia College & State University

1.1 Introduction

The Georgia College & State University (GCSU) middle grades mathematics faculty believe that teaching mathematics is a complex task and it is not sufficient for effective teaching for teachers only to know their content. Instead, we believe that teachers need a specialized knowledge of mathematics that includes a profound understanding of the content they teach and knowledge of how students learn mathematics [25]. In addition, we believe they need to be aware of the mathematical content in the previous grades and the grades to follow and how they are interwoven. We refer to this specialized content knowledge as pedagogical content knowledge [23]. We believe pedagogical content knowledge is necessary for teachers to make productive use of their content knowledge within the classroom. In all mathematics education courses at GCSU the topics, tasks, activities, technology, and manipulatives are chosen to enhance both content knowledge and pedagogical content knowledge.

1.2 Georgia College & State University

Georgia College is a small liberal arts college with about 6000 students located in a small town in central Georgia. It was founded in 1889 and is accredited by the Commission on Colleges of the Southern Association of Colleges and Schools to award associate, baccalaureate, masters, and specialist degrees. The student population at GCSU is among the best in the state academically with students ranking third highest in average SAT scores. GCSU has a rich history in teacher preparation and has always been an innovator in middle grades education. The college is home to Dr. John H. Lounsbury, Dean Emeritus, who is well known as one of the founding fathers of the middle school movement.

The John H. Lounsbury College of Education (JHL COE) completed the transition from a traditional model of teacher education to an innovative teacher preparation program in 1996 and stepped to the forefront in the preparing of teachers for certification in middle grades education. Recognizing that the educator of the future must be prepared to take a leadership role in the community and have the skills to collaborate with others to meet the needs of students, the school initiated an intensive field-based teacher preparation program, described below, with the goal of producing teachers who would be *Architects of Change* in their schools and communities. In contrast, students at GCSU preparing to become secondary teachers follow a much different path. After completing a bachelor's degree in their field, such as mathematics, secondary pre-service teachers enter a fifth year of school in which they take a year of education courses and spend a semester student teaching. Then they are awarded a master's degree in education and teacher certification in their field of study.

GCSU has long been recognized as a national leader in middle grades education and has had a program that has been preparing teachers specifically to teach middle grades students for over 25 years. While the middle grades program primarily serves the state of Georgia, students from throughout the nation participate in it. The number of graduates varies from year to year with the average number of graduates in all areas of middle grades education being approximately 20 per year while in middle grades mathematics it varies from 5 to 10 teacher candidates a year.

1.3 Middle Grades Program

Students apply for admission to the middle grades program in the JHL COE during the spring of their sophomore year. Criteria for admission to the program include:

- Completing all areas of the Core curriculum, which includes two entry-level mathematics courses, at GCSU or at another accredited higher education institution, with a minimum 2.5 grade point average (GPA).
- Completing six pre-education courses that include three introductory education courses and three content-related courses.
- Passing the reading and writing subtests of the Georgia Regents' Examination.
- Passing the Georgia Certification Exam (GACE) I Academic Skills Assessment (or having a combined SAT Verbal and Math score of 1000 or combined ACT score of 43 in English and Math).
- Submitting three acceptable letters of recommendation.
- Completing a formal interview with representatives from the middle grades program.

Students who are admitted to the program understand from the beginning the intensity of commitment it demands. They are provided with a Student Handbook that explains the expectations of the program, provides guidelines for field experiences, and outlines requirements that must be met at decision points along the way. They enter the program knowing that, each week, they are expected to commit approximately 40 hours to taking classes and their field placements. A brief description of the sequence of the current program is outlined below:

Freshman and Sophomore	General Studies Courses Four Mathematics Content Courses Field-based Investigating Critical Issues in Education Field-based Exploring Socio-Cultural Perspectives Field-based Exploring Learning and Teaching	
Junior	Mathematical Investigations Cultural Geography Integrated Science English/Language Arts Developmental Learning Instructional Technology for Teachers I Field Placement 1	Concepts in Algebra Concepts in Geometry English/ Language Arts Reading Instructional Strategies Middle Grades Curriculum Instructional Technology for Teachers II Field Placement 2
Senior	Literature, Reading, and Writing in the Content Areas Developing and Assessing Logical Thought Sociology of Middle Schools/ Responsive Classrooms Legal Issues in SPED Adolescent Literature Field Placement 3	Advanced Instructional Technology for Teachers Ethics/Professionalism/School Law Capstone Experience Student Teaching Internship

1.3.1 Field-Based Teacher Preparation Program

A cohort of teacher candidates directed by a faculty mentor engage in two years of intensive field-based work with a regular course load of arts and sciences are the unique and essential characteristics of Georgia College's initial preparation of middle grades mathematics teachers. Professional courses, for the most part, are taken in the third and fourth years. Cohorts of 15–25 students are formed as students enter the junior year. Then throughout the two-year process students in a cohort take all their classes together except when they break into sub-cohorts to take their content concentration courses (mathematics, science, or social science) in the College of Arts and Sciences (CoAS). Providing program offerings in cohort groups enhances the opportunities for developing collaboration that eventually applies to schools and communities.

The variety of experiences and time spent in classrooms is a critical element of this program. During the junior year, students are placed in four different school settings. Most of the placements occur in schools that are predominately rural but which have widely varying populations. They spend 16–20 hours each week in a mathematics classroom under the guidance of a host teacher who has been selected to supervise the students. In their senior year, students spend fall semester in a single placement. They begin the semester with their host teachers in pre-planning so that they can experience the beginning of the school year, and before the semester is over, they engage in full-time teaching for a minimum of three weeks. By the time they graduate, they have over 1000 hours of classroom experience and an understanding of how to apply theory to practice.

Another unique element of this innovative program is the ethic of collaboration that permeates it. This field-based program is the result of consistent collaborative planning between CoAS faculty and faculty from the JHL COE over many years. These collaborative meetings have resulted in innovative problem solving, consistent evaluation standards, and a sharing of ideas that are incorporated into classes and field experiences. As a result of this successful collaboration, graduates of the JHL COE middle grades program are recognized statewide as effective teachers. Consistently, 100% of our teacher candidates pass our state certification test in two content area concentrations (English/language arts and either mathematics, science, or social science). All our graduates are recruited heavily by principals from across the state. In a 2003 survey of administrators, principals rated 91% of our beginning teachers as "good" or "excellent."

More importantly, and in contrast to national and statewide trends, graduates of the GCSU middle grades program elect to remain in the profession. When nationally 30% of beginning teachers exit the classroom in three years and 50% in five years [12] and in Georgia 33% of beginning teachers leave the classroom by the end of their third year [10], graduates of our middle grades program remain committed to their profession. In the fall of 2006, over 90% of each year's graduates from 1999 to 2003 remained in education [15]. After seven years, 92% of the 1999 graduates were still in education; after five years 95% of the 2001 graduates were still in education; and after four years 100% of the 2003 graduates continued in their chosen profession.

1.3.2 Math Concentration

Undergraduate middle grades majors with a concentration in mathematics are required to complete successfully four mathematics courses and three mathematics education courses. The elementary preparation program requires preservice early childhood majors to take two mathematics content courses and three courses in mathematics education. The first two mathematics content courses, taken by both middle grades and early childhood majors, are required for completion of the Core curriculum at GCSU. Students may choose from a variety of entry-level courses to fulfill the requirements, including mathematical modeling, precalculus, survey of calculus, calculus, linear algebra, or elementary statistics. The courses are taught by both mathematicians and mathematics educators from the math department. Decisions about them are made through consultation with an advisor in the JHL COE and are based on the student's mathematical background and declared interest. Because of the focus on statistical ideas in the middle grades classroom, students are encouraged to select elementary statistics as one of their classes.

Once students have declared an interest in teaching middle grades mathematics, they are required to take two additional mathematics courses. Students generally take them in their sophomore year prior to entering the cohort. They are allowed to select them based on their interests and backgrounds. Most select courses from those already mentioned, but a few elect to take more advanced courses such as abstract algebra, differential equations, or an introduction to higher-level mathematics course called foundations. These courses are generally taught by mathematicians.

Upon entering the cohort in their junior year, students are required to take three mathematics education courses taught by mathematics educators in the mathematics department. The courses are designed specifically for middle school pre-service teachers. Occasionally mathematics majors who are interested in teaching choose to take the courses as electives. Since two of the courses are cross-listed as graduate level courses, students in the Masters of Art in Teaching, Masters in Education or Specialist programs often take them to satisfy their mathematics requirement. Some of the graduate students are practicing teachers. The mix of practicing and pre-service teachers provides an interesting dynamic. The practicing teachers often gain from the enthusiasm of the pre-service teachers while the pre-service teachers benefit from the wisdom of the practicing teachers.

The first of the courses is an introduction to mathematical problem solving. All middle grades majors are required to take this course, regardless of their concentration. There are typically 20 to 25 students enrolled. It is designed to expose prospective middle grades teachers to a variety of problem solving strategies. It is the goal of this course, and of all mathematics education courses at GCSU, to foster an in-depth understanding of middle grades mathematics, to encourage cross-disciplinary thinking, and to promote effective mathematical communication orally and in writing. This is accomplished by exposing students to a wide range of mathematical topics and a variety of pedagogical strategies that enhance the development of their mathematical thinking, as well as the mathematical thinking of their future students. The National Council of Teachers of Mathematics standards and the state of Georgia curriculum standards, which are called the Georgia Performance Standards (GPS), serve to guide the content choices for the course. The text for this course is John A. Van De Walle's *Elementary and Middle School Mathematics: Teaching Developmentally*. The text, which is typically used in mathematics methods courses, is heavily supplemented with activities and tasks that focus on developing mathematics content. Inquiry-based activities that allow students to discover various mathematical concepts are used. An example of one such activity is the chords of a circle problem (see Appendix A) in which students investigate inductive versus deductive reasoning and the power of the counterexample.

The chords of a circle problem asks students to determine a rule for the number of chords on a circle with n points. Often students notice the recursive pattern indicated by the table in Section 1.1. For instance, for a circle with 2 points there is 1 chord, a circle with 3 points has 3 chords, and a circle with 4 points has 6 chords. Many students will acknowledge that the number of chords is increasing by the number of points at that stage, or they may suggest that the number of chords is increasing by consecutive integers. While these students correctly identify a recursive rule that is nonlinear, they do not usually find a way of symbolically representing this rule without prompting. The rule could be written as $C_n = C_{n-1} + (n-1)$, where C_n represents the number of chords for n points on the circle. Other students aren't satisfied with a rule where an output depends on the previous output; they want a rule that maps directly from input, the number of points, to the output, the number of chords. Those who find a closed formula for the number of chords for n points on the circle often notice that when there are 5 points on the circle, there are 4 chords connected to each point. When counting these chords, a shorter way of producing this count would be 5(4). However, when counting one by one, they notice that there are not 20 chords on a circle with 5 points. Instead, there are half as many because each chord has been counted twice. Thus, when determining the number of chords, C_n, on a circle with n points, a closed rule is $C_n = \frac{n(n-1)}{2}$. When we determine that the rule continues to work for 6 points and 7 points, the students are generally convinced that it will work for any number of points. Often a few more ways of determining the rule are discussed. For instance, rather than seeing that there are four chords at each point, some students see that there are four chords at the "first point," three chords at the "second point," two chords at the "third point," one chord at the "fourth point", and zero chords at the "fifth point."

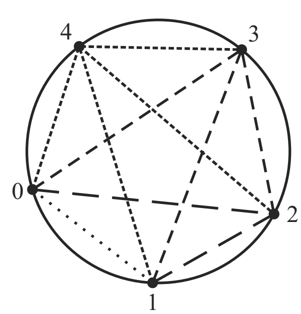

Figure 1.1.

This results in the determining the number of chords, C_5, on a circle with 5 points by the sum $1 + 2 + 3 + 4$. With some prompting, students come to determine that these addends could be paired as $(1 + 4) + (2 + 3)$. This results in two groups of five. If looking at a circle with an odd number of points, this will always result. For instance, a circle with 7 points will result in the sum: $1 + 2 + 3 + 4 + 5 + 6$. Pairing generates a product of $3(7)$. This generates a closed rule of $\frac{n-1}{2}(n)$. Of course, students wonder if this rule works for the circles with an even number of points. Upon encouraging the students to try it, they determine that a new, but equivalent, rule is necessary. When there is an even number of points, such as 6, the sum they generate is either $0 + 1 + 2 + 3 + 4 + 5$, which results in the product $3(5)$, or for n points, where n is even, $\frac{n}{2}(n - 1)$. However, many see the sum as $1 + 2 + 3 + 4 + 5$, which is more difficult to determine a corresponding product by pairing from the "outside in." The method generates a sum of $(1 + 5) + (2 + 4) + (3)$. We can see that the first two addends result in a sum of 6, and the last term is half of six. Thus, we are generating a product of $2\frac{1}{2}(6)$, or $\frac{5}{2}(6)$. For n points, where n is even, this still corresponds to a product of $\frac{n-1}{2}(n)$. All the closed rules are equivalent.

The chords example provides a case where inductive reasoning works well for determining a rule. Now that the students are comfortable determining patterns by using inductive reasoning, the next example is placed there to challenge them. When students are asked to determine a rule for counting the maximum number of interior regions formed in a circle with n points, they see that the number of regions for 2 points is 2, the number of regions for 3 points is 4, with 4 points there are 8 regions, and for 5 points there are 16 regions. Because the numbers are powers of two, students usually conclude that , 2^{n-1} will be the maximum number of regions when n points are on the circle. When asked the number of regions for 6 points, most students assume the closed formula that they generated will produce a correct output. They see no need to count regions. When forced to count, students believe that they counted wrong because they found 31 regions, not 32. This activity shows the limitations of inductive reasoning and to introduce counterexamples. The rule they generated, 2^{n-1}, was a conjecture and that a circle with 6 points provides a counterexample. This problem is often given on the first and second day of class and helps to provide a framework for the rest of the course.

Some additional resources for mathematical tasks include:

- *Mathematics Activities for Elementary School Teachers: A Problem Solving Approach* by Dolan, Williamson, and Muri
- *Mathematics Activities for Teaching & Learning* by Barnard and Wheeler
- *Connected Mathematics (CMP) for grades 6 through 8* by Lappan, Fey Fitzgerald, Friel, and Phillips
- *A Survey of Mathematics with Applications* by Angel, Abbott, and Runde

During the second semester of their cohort experience students are separated into sub-cohorts focusing on science, social studies, or mathematics. The middle grades mathematics students take two courses: one in geometry and one in algebra. Both are designed specifically for middle grades teachers.

1.3.3 Concepts in Geometry

The first course specifically designed for pre-service teachers concentrating in mathematics is Concepts in Geometry. It is cross-listed, and there have been enrollments from 10 to 23 students. The course is offered every spring along with Concepts in Algebra. The middle grades cohort students concentrating in mathematics take both at the same time. The goal of the geometry course is to help students to develop a conceptual understanding of Euclidean geometry and of real number measurements of perimeter, area, and volume. Topics involve many of the ideas from the Georgia Performance Standards, including classes of geometric figures, symmetry, transformations in geometry, similarity and congruence, and constructions.

An essential element of the middle grades geometry course is the frequent use of The Geometer's Sketchpad (GSP) as an exploration tool. Many students at GCSU have never thought of mathematics as a topic to explore, instead they think of it as simply a toolbox of procedures to apply. Therefore, introducing exploration, students need guidance and suggestions, which include leading questions, tables to fill out, organizational strategies, etc. As students progress throughout the semester, fewer suggestions are needed. Through our experiences with both pre-service and in-service teachers, we have found this method, in which assistance is lessened throughout the semester, to be necessary and effective.

As an example of how GSP is used within the course, one of the first tasks that students are given is an activity called the Incomplete Boat. On the first day of class students are given a projection of a boat that has been previously constructed and animated to sail along a sine curve, as shown in Figure 1.2. They are fascinated by this.

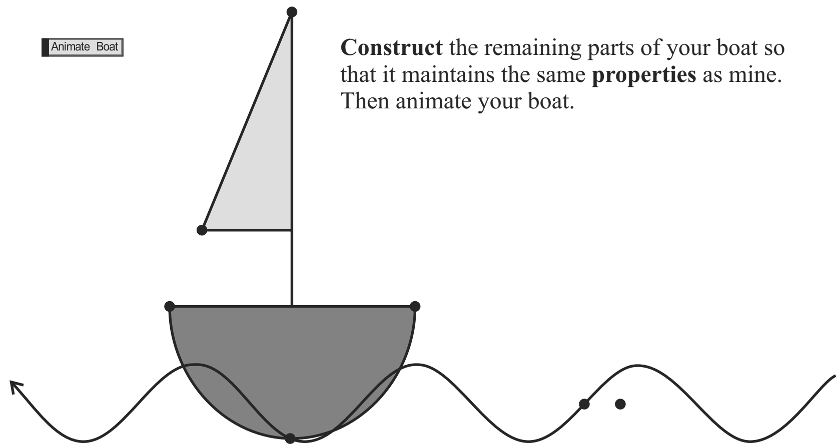

Figure 1.2.

After an introduction to GSP, students are given a short period of time to explore and familiarize themselves with the program. The students are then given a partially complete GSP sketch, as shown in Figure 1.3, with the construction of the boat. The only instructions given to the students are to complete the boat by constructing it to have the same properties as the projected boat. As expected, most of the students draw rather than construct the boat to look like the one projected. When animated, the sail on their boat generally stays fixed or moves randomly as the rest of the boat follows the path of a sine wave, making for some interesting designs and reactions. Immediately, students want to know why their boat does not move nicely along the wave like the one projected. This leads to a discussion about the difference between drawing and constructing: an important concept explored throughout the semester.

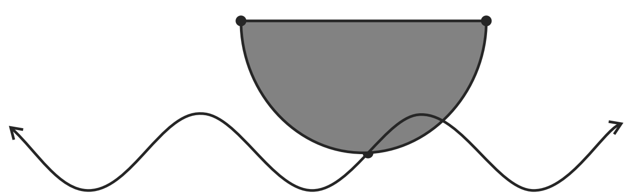

Figure 1.3.

Along with technology, our geometry course is designed to emphasize the importance of hands-on experiences to develop conceptual understandings. Manipulatives such as patty paper, pattern blocks, pentominoes, square tiles, clear plastic three dimensional objects, and D-Sticks are used throughout the course to explore different geometric concepts. For instance, students explore what it means for two figures to be similar using pentominoes or patty paper, discover the sum of the interior angles in a polygon using pattern blocks, construct the relationships between and among classes of quadrilaterals using D-sticks, and derive formulas for the volume and area of common geometric figures using tiles, nets and 3-D objects.

The Concepts in Geometry course uses the Van De Walle text and the CMP materials mentioned earlier. In addition, problems, tasks, and articles for this course are taken from:

- *Fostering Geometric Thinking: A Guide for Teachers, Grades 510* by Driscoll
- *Patty Paper Geometry* by Serra
- *Geometry Activities for Middle School Students* by Wyatt, Lawrence and Foletta
- *Rethinking Proof with Geometer's Sketchpad* by De Villiers
- *Developing Mathematical Ideas: Examining Features of Shape* by Schifter, Bastable, Russell

- *Developing Mathematical Ideas: Measuring Space in One, Two, Three Dimensions* by Schifter, Bastable, Russell
- *Discovering Geometry: A Investigative Approach* by Serra

1.3.4 Concepts in Algebra

The second mathematics education course is Concepts in Algebra. It is also cross-listed and typically has from 10 to 23 students. Topics include prealgebra concepts, linear and quadratic equations, inequalities, and applications of algebra to real-life situations. The topics are not altogether new to our students. Many of them enter this course with what they believe to be a broad understanding of algebraic topics, but what most really enter with is a procedural knowledge of how to solve equations and the notion that algebra involves only the manipulation of symbols. While we devote some time to discussing meaningful use of symbols as well as common misconceptions, one of the goals of the course is to extend students' view of algebra beyond a procedural manipulation of symbols. We believe that the study of algebra includes generalizing from patterns and arithmetic that can be found in all areas of mathematics [25]. By carefully developing lessons and activities, we help students determine ways to make explicit algebra's connections to geometry, measurement, data analysis, and numbers and operations.

The study of patterns and functions is one of the most important forms of algebraic reasoning and is therefore a major focus of the course. Our students have some experience extending patterns and building rules for functions [8]. However, we have found that their reasoning in pattern activities typically involves looking at the numbers in a sequence and then manipulating them until they are able to express the pattern using variables. To broaden their understanding of patterns, we engage them in activities involving visual patterns without numbers. They often involve physical materials such as toothpicks, tiles, or counters. They allow students to rearrange the materials, to build on to one step to make a new step, and to see the connection between algebra and other topics. Through this process the students begin to recognize recursive relationships as well as functional relationships with the term number. Building functional relationships from patterns leads naturally into mathematical modeling, which helps bridge the gap to real-life situations [25]. Here is an example of a typical geometric pattern used in the course.

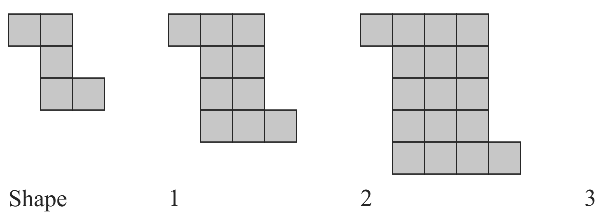

Figure 1.4.

In predicting the total number of squares in the nth term, where n is the shape number, the students taking the Concepts in Algebra course are challenged to look at the generalization geometrically using one-inch tiles. One common student solution to generalizing this pattern is in Figure 1.5.

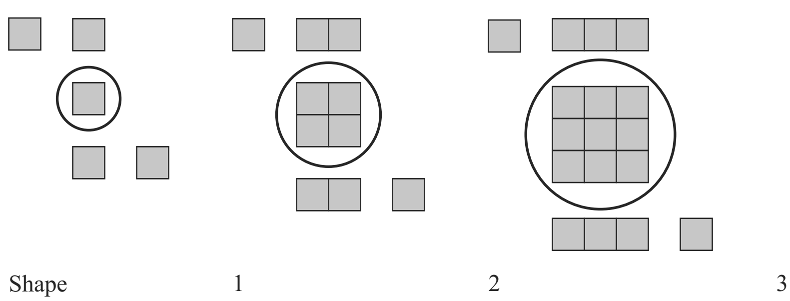

Figure 1.5.

The circled part of each figure shows the shape number squared while the two rectangles directly above and below the circle show twice the shape number. Each shape has two remaining blocks so in generalizing this pattern to the nth term, where n is the shape number, we clearly have $n^2 + 2n + 2$.

The Concepts in Algebra course also focuses on the structure in our number system. This area of algebra involves the exploration of properties, making conjectures about these properties, expressing them in general terms, and proving that these properties are true. At the beginning of the course, when students are asked how do they know that something is always true, they will often respond that they learned it in high school or that someone told them it was true. Our goal is to have them verify their own results rather than accepting the word of others [25]. Along with resources used in previous courses such as Van De Walle's text and the CMP materials, we also use the following sources:

- *Fostering Algebraic Thinking: A Guide for Teachers Grades 6–10* by Driscoll
- *Algebra Tiles Workbook* by Burgdorf and Robinette
- *Implementing Standards-Based Mathematics Instruction: A Casebook for Professional Development* by Stein, Smith, Henningsen, and Silver
- *Thinking Mathematically: Integrating Arithmetic and Algebra in Elementary School* by Carpenter, Franke, and Levi
- *Connecting Mathematical Ideas: Middle School Video Cases to Support Teaching and Learning* by Boaler and Humphreys

1.4 Assessment

A method for evaluating student understanding and work had to be developed. In addition to traditional assessments, all the courses require students to submit written solutions to posed problems. We define a problem as a task for which the student does not have a set procedure to solve. The students organize their work similar to Pólya's [19] heuristic, where the first section is directly stating the problem and then paraphrasing it in their own words. This section is called Problem Set Up and corresponds to Pólya's first step of Understanding the Problem. In the second section of students' problem write-ups, they are expected to discuss connections to previous problems they've worked, solve a simpler but related problem if needed, and to discuss how the problem can be respresented (algebraically, graphically, numerically, and/or verbally). This section, called Plans to Solve/Investigate the problem, corresponds to Pólya's second step, Devise a Plan. The third section is called Investigation/Exploration of the Problem. In this section students convey how they solved the problem, explaining how they made sense of it, showing multiple representations, at least one of which must be visual. Students are encouraged to use technology or manipulatives to explore the problem rather than using them to show their final product after solving. The last section is the Extension of the Problem, which corresponds to Pólya's fourth step, Looking Back. In this section they are encouraged to reflect on what they found in the previous section, making sure that their solution makes sense. Students are encouraged to generalize or prove any conjectures made during the investigation of the problem. Some of the problems the students use for these write ups come from Project Intermath, a website developed by University of Georgia with recommended problems created specifically for middle school teachers.

1.5 Teaching Certification

In 2006, GCSU moved to adopt and currently uses as standards the Georgia Systematic Teacher Education Program (GSTEP) Framework for Accomplished Teaching. This framework and the Association for Middle Level Education (AMLE), formerly the National Middle School Association, standards for Initial Middle Level Teacher Preparation are used currently to assess teacher candidates as they progress through the middle grades program. Before graduation and being recommended for certification, teacher candidates are expected to satisfy each standard as determined by JHL COE Middle Grades faculty. The GSTEP and NMSA Standards are closely aligned with the Interstate New Teacher Assessment and Support Consortium Principles but are written in detail to describe the knowledge, dispositions, and performances that we expect students to learn in GCSU's middle grades teacher preparation programs. The result is an assessment of candidates in the areas of student development, particularly young adolescent; learning environments/philosophy and school organization; curriculum, instruction, and assessment; family and community involvement; and professional roles. Students who successfully meet all areas assessed and pass the Georgia Certification Exam (GACE®) II in their area of concentration are recommended to the State of Georgia Professional Standards

Commission for middle grades certification in that area (grades four through eight). In comparison, teacher candidates who successfully complete the early childhood program and pass the GACE II exam specific for early childhood are recommended for certification in all content areas (grades prekindergarten through five).

1.6 Conclusion

All the middle grades mathematics education courses at GCSU have the goal of providing our students with a profound understanding of fundamental mathematics and knowledge of how students learn mathematics, and exposing them to a variety of student-centered teaching strategies. All the mathematics education courses at GCSU provide students with a variety of experiences that include applying problem solving skills to meaningful mathematical problems, exploring mathematics using manipulatives and technology, communicating mathematically through written work and presentations, focusing on the “why” of middle grades mathematics and not the “how to”, reading current research articles in the field of mathematics education, and working with other disciplines to integrate mathematics throughout the curriculum. This focus in mathematics education combined with other education courses and the field-based experiences has proved to produce successful, knowledgeable, and committed middle grades mathematics teachers.

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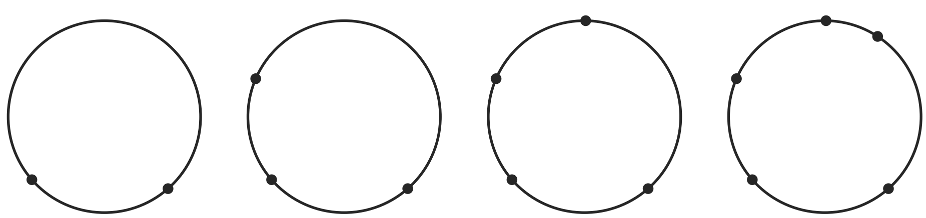
Appendix A
Chords of a Circle

Definition of a chord A line segment connecting any two points on the circumference of a circle.

Part 1

1. Using the circles drawn below, connect each point on the circle to every other point on the circle with a line segment (chord). Record the number of chords needed for each circle in the table and the number of interior regions formed by the chords.



Number of points	2	3	4	5
Number of chords drawn				

2. On the basis of the pattern you observe, what is the largest number of chords that will connect six points on the circumference of the circle? _____

3. Write an equation for determining the largest number of chords that will connect n points on the circumference of a circle. _____

Part II

1. Using the circles above, count the maximum number of interior regions formed in each circle and record your results in the table.

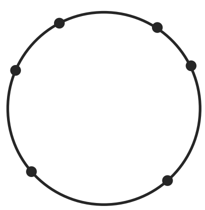
Number of points	2	3	4	5
Number of interior regions				

2. On the basis of the pattern you observe, what is the largest number of regions into which you can divide the interior of a circle by connecting six points on the circumference? _____

3. Write an equation for determining the largest number of regions into which you can divide the interior of a circle by connecting n points on the circumference? _________

Part III

1. Using the circle drawn below, connect each of the six points on the circumference of the circle to every other point on the circle with a line segment (chord). Record the number of chords needed and the number of interior regions formed.



Number of points _____

Number of chords _____

Number of interior regions _____

2. How do these solutions compare with the results found in parts one and two?

3. What can you determine about the generalizations found in parts one and two?

[1] Problem adapted from *Modern Mathematics*, ninth edition by Wheeler and Wheeler.

2

The Mathematics for Middle School Teachers Program at Western Oregon University

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Western Oregon University *Southern Connecticut State University*

2.1 Introduction

Although there is some common pedagogical ground for all teachers of mathematics, there is a fundamental difference in the topics and depth of content knowledge required for students preparing to teach elementary, middle, or high school mathematics. The middle school teacher preparation program at Western Oregon University (WOU) seeks to develop foundational content for middle school teachers while exploring best practices such as active learning, appropriate use of technology, and hands-on exploration. WOU offers many courses specifically for middle school teachers that are designed to develop mathematical maturity and content knowledge while connecting the subject matter to the middle school curriculum and standards. This article describes the structure of WOU's middle school mathematics program and the courses designed specifically for middle school mathematics teachers. We point out the difference in the mathematical preparation and requirements for middle school mathematics teachers compared to elementary teachers and high school mathematics teachers and explain the licensure requirements for middle school mathematics teachers in Oregon.

2.2 Background and Philosophy of the Program

Western Oregon University's math curriculum for K–8 teacher preparation was among ten programs singled out as meeting critical coursework needs by the National Council on Teacher Quality [5]. The teacher preparation program as a whole was named the 2010 recipient of the Christa McAuliffe Award for Excellence in Teacher Education by the American Association of State Colleges and Universities (AASCU). At WOU, students wishing to teach only mathematics at the middle school level and who are not currently licensed can earn an initial license with an elementary/middle level authorization, which allows them to teach grades three through nine in an elementary or middle school only, or a middle/high level authorization, which allows them to teach grades six to twelve. Students seeking elementary/middle level licensure must choose a focus area; one of which is mathematics. An early childhood/elementary authorization is also available that allows teachers to teach children age three through grade eight in an elementary school only. Students wishing to earn an initial license and teach mathematics in the middle school choose the elementary/middle level licensure with a focus area of mathematics. If someone already has a teaching license and wishes to teach mathematics at the middle school level, then he or she pursues a Master of Science in Education with mathematics as the content area. Currently, of the 6,000 students enrolled at WOU, approximately thirty undergraduate students are pursuing an initial license to teach middle school with a mathematics focus area, and ten graduate students are pursuing a Master of Science in Education with mathematics as the content area.

Western Oregon started offering an elementary/middle authorization in mathematics in the mid 1990's. Two of the ideas that have guided the development and evolution of the middle school program and courses are: (1) A middle school mathematics teacher must have advanced content knowledge in the areas they will teach, and also in the subject matter their students will encounter [1]; (2) The content courses designed for mathematics majors do not always prepare future middle school mathematics teachers to connect the topics to the middle school classroom, nor do they always model appropriate pedagogical techniques for presenting the material to young learners. We have developed courses specifically designed to prepare students to be middle school teachers that address these issues.

The middle school mathematics focus courses at WOU include algebra, geometry, calculus, probability and statistics, discrete mathematics, and abstract algebra (see Table 2.1). The College of Education and the Mathematics Department chose these courses for several reasons. First, discrete mathematics, high school level geometry, and probability and statistics have become standard in the middle school curriculum across the nation and in Oregon (see [7]). Second, effective middle school mathematics teachers should help students develop algebraic problem solving skills, prepare students for calculus and help learners develop intuition about subjects like probability and statistics. The concepts of instantaneous and average rates of change taught in the calculus course for middle school teachers can be important when teaching algebra or looking at graphs. Third, a course in abstract algebra can solidify a student's knowledge of the properties of various sets of numbers, which can be helpful when teaching algebra. In particular, associativity and the concept of an inverse are helpful when teaching algebra to middle school students. Finally, students wishing to teach middle school mathematics in Oregon must pass the National Evaluation Series Middle School Mathematics Test (`www.nestest.com/`) and WOU makes sure the students are prepared for passing the exam by offering courses with the appropriate content.

All mathematics focus courses are designed specifically for pre-service or in-service teachers. They are taught by faculty members who have a doctorate in mathematics, but whose research involves mathematics education. The classes are being conducted from both a mathematician's and an educator's point of view. Classes are taught in a cooperative-group, exploratory learning environment. Students present their work and ideas to the class regularly. We believe that when students learn through guided discovery, it enhances their mathematical ability to stand in front of a middle school classroom confidently. Presenting their work and ideas in front of their peers instills confidence, as they are able to practice teaching in an environment where they have to justify their ideas in a clear manner, while being mathematically precise for the instructor and the other students in the class. The classes generally require a final project where students write a lesson plan using the course material in a middle school lesson and present the lesson plan to the class. The final exams are sometimes one-on-one oral exams, where students demonstrate their knowledge and their ability to communicate. Many of the courses use coursepacks written and designed by WOU faculty (materials are available, see [12]).

2.3 Mathematics Course Requirements for Middle School Teachers

There are three options at WOU for earning an initial teaching license: an early childhood/elementary authorization, an elementary/ middle level authorization, and a middle/high level authorization. Pre-service students intending to teach middle school mathematics take the elementary/middle level authorization with a focus in mathematics. Our program for in-service middle school mathematics teachers is a Master of Science in Education degree with a focus in mathematics.

2.3.1 Pre-service Mathematics Course Requirements

All students (regardless of focus area) seeking the initial licensure with an elementary / middle level authorization are required to take six mathematics courses taught by the Mathematics Department. There is a core of five courses: Foundations of Elementary Mathematics I, II, and III (a three-term sequence), Elementary Problem Solving, and College Algebra for Elementary and Middle School Teachers; the sixth course can be any of the 300 or 400 level courses listed in Table 2.1. A standard College Algebra course may be substituted for College Algebra for Elementary and Middle School Teachers for students pursuing the elementary/middle level authorization. Students in the elementary / middle level program pursuing a mathematics focus area are required to take all ten courses listed from Table 2.1. For a description of the courses designed for middle school teachers, see Appendix A.

MTH 211-212-213	Foundations of Elementary Mathematics I, II, and III
MTH 396	Elementary Problem Solving
MTH 392	College Algebra for Elementary and Middle School Teachers
MTH 393	Probability and Statistics for Elementary and Middle School Teachers
MTH 398	Discrete Mathematics for Elementary and Middle School Teachers
MTH 492/592	Abstract Algebra for Middle School Teachers
MTH 494/594	Geometry for Middle School Teachers
MTH 495/595	Calculus Concepts for Middle School Teachers

Table 2.1. Course Offerings in the Mathematics Department for Education Majors (MTH 211-212-213 are 4 credits each, all others are 3 credits. All credit hours are quarter credits.)

The College of Education requires two three-credit courses in mathematics pedagogy. These courses, ED 434: Content Pedagogy I and ED 436: Content Pedagogy II are taken by all students with a mathematics focus or who are seeking to teach the middle/high levels. The College of Education requires students to finish most of their course work before entering the education program, but some students need to take a class while they are busy with student teaching or other requirements that take them off campus during regular school hours. So all the classes labeled "for Elementary Teachers" start at 2 P.M. or later, and the classes labeled "for Middle School Teachers" start at 5 P.M. Occasionally pre-service teachers choose to substitute one of the courses offered during the summer, mentioned in the next section, to satisfy a course requirement.

2.3.2 In-service Course Requirements

Students pursuing a Master of Science in Education at WOU with mathematics as the content area are required to take 18 credits of mathematics. In addition, they take ED 637: Advanced Content Pedagogy-Mathematics through the College of Education. Some of the students receive professional development credits from their districts that can apply towards the 18 credit requirement. Courses with a 500 level designation are considered graduate level courses. Besides the three dual listed courses listed in Table 2.1, students may choose from the six courses listed in Table 2.2. These courses are offered during the summer (two of the courses are offered each summer) on a rotating basis. In Appendix A, we provide a description of the courses, along with references for materials and resources for these courses.

MTH 496/596	Problem Solving for Middle School Teachers
MTH 493/593	Experimental Probability and Statistics for Middle School Teachers
MTH 499/599	Algebraic Problem Solving for Middle School Teachers
MTH 497/597	Discrete Mathematics for Middle School Teachers
MTH 491/591	Historical Topics in Mathematics for Middle School Teachers
MTH 489/589	Algebraic Structures for Middle School Teachers

Table 2.2. Additional Courses for Middle School Teachers seeking an MS in Education

Each of the courses designed for middle school mathematics teachers can be taken for either undergraduate or graduate level credit. In the dual-numbered courses, those seeking graduate credit have additional requirements, which include extra assignments, additional lesson plans in the final project, a more extensive presentation requirement for the final project, and more difficult and additional exam questions.

2.4 University and State Requirements

Besides taking the courses taught by the Mathematics Department, students pursuing an initial licensure to teach middle school mathematics enter the Teacher Education Program, a three-term program offered by the College of Education (see www.wou.edu/education/). As part of the Teacher Education Program, students fulfill the following requirements:

- Forty-one credits in education while in the program, including the two courses in mathematics pedagogy taught in the College of Education.
- Two letters of recommendation documenting their supervised experience with children within the three years prior to entering the program.
- Passing scores for the following examinations before entering the program:
  - NES EAS (Reading, Writing and Mathematics)—Praxis I, CBEST, and West-B are also accepted
  - Oregon Educator Licensure Assessments (ORELA) Multiple Subjects Test I and II
  - Oregon Educator Licensure Assessments (ORELA) Protecting Student and Civil Rights in the Educational Environment
- A passing score for the National Evaluation Series Middle School Mathematics test before beginning Term Two in the Education Program.

Passing the National Evaluation Series Middle School Mathematics test is needed to add a Middle School Mathematics Endorsement to an Oregon Teaching License. If an out-of-state teacher has passed the Praxis: Middle School Mathematics, the scores on this may be used to satisfy the federal definition for "highly qualified teacher". Out-of-state teachers with a mathematics endorsement may be eligible for waiver of the National Evaluation Series Middle School Mathematics test if they meet Oregon's minimum academic preparation (see [6] www.ode.state.or.us/ for more information).

2.5 Comparison to Elementary and High School Requirements

Students pursuing the early childhood / elementary licensure are required to take five mathematics courses from the Mathematics Department and one course on mathematics (and science) pedagogy from the College of Education. They take the three-course Foundations of Elementary Mathematics sequence (MTH 211-212-213) and the Elementary Problem Solving course (MTH 396). The students also must choose one of the 300 level mathematics courses from Table 2.1 or they may take MTH 394: Geometry for Elementary Teachers.

Students wishing to teach high school mathematics pursue the middle/high level licensure option. The courses for elementary or middle school teachers do not satisfy the degree requirements. These students take the classes designed for mathematics majors. In fact, their required coursework in mathematics differs by only eight credits from the coursework for mathematics majors. Anyone wishing to teach high school mathematics must pass the ORELA NES: Mathematics test.

2.6 Assessment

There are a number of ways that a teacher preparation program can assess its level of success. One measure is to examine job placement and retention statistics. Because of the weak economy in Oregon there are few jobs available for new graduates and many former graduates have been laid off. Employment statistics are thus a poor measure of program quality.

A second way is to look to the results of outside evaluations. The Western Oregon University teacher preparation program has been continuously accredited by National Council for Accreditation of Teacher Education (NCATE) and the Oregon Teachers Standards and Practices Commission (TSPC) since 1954 and is one of only eight nationally accredited teacher preparation programs in Oregon. The mathematics teacher preparation program was recognized by the National Council on Teacher Quality and the program as a whole was named the 2010 recipient of the Christa McAuliffe Award for Excellence in Teacher Education by the American Association of State Colleges and Universities (AASCU). The accreditation history and award recognition indicate that the program is successfully preparing middle school mathematics teachers.

Third, we can look to the research and compare our program with the guidelines set forth for effective teacher preparation programs. Researchers have argued that teacher candidates need a strong background in mathematics—but not any mathematics will do [2]. The research tells us that pre-service teachers need mathematics that is specific to how children learn and that explores K–12 mathematics content at a deep level. With this in mind, we have designed the coursework at Western Oregon University to align with this research and feel that we are meeting these criteria.

2.7 Conclusion

We have discussed the implementation of Western Oregon University's successful program for the training of middle school mathematics teachers. The idea that middle level teachers need specific preparation is supported by the Conference Board of Mathematical Sciences' recommendation in the Mathematics Education of Teachers document [4]. We hope that readers can use this information to aid in the development of their own middle school specific program. Many resources from our courses developed at WOU especially for the preparation of middle school teachers are available at our website [12]. Contact information is available on there and the authors would be happy to answer any questions about the program or materials.

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Appendix A
Courses for Middle School Teachers

Below are the descriptions for the courses specifically designed for middle school teachers. The WOU course materials website [12], `www.wou.edu/math/msmathed`, has links to many of the instructional materials referenced below.

MTH 392: College Algebra for Elementary and Middle School Teachers and MTH 489/589: Algebraic Structures for Middle School Teachers These courses emphasize the material that a future teacher needs to understand to teach an introductory course in algebra, using the algebra pieces designed by the Math Learning Center (www.mathlearningcenter.org). In the first part of the courses, students use black and red tiles to model and perform integer operations. Students then use black and red tiles as well as algebra pieces to model algebraic patterns, solve basic algebra problems, and investigate features of linear functions such as intersections, intercepts and slope.

The next portion of the courses looks at the graphs and equations of absolute value, quadratic functions, and inequalities involving linear and quadratic terms. Students learn how to distinguish quadratic functions from linear functions, to graph quadratic functions, and to use algebra pieces to find the key features of quadratic graphs and their points of intersection with other quadratic and linear graphs. Students use squares and square roots while solving quadratic equations and learn how to complete the square to find the quadratic formula. This is the last portion of the *College Algebra for Teachers* course. Students taking the *Algebraic Structures for Middle School Teachers* course also work with polynomials of higher degree and with complex numbers.

Throughout the courses, after exploring a topic using algebra pieces, students discuss techniques in solving standard exercises and word problems without the use of algebra pieces. The activity book *Visual Algebra for College Students* was written by Laurie Burton, WOU (see [3] for a paper on these materials).

MTH 491/591: Historical Topics in Mathematics for Middle School Teachers

This course offers a survey of the historical development of topics in mathematics from ancient to modern times, with emphasis on topics in arithmetic, algebra, and informal geometry. The textbooks for this course are *Historical Connections in Mathematics, Volumes I, II and III* [8] and *Mathematicians are People, Too, Volumes 1 and 2* [9, 10].

MTH 492/592: Abstract Algebra for Middle School Teachers

This course covers numbers sets and properties, elementary group theory, with applications to the middle school classroom. It starts with an in-depth exploration of the properties of commutativity, associativity, closure, identity, and inverse, studied with respect to the operations of addition, subtraction, multiplication, and division in the setting of the whole numbers, integers, rational numbers and real numbers. This leads into the study of groups and subgroups using these familiar sets and operations as examples and non-examples. Modular groups, matrix groups, and the dihedral groups are also considered.

Two applications of the group theoretic topics suitable for both college and middle school students are cryptography and math art posters. The unit on secret codes covers simple ciphers such as shift ciphers that use modular arithmetic. The students can connect the ideas of identity and inverse to encryption and decryption. The students create math art posters using Cayley tables from modular groups (see www.wou.edu/~burtonl/apps.html). The course uses a coursepack developed by WOU faculty members Laurie Burton and Cheryl Beaver. Sample materials are available [12].

MTH 493/593: Experimental Probability and Statistics for Middle School Teachers

This course is designed to help future teachers develop insights into probability and statistics to help them make the topics accessible to their students. The subject matter includes data types and random variables, data collection and sampling techniques, bias, displaying and describing distributions, measures of center and variability, correlation versus causation, experiments and simulations, elementary probability, the normal and binomial distributions, and the Central Limit Theorem. Topics covered in probability include permutations and combinations, counting techniques, and the addition and multiplication rules.

The material is taught using real-life situations and real data whenever possible. Students are challenged to think critically about displays and interpretations of data especially as commonly seen in politics and advertising. About 40 percent of the class is taught using computer applications. The software package Excel is used to look at large data sets and graphs, to analyze and interpret data, and to perform statistical tests on the data. Sample course materials developed by Scott Beaver of WOU are available [12].

MTH 494/594 Geometry for Middle School Teachers

This course begins by reviewing students' knowledge of geometric terms and proof techniques, including using triangle congruency theorems, deriving the formula for the sum of the measures of the interior angles of an n-sided polygon, and proving that there are only three regular polygons that can tessellate the plane. Students then explore straightedge and compass constructions, and the proofs that they are correct. Primarily using a straightedge and compass, students learn about the four isometries (translation, rotation, reflection, and glide reflection) and determine whether composition of two isometries is commutative. Using geoboards and solids, students derive various area and volume formulas. They then use them, when working with dilations, to determine the relationships of a figure and its similar figure with respect to perimeter, (surface) area, or volume. During the last day or two of class, each student does research on and presents a brief lesson to the class on a topic not previously discussed. Examples of such topics include the history of π, the nine-point circle, taxicab geometry, and M.C. Escher. Students really seem interested in these topics unfamiliar to them.

About 50 percent of the class is taught using *The Geometer's Sketchpad*. Students use it to help with their conjectures and problem solving. Materials from this course are an updated version of a coursepack originally from Portland State University that focuses on visual learning. Sample course materials are available [12].

MTH 495/595: Calculus Concepts for Middle School Teachers

Students in this course explore the traditional calculus concepts of slopes, limits, derivatives, and anti-derivatives. The topics are introduced and developed using tables, graphs, and real-life examples. Formulas are derived and applied in examples. The class is taught using group work and exploratory worksheets allowing the students to develop intuition about calculus concepts and develop their problem solving skills. Other topics covered include higher derivatives and how they relate to the shape of a graph, optimization problems, and area and volume applications.

The WOU faculty has written a coursepack for this course using hands-on examples, explanations, and applications drawn from a Portland State University coursepack and traditional calculus textbooks. This coursepack is available [12].

MTH 496/596 Problem Solving for Middle School Teachers
MTH 499/599 Algebraic Problem Solving for Middle School Teachers

These courses offer students the opportunity to master a variety of problem solving strategies. They focus on writing word problems appropriate for middle school students and on developing and using scoring rubrics. The *Problem Solving for Middle School Teachers* course considers strategies such as guess and check, tables, diagrams and pictures, exhaustive search, logic, looking for patterns, Venn diagrams, and algebra. The *Algebraic Problem Solving for Middle School Teachers* course considers problem solving strategies that lead to algebraic techniques and notes the connection between various problem solving strategies and algebra. There is an emphasis on ratio and percentage problems, rate problems, and work problems as these arise frequently in the middle school classroom. Both courses use a coursepack developed by Cheryl Beaver of WOU. Sample course materials are available [12].

MTH 497/597: Discrete Mathematics for Middle School Teachers

Students in this course are introduced to discrete mathematics through a variety of hands-on activities. For the first few weeks of the course, students learn about the pigeonhole principle, set theory (including the inclusion-exclusion principle using attribute pieces), and arithmetic series. The next portion of the course focuses on counting: the multiplication rule, and permutations and combinations. Pascal's triangle and its connections to combinations and the binomial theorem are discussed. An emphasis is placed on the derivation of the formulas for permutations and combinations and their relationship to the binomial theorem. Students use their knowledge of basic counting techniques to get an insight into game show games, lotteries, poker, and Tic-Tac-Toe in three dimensions. The last part of the course has the most hands-on activities, where students learn about fractals like the Sierpinski gasket and Koch's snowflake, and about graph theory. Topics in graph theory include complete graphs, bipartite graphs, trees, vertex coloring, and map coloring. As with the Geometry for Middle School Teachers class, the last days of class have students presenting lessons on topics not previously discussed, like the game of Nim and magic squares. In the graduate level course, there is a more emphasis on proofs.

Sample activities in counting and graph theory used in this course are available [12].

ED 434 and ED 436: Content Pedagogy I and II

This is a two-term sequence of courses that focuses on the process of teaching mathematics. While content is included in the course, it is not the focus. The bulk of the course sequence discusses the NCTM Principles: Equity, Curriculum, Teaching, Learning, Assessment, and Technology. The course begins with work comparing the national standards (NCTM) and the state standards (Common Core State Standards). Classroom activities incorporate discussions, exploration of mathematical tasks, student presentations, and reading activities. Assignments include designing activities including a problem based lesson, analyzing video of practicing teachers, creating proficiency based assessments, reflecting on mathematics education research, analyzing individual teaching practice, and modifying curriculum to raise the cognitive demand.

Appendix B: Description of Foundation of Elementary Mathematics Series and Elementary Problem Solving courses

MTH 211–MTH 213: Foundations of Elementary I–III
In MTH 211, the first of the three-course sequence, students are introduced to problem solving, whole number and integer operations, and number theory. Fractions, decimals, and probability and statistics are the main areas studied in MTH 212. Students explore a variety of topics in geometry in MTH 213. Although some of the material may be review for the students, the goal of the courses is to give students a deeper understanding of the material, in particular, why the rules they have learned are true and how to get students to correct their misconceptions. Each course uses a group work / discovery-based learning style and makes extensive use of manipulatives.

MTH 396: Elementary Problem Solving
The focus of this class is problem solving. Students solve problems using a variety of techniques, practice creating their own word problems to be solved using a given strategy, and assess student problem solving skills through a mentoring program with local elementary school children.

Appendix C
Comparative Summary of Mathematics Requirements

Authorization level	Eligible to teach	Mathematics Courses Required from Mathematics Department	Mathematics Focus Courses Required from Education Department	Required state tests
Early Childhood /Elementary	Age three through grade 8 in an elementary school only	MTH 211-212-213, MTH 396, one elective from {MTH 392,393, 394,398}	ED 453	CBEST[1] ORELA[2]
Elementary/Middle with a Math Focus	Grades 3 through 9 Mathematics in an elementary or middle level school only	MTH 211-212-213, MTH 396, MTH 392, MTH 393, MTH 398, MTH 492, MTH 494, and MTH 495	ED 434, ED 436	CBEST ORELA National Evaluation Series Middle School Mathematics test
Elementary/Middle	Grades 3 through 9 in an elementary or middle level school only	MTH 211-212-213, MTH 396, MTH 392, and one elective from {MTH 393, 398, 492, 494, 495}	ED 434, ED 436	CBEST ORELA
Middle/High with a Math Focus	Grades 6 through 12	The required mathematics courses differ from those of a mathematics major by only two courses. No courses are specifically designed for pre-service teachers.	ED 434, ED 436	CBEST ORELA National Evaluation Series Middle School Mathematics test

Table 2.3. Initial License Mathematics Requirements at WOU

[1] California Basic Educational Skills Test
[2] Oregon Educator Licensure Assessments Multiple Subjects Test

3

Connecting Middle School Mathematics with College Mathematics

A Core of Mathematics Courses for Middle Grade Mathematics Teachers

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3.1 Introduction

A common response when questioning in-service mathematics teachers about their mathematical preparation for teaching is that their college mathematics courses did not adequately prepare them to teach school mathematics because the college courses often failed to connect the mathematics they were learning with the school mathematics that they would be teaching. Though the courses were rich in mathematical ideas, their connections to important concepts in school mathematics were not always explicitly detailed. Several factors contributed to this deficiency, but the most prominent one was the lack of high quality textbooks that identify and explain the critical connections. Without such materials, it is challenging and time consuming for mathematicians, who primarily teach content courses for pre-service teachers and who are typically unfamiliar with school mathematics curricula, to make these critical connections.

To help address the need for specialized courses and materials for pre-service mathematics teachers, the Conference Board of Mathematical Sciences, and the Mathematical Association of America (with funding provided by the United States Department of Education), developed the *Mathematical Education of Teachers Report* (*MET*), [2]. It gives a framework for mathematics content courses for prospective teachers that is built on the premise that “The mathematical knowledge needed for teaching is quite different from that required by college students pursuing other mathematics-related professions. Prospective teachers need a solid understanding of mathematics so that they can teach it as a coherent, reasoned activity and communicate its elegance and power.” The report asks mathematicians to rethink courses for prospective teachers and provides curriculum and instruction recommendations for the mathematical preparation of K–12 teachers:

1. Prospective teachers need mathematics courses that develop a deep understanding of the mathematics they will teach.
2. Although the quality of mathematical preparation is more important than the quantity, the following amount of mathematics coursework for prospective teachers is recommended.
   (a) Prospective elementary grade teachers should be required to take at least 9 semester-hours on fundamental ideas of elementary school mathematics.

(b) Prospective middle grade teachers of mathematics should be required to take at least 21 semester-hours of mathematics, that includes at least 12 semester-hours on fundamental ideas of school mathematics appropriate for middle grade teachers.
(c) Prospective high school teachers of mathematics should be required to complete the equivalent of an undergraduate major in mathematics that includes a 6-hour capstone course connecting their college mathematics courses with high school mathematics.

3. Courses on fundamental ideas of school mathematics should focus on a thorough development of basic mathematical ideas. Along with building mathematical knowledge, mathematics courses for prospective teachers should develop habits of mind of a mathematical thinker and demonstrate flexible, interactive styles of teaching.
4. Teacher education must be recognized as an important part of mathematics departments' mission at institutions that educate teachers. More mathematicians should consider becoming deeply involved in K–12 mathematics education.

3.2 Developing a Core Mathematics Curriculum for Middle Grade Teachers

Using the *MET Report* as a framework, a group of research mathematicians and mathematics educators at the University of Missouri-Columbia developed four foundational college-level mathematics courses for pre-service middle grade teachers and accompanying textbooks as part of the NSF funded project, *Connecting Middle School and College Mathematics* [$(CM)^2$], 2001–2006. The courses and materials were designed to provide pre-service and in-service middle grade mathematics teachers with a strong mathematical foundation and connect the mathematics they are learning with the mathematics they will be teaching.

The four undergraduate mathematics courses focus on algebra and number theory, geometric structures, data analysis and probability, and the mathematics of change (cf., Math 4060, Math 4070, Math 4080, and Stat 4050). These courses (12 credit hours) serve as the core of the 29 credit hour mathematics content area of the College of Education's middle school mathematics certificate program at the University of Missouri-Columbia and are offered on a staggered semester schedule (with 10–30 students per course, including mathematics education graduate students registered with cross listed course numbers). The other mathematics courses in the program (17 credit hours) are standard (general population) mathematics courses and have not been tailored for middle grade teachers. In addition to the 29 credit hour mathematics content requirements, all students at the University of Missouri must take College Algebra, or qualify or test out of this general education requirement.

Mathematics Content Requirements for a Middle School Certificate (29 hours)
Stat 1200: Introduction to Statistical Reasoning (3 credits)
Math 1160: Pre-Calculus (5 credits)
Math 1300: Finite Mathematics (3 credits)
Math 1360: Geometric Concepts (3 credits)
Math 4060: Connecting Geometry to Middle School and Secondary School (3 credits)
Math 4070: Connecting Algebra to Middle School and Secondary School (3 credits)
Math 4080: Calculus for Teachers (3 credits)
Math 2320: Discrete Mathematics (3 credits)
Stat 4050: Connecting Statistics to Middle School and Secondary School (3 credits)

Mathematics Methods and Associated Field Experience
TDP 4360: Introduction to Teaching Mathematics in Middle and Secondary School (3 credits)
TDP 4364: Field Experience for TDP4360 (1 credit)
TDP 4370: Teaching and Modeling Middle School Mathematics (3 credits)
TDP 4374: Field Experience for TDP 4370 (1 credit)

See `registrar.missouri.edu/degrees-catalogs/2010-2012a/index.php` under the University of Missouri College of Arts and Sciences and the College of Education for current and detailed course descriptions.

Also see dese.mo.gov/divteachqual/teachcert/certclass.html for a description of the Missouri Department of Elementary and Secondary Education requirements for middle school mathematics teaching certification.

Mathematics and statistics faculty and graduate assistants in the College of Arts and Sciences teach the mathematics courses in the program, and mathematics education faculty and graduate assistants teach and supervise the methods and field experience components. For in-service elementary or middle grade teachers seeking graduate mathematics experiences to improve their mathematics content knowledge, cross listed extended versions of the four core courses developed under the $(CM)^2$ Project are offered for graduate credit. The middle school mathematics teacher education program at the University of Missouri-Columbia graduates on average ten teachers per year.

The College of Education mathematics content requirements for the middle school mathematics certificate program at the University of Missouri-Columbia are not the same as those for elementary or secondary certificates. Prospective elementary teachers are required to take College Algebra, Elementary Statistics, and a yearlong methods and content course on number, operation, and geometry (a total of 12 credit hours). Secondary mathematics education majors have a 40 credit hour content requirement consisting of Calculus I, II, III, Discrete Mathematics, Connecting Geometry to Middle School and Secondary School (one of the $(CM)^2$ Project courses), Matrix Theory, Higher Algebra (a course in Abstract Algebra designed for secondary mathematics teachers), non-Euclidean Geometry, Connecting Statistics to Middle School and Secondary School (one of the $(CM)^2$ Project courses), Introduction to Problem Solving and Programming (Computer Science), and 6 credit hours of upper level mathematics courses. See www.math.missouri.edu/degrees/undergraduate/courses.html for undergraduate mathematics course descriptions.

To help students explore and learn mathematics in greater depth, the textbooks that were developed as part of the $(CM)^2$ Project have a unique design feature that uses current middle grade mathematics curricular materials in the following multiple ways:

- As a springboard to college-level mathematics
- To expose future (or present) teachers to current middle grade curricular materials
- To provide motivation to learn more and deeper mathematics
- To support curriculum dissection—critically analyzing middle school curriculum content—developing improved middle grade lessons through the lesson study approach
- To use college content to gain new perspectives on middle grade content and vice versa
- To apply middle grade instructional strategies and multiple forms of assessment to the college classroom.

The $(CM)^2$ materials were used in Summer Institutes for in-service elementary and middle grade teachers, in courses offered at school-based sites for in-service elementary and middle grade teachers, in professional development programs, and in undergraduate and graduate semester courses offered at a number of universities throughout the nation. The mathematical content of the materials was shaped by the thoughtful and insightful comments of those piloting them. The four textbooks were improved by the shared expertise of the classroom teachers, mathematics educators, statisticians, and mathematicians who examined and used them.

The $(CM)^2$ college-level mathematics texts are now in print (published by Prentice Hall, 2005, 2006) and we give a brief content description of each (for a more detailed description of the textbooks, go to: vig.prenhall.com/ and search for Connections in Mathematics Courses for Teachers):

- *Algebra Connections* by Ira Papick [6]. The Fundamental Theorem of Arithmetic and related notions such as greatest common divisor and least common multiple are fundamental in algebra and number theory. It is important that middle grade teachers reach a firm understanding of them through various problem-solving experiences in mathematical contexts, and real world settings. Course topics include a detailed study of the integers with special emphasis on the division algorithm, least common multiple, greatest common divisor, Euclid's algorithm, prime numbers, The Fundamental Theorem of Arithmetic, Pythagorean triples, modular arithmetic and algebra, rational and irrational numbers, and linear systems of equations and matrices.

- *Calculus Connections* by Asma Harcharras and Dorina Mitrea [4]. Patterns, repeating decimals, motion, length, surface area, and volume are fundamental topics in middle school mathematics curricula. An understanding of them by middle school mathematics teachers can be attained by a study of the key elements of the mathematics of change. The book is designed for this purpose; its focus is on important concepts of calculus and their connections to the middle grades curricula. Course topics include concepts of calculus such as sequences, series, functions, limits, continuity, differentiation, anti-differentiation, and other mathematics of change with connections to the middle grades curricula.
- *Geometry Connections* by John K. Beem [1]. Geometry in middle schools is increasingly being taught with more ties to other parts of mathematics and with more real world applications. Middle school mathematics teachers must have a fundamental understanding of the structure of geometry and of how it is used in applications that middle school students can understand. In this book, students are encouraged to use Geometer's Sketchpad and to think about how they might use it and other tools to present mathematics in the middle grades. Course topics include logic, similarity, surface area, volume, coordinate geometry, vectors, transformations, and the use of matrices in geometry. Some comparisons of Euclidean geometry with spherical geometry and taxicab geometry are included.
- *Probability and Data Analysis Connections* by Debra and Michael Perkowski [8]. Future middle school teachers are often required to take a statistics course, but many see little connection between it and their future in teaching. This book addresses this problem by explicitly relating college-level statistics to middle school curricula. It also mirrors the middle school classroom by including explorations with technology and activities for group work and discussion. Course topics include: collecting and displaying data, describing data with numbers, correlation with regression, probability, methods for counting outcomes, random variables and probability distributions, estimation, and hypothesis testing.

Each book contains *Classroom Connections, Classroom Discussions,* and *Classroom Problems*. They are designed to deepen the connections between the college-level mathematics students are studying and the mathematics they will be teaching. The *Classroom Connections* are middle grade investigations that serve as launch pads to the college-level *Classroom Discussions, Classroom Problems,* and other collegiate mathematics. The *Classroom Discussions* are mathematical conversations between a college teacher and pre-service middle grade teachers, and are used to introduce and explore a variety of important concepts during class periods. The *Classroom Problems* are a collection of problems with complete or partially complete solutions that are meant to demonstrate and engage pre-service teachers in various problem solving techniques and strategies. The process of connecting what they are learning in the college classroom to what they will be teaching in their classrooms provides teachers with motivation to strengthen their mathematical content knowledge.

3.3 Connecting Middle School Mathematics Curriculum with College-level Mathematics Content: Examples from Algebra and Geometry

To illustrate the $(CM)^2$ structured relationship between middle grade mathematics material and college-level mathematics content, we will discuss how some important mathematical ideas might unfold in an Algebra Connections and Geometry Connections classroom. For a discussion concerning a typical lesson in a Probability and Data Analysis Connections classroom, see the Tarr, Papick paper [9].

3.3.1 Algebra Connections Classroom

A class period begins by dividing the students into small working groups (2 to 4 students per group) that investigate some important ideas in mathematics as they are laid out in a specific lesson from a middle grade curriculum. For example, the class might examine the sixth grade unit in *Connected Mathematics Project* (1st Edition) [3] called *Prime Time*. It considers prime numbers, factorization, least common multiple (LCM), and greatest common divisor (GCD), and sets the stage for a college-level study of the Fundamental Theorem of Arithmetic.

A lesson from *Prime Time,* (Investigation 4, Common Factors and Multiples) begins with a discussion factors, multiples, common factors, and common multiples, and then gets the students to investigate them through real world problems. Working through the problems in this unit gives the pre-service middle grade teacher experiences with concepts they will be teaching. Using middle school materials provides them with opportunities to think about how they might solve the problems, how sixth grade students might comprehend and solve the problems, and how they might help their students understand and learn the mathematical ideas in the lesson.

After the groups report their findings and methods to the rest of the class, it is instructive to reflect on the fundamental mathematical notions that were encountered while working on the problems (multiple, common multiple, least common multiple), and begin to formalize some definitions and terminology. This formalization is critical, since the validity of mathematical conclusions can be verified only when tested against precise definitions or valid deductions.

In addition to investigating the mathematics in the middle grade unit, the pre-service teachers extend their knowledge through a study of:

- Expressing the LCM and the greatest common divisor (GCD) of two integers in terms of their prime factors,
- Determining the relationship between the LCM and GCD,
- Using and justifying Euclid's algorithm to compute the GCD (and the LCM) of two integers,
- Using Euclid's algorithm to write the GCD of two integers as a linear combination of them,
- Deriving elementary number theoretic facts by exploiting the linear combination identity.

The development of these ideas would span several class periods and would be integrated into other topics of study.

3.3.2 Geometry Connections Classroom

Prospective teachers are motivated when the mathematics covered in a college-level mathematics course is directly related to materials that they might use as teachers. A good method is to have a team of two to four students prepare and deliver a lesson from a geometry unit in a middle grades curriculum. The presentation should explain the fundamental mathematical ideas. This process requires that the teachers thoroughly understand the middle grade materials, and they understand how the middle school mathematics concepts are connected to the college-level content they have studied.

For example, students might present part of the unit from *Connected Mathematics Project* (1st Edition) [3], called *Stretching and Shrinking*. In this unit, middle school students examine similarities in the xy-plane using graph paper. Given a figure defined by coordinates for corners, middle school students use simple rules to construct related figures and decide which new figures are similar to the original figure and which are not. Future middle school teachers quickly come to understand the need for them to have a strong mathematical background in coordinate geometry and transformations. It also provides an opportunity to use proportional reasoning, important for middle school students. The investigation establishes a foundation for the study of groups of geometric transformations encountered later in the course, and the role of similarity in Euclidean geometry and other geometries.

Similarity is further studied in the Geometry Connections course when The Geometer's Sketchpad (GSP) is introduced. Sketchpad is useful in studying similar figures and similarity transformations. The prospective middle grade teachers readily grasp how easy it is to use Sketchpad as an effective learning and teaching aid. It is instructive in the Geometry Connections classroom to have a team of pre-service teachers present a lesson to the class illustrating how Sketchpad can be used to teach some aspect of geometry such as similarity.

3.4 Effectiveness of Courses and Materials

As part of the required NSF evaluation of the $(CM)^2$ Project, the project evaluator collected and analyzed evaluation data from the teachers who participated in the $(CM)^2$ Summer Institutes and the core courses offered at school-based sites. They were uniformly positive and showed that the courses and materials helped the teachers to expand and deepen their mathematical content knowledge, facilitate their learning, improve their teaching practice by using collaborative learning strategies, lesson study models, and student centered learning, etc., and provide them with resources that they could use in their classrooms. They appreciated seeing and understanding the connections between what they were learning and what they would be teaching to their future students.

3.5 Concluding Remarks

The $(CM)^2$ core mathematics courses and materials were developed to help middle grade teachers gain a deep understanding of the key mathematical ideas that they will be teaching so that they can help their students learn important mathematics. The underlying principle was that prospective mathematics teachers should learn significant college-level mathematics and understand the connections between what they are learning and what they will someday be teaching. We hope that our work contributes to meeting this goal. In an effort to improve mathematics teacher education and to sustain our work at the University of Missouri-Columbia, fifty percent of the royalties from the four Connections textbooks go to a scholarship fund for future middle and secondary mathematics teachers.

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4

The Middle School Program at the University of Wisconsin Oshkosh

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4.1 Introduction

"The key to distinguishing the knowledge base of teaching lies at the intersection of content and pedagogy, in the capacity of a teacher to transform the content knowledge he or she possesses into forms that are pedagogically powerful and yet adaptive to the variations in ability and background presented by the students."
— Lee S. Shulman, 1986 [20]

As mathematicians and educators, we are faced with the challenge of preparing future generations of middle grades teachers of mathematics. This is a daunting task indeed, given that many prospective and practicing teachers reveal a paucity of mathematics content knowledge [1, 11] and have overtly authoritarian beliefs about the nature of mathematical behavior [18, 19]. We are left to wonder how to best design our programs for prospective middle grades teachers. Should we emphasize mathematics content or stress mathematics methods and connections to the school curriculum? What forms should our courses take and which faculty members are best qualified to teach them?

At the center of this problem lies a tension between the education and the mathematics communities over the types of mathematical knowledge that teachers need. The research literature distinguishes several components: content knowledge, knowledge of learning and the learner, general pedagogical knowledge, and pedagogical content knowledge. A synthesis of this literature can be found in [6], where the authors propose that teacher knowledge is both situated and changeable. "Within a given context, teachers' knowledge of content interacts with knowledge of pedagogy and students' cognitions and combines with beliefs to create a unique set of knowledge that drives classroom behavior" (p. 162). That is, teachers must mediate between the understanding and preconceptions brought by their students and the mathematical demands of the content.

In our program we attempt to support prospective middle grades teachers in three ways. First, we want our students have a deep understanding of middle school mathematics, its definitions, representations, and connections to the broader mathematical landscape. Second, we want them to view mathematics as the human activity of making sense of patterns, seeing it as something that they and their future students can do together. Third, we aim to help them to become sufficiently mathematically sophisticated so they can make judgments about their future students' conjectures, make mathematical arguments and construct counterexamples, and create examples and non-examples of mathematical objects.

While our thinking about middle grades teacher education has evolved during the past fifteen years, we should mention that impetus for the program was the 1995 Middle Math Program Conference at East Carolina University, which focused on disseminating NSF-sponsored middle grades curriculum projects: *MathScape* (Glencoe/McGraw-Hill), *MATH Thematics* (McDougal Littell), *Connected Mathematics Project* (Pearson), and *Mathematics in Context* (Brittanica Inc.). Based on what we learned at the conference and our study of these middle grades curriculum materials we designed our middle school program to model and make explicit the mathematical processes that will be expected of our students as future teachers.

In this chapter, we will describe our program: our educational philosophy, our courses, our pedagogy, and our materials. We are fortunate to have relatively good students and the departmental support to offer a sequence of courses specifically for future middle grades teachers. We hope that some of our work will translate to other settings and provide ideas for creating a program that could work elsewhere.

4.2 The Context

"Wisconsin's tradition of quality public education for all has paved the way for the achievements of those who grew up here. There are countless Wisconsin success stories that reveal the true value of our public educational system." — Elizabeth Burmaster, Wisconsin State Superintendent, 2008

In Wisconsin, middle school teachers are either elementary education majors with a Grades 1–8 license or secondary education mathematics majors with a Grades 6–12 license. We have no certification specifically for the middle school but the Wisconsin Department of Public Instruction does confer a Mathematics Minor endorsement.

The University of Wisconsin Oshkosh (UWO) is a comprehensive state university with approximately 11,000 undergraduate and 1000 graduate students. Elementary education serves almost 1000 students and is the largest major at UWO. All elementary education majors (with the exception of special education majors) are required to complete three 3-credit mathematics content courses: Number Systems, Geometry and Measurement, and Data Exploration and Analysis. In addition to the above coursework, the students complete a 3-credit mathematics methods course and a practicum. At UWO, a passing score on the PRAXIS I is required for admission to the elementary education major.

We prepare middle school teachers from the population of elementary education majors by providing them with a mathematics minor program, which we refer to in this paper as the middle school program. Students who enter the program typically have done well in the required courses mentioned, and we actively recruit many of them. Approximately 8% of the elementary education majors complete our middle school program, although some who do will teach elementary school rather than middle school. This means that we are able to offer four courses in the program each academic year with an average of enrollment of about 24 students per course. Annually we prepare approximately 20 students to serve as middle grades mathematics teachers or elementary mathematics specialists. Secondary education students complete a mathematics major composed of courses entirely disjoint from that of the middle school program.

4.3 The Middle School Program at UWO: An Overview

"This [program] showed me that math was not just a complicated, rule-filled, difficult subject, but a beautiful one. I learned to be a resourceful teacher, an attentive teacher, a constructivist teacher, and a math advocate." — Michelle Warzyn (student), 2008

In addition to completing Number Systems, Geometry and Measurement, and Data Exploration and Analysis, students in our middle school program complete three of four mathematics content electives and a 3-credit capstone Senior Seminar for a total of 24 credit hours of mathematics. Each of the four 4-credit electives Modern Algebra, Modern Geometry, Infinite Processes, and Probability and Statistics is designed specifically for future middle grades teachers and elementary mathematics specialists, and only the students in the middle school program take them. A grade of C or better in Intermediate Algebra is prerequisite to the Number Systems course and is thus a prerequisite for the program.

To summarize, our program consists of three courses taken by all elementary education majors (for sample syllabi, follow the links below)

Number Systems (3 credits) `www.uwosh.edu/mathematics/MAA/NumberSystems.Syllabus.pdf`

Geometry and Measurement (3 credits)
`www.uwosh.edu/mathematics/MAA/GeometryMeasurement.Syllabus.pdf`

Data Exploration and Analysis (3 credits)
`www.uwosh.edu/mathematics/MAA/DataExplore.Syllabus.pdf`

three of four mathematics electives taken only by students in the middle school program

Modern Algebra (4 credits) `www.uwosh.edu/mathematics/MAA/ModernAlgebra.Syllabus.pdf`
Modern Geometry (4 credits) `www.uwosh.edu/mathematics/MAA/ModernGeometry.Syllabus.pdf`
Infinite Processes (4 credits)
`www.uwosh.edu/mathematics/MAA/InfiniteProcesses.Syllabus.pdf`
Probability and Statistics (4 credits)
`www.uwosh.edu/mathematics/MAA/ProbabilityStatistics.Syllabus.pdf`

and a required capstone course:

Senior Seminar for Elementary and Middle School Programs (3 credits)
`www.uwosh.edu/mathematics/MAA/SeniorSeminar.Syllabus.pdf`.

In Wisconsin, our program constitutes a licensable minor. Students must earn a 3.0 GPA in the courses in order to obtain that endorsement on their teaching licenses.

4.4 Philosophy

"It is impossible to distinguish and thus contrast the interpretation of a thing from the thing itself . . . because the interpretation of the thing is the thing." — Mehan & Wood [15]

The middle school program is housed in the mathematics department, and the courses are taught by five faculty members who are both mathematicians (as evidenced by our doctorates) and mathematics educators (as evidenced by our research programs). Thus our pedagogy is informed by our training as mathematicians and our work as educators.

First, we are mathematicians and representatives of the mathematical community. Our community holds a set of values, mathematical tools, and distinctions about language that allow us to learn and develop new mathematics and to solve problems. Mathematicians value problems, careful definitions of objects, deductive arguments, and shared notations. They use logic, create examples and counterexamples, consider extreme or trivial cases, and make models for problems. They distinguish necessary from sufficient conditions, pay close attention to quantifiers, and are sticklers for careful language. Our program is designed to help make this culture transparent to our students both in the way we speak about mathematics and in the way we do mathematics with them in class.

Second, we are constructivists. We believe that knowledge must be conceived by the learner, who will create conceptions based on prior meanings he or she has made and considers to be relevant. In a larger sense, we believe that knowledge is created by a culture, and that learning is a process of acculturation. "[T]he understanding of learning and teaching mathematics support[s] a model of participating in a culture rather than a model of transmitting knowledge. Participating in the processes of a mathematics classroom is participating in a culture of using mathematics, or better: a culture of mathematizing as a practice" [3, p. 4]. We have designed our courses to support mathematizing.

Third, we are teacher educators concerned with preparing our students for their roles as teachers and for the demanding mathematical work of teaching. The work of the elementary or middle grades teacher has been described as negotiating meaning for language about doing mathematics [9] and framing paradigm cases and initiating whole class discussions of student obligations and expectations with respect to them [22]. Cobb et al describe the teacher's role as ". . . giving commentary from the perspective of one who could judge which aspects of the children's activity might be mathematically significant" [5, p. 262]; initiating ". . . shifts in the discourse such that what was previously done in action can become an explicit topic of conversation" (p. 269); and developing symbolic records of participant contributions to the discourse. In the program, we model these aspects of teaching practice and we draw explicit connections between the content of our courses and the middle grades curricula. Furthermore, we demand that our students practice the mathematical work of teaching through their daily participation in class.

4.5 The Content Electives

"This class went above and beyond my expectations. Mathematics was seen as not just rules, computing, and memorization, but a magnificent world of patterns that have meaning behind every page. This class showed me that math did not need to be complicated and filled with rules and formulas, but a subject that is beautiful and full of logic and reasoning." — Amy Herman (student), 2008

The elective content courses are activity-based and designed to encourage students to develop deep understandings of mathematics and the mathematical thinking of children. Each course requires that students practice mathematically powerful ways of thinking, speaking, and behaving. In order to show students the connections and the logic in mathematics, we allow them to really do mathematics in small groups in class and then to discuss ideas in a large group. This concentration on the process of mathematics rather than on the finished product encourages students to make sense of mathematics, and the format supports significant mathematical communication. Because the students have often memorized mathematics in the past, we are careful as instructors to refrain from giving answers or participating overtly in the problem-solving process. We watch and listen as they work. Only after they have tried to understand a definition, derive a formula, prove a theorem, solve a problem, or analyze children's thinking about a piece of mathematics will we have a whole-class discussion about the content that arises from the work. This way even when students have been unable to successfully derive, prove, solve, or analyze in the small group, they are, after the attempt, ready to hear and make sense of the mathematical contributions of their peers.

To support this pedagogy we have written our own materials for each of our content electives based on our constructivist philosophy, the research literature, and more than a decade of work with future middle grades teachers. As mentioned, we have also studied activity-based and Standards-based curriculum projects designed for middle school students: *Mathematics in Context, Connected Mathematics* and *MathScape*. Our students meet the big ideas in our texts through class activities. For example, follow the link www.uwosh.edu/mathematics/MAA/AlgebraActivity.pdf to explore an activity from the algebra course designed to help students learn a specific definition (that of a binary operation on a set), and how to read and make a sense of mathematical definitions in general. The ability to read and then use a new mathematical definition is part of learning how *to learn* mathematics, and fostering such skills is an explicit aim of our program. Each class activity is followed by a "Read and Study" section meant to be read with pencil in hand. The mathematical topic is then tied to the middle school curriculum through a "Connections" section. In many cases, sample pages from middle school curriculum materials are included. Finally, the texts provide a "Homework" section in which students are asked to tackle related problems (not simply exercises) and to think about the content from the perspective of a teacher. More detailed information about, and examples from, these texts are presented in our chapter in Section Two of this volume.

Our courses require significant written work. Assignments typically ask students to describe their problem-solving strategies as well as their solutions, and to make mathematical arguments that their solutions are correct. Each course requires a project (chosen by the instructor) in which students relate the course content either to its historical development or to educational issues. For example, in the geometry course, students have been asked to read the "Geometry and Spatial Reasoning" chapter in the *Second Handbook for Research on Mathematics Teaching and Learning* [11], and then to interview children and categorize their thinking using a theoretical framework from the chapter. We typically require students to present their work to the class, thus providing them with opportunities to practice preparing, explaining, and listening to mathematical ideas. The students are also assessed using written exams, quizzes, and homework assignments. We now summarize the content of each elective course.

Modern Algebra is centered on the structure of operations and the study of symbolizing and generalizing patterns. The National Council of Teachers of Mathematics recommends that algebraic and geometric ideas be the instructional focus for grades six through eight. "Students need to understand the concepts of algebra, the structure and principles that govern the manipulation of symbols, and how the symbols themselves can be used for recording ideas and gaining insights into situations" [16, p. 37]. In this course we ask students to understand the language of algebra and logic; the structure of operations on sets; elementary group theory; the power of pattern-seeking, modeling, and functional representations (including polynomials, exponentials and trigonometric functions); and the nature of building and solving equations and systems of equations.

Modern Geometry demands that students think spatially, see structure in art and form, and create and visualize new worlds with different rules. Geometry is a domain for action and activities. The NCTM advocates that middle grades students draw, measure, visualize, compare, transform, and classify geometric objects [16]. In our course students explore axiom systems through finite geometries; classical Euclidean geometry through synthetic, analytical, and transformational approaches; non-Euclidean geometries such as Taxicab, spherical, hyperbolic, and fractal geometry; and the elementary topology of spaces like the Möbius strip and the torus.

Infinite Processes is an intuitive approach to the ideas of calculus: infinity, limits, continuity, the derivative, the definite integral, and the Fundamental Theorem of Calcuulus. While our students will not teach calculus, many of the ideas that lead to calculus will be a part of their middle grades curriculum. For example, decimal representations of rational numbers are infinite series. Furthermore, calculus contains some of the greatest ideas in mathematics, and as such, provides our students with an important piece of cultural knowledge. In this course students explore paradoxes regarding infinity, countable and uncountable sets, infinite sequences and series; rates of change, and accumulation functions.

Probability and Statistics focuses on an area that has received attention in the middle grades during the past decade as statistics has become increasingly important for understanding politics, science, and economics. The NCTM asserts, "Students need to know about data analysis and related aspects of probability in order to reason statistically—skills necessary to becoming informed citizens and intelligent consumers" [16, p. 48]. Students in this course learn the vocabulary and conceptual foundations of probability, data analysis, counting, and statistical inference. A history of the ideas is woven throughout the course and real data is used whenever possible.

4.6 The Capstone Course

"[This course] helped me understand the importance of knowing what you are teaching. It also taught me that there is more to teaching math than just being good at it. It also takes a great deal of understanding [of] how students think about math. Through giving the unit presentations, I learned that I must be prepared to think on my feet, and that it will not always work to follow a strict lesson plan. I need to follow the direction that my students are going while still focusing on the important mathematics."

— Lisa Benzschawel (student), 2008

Our capstone course, *Senior Seminar for Elementary and Middle School Programs*, is a three-credit seminar course on issues in mathematics and mathematics education. Much of the class time is spent discussing selected readings on these topics, including the nature of mathematics, learning theory, an analysis of mathematics pedagogy, editorials on mathematics education, and research literature on student learning of elementary and middle grades mathematics. The course is designed around three questions: (1) What is mathematics and mathematical behavior? (2) How do humans learn mathematics? and (3) How should mathematics be taught?

We study the mathematician's perspective on mathematics and contrast this with the layperson's attitudes and beliefs about mathematics. We read papers by mathematicians such as G. H. Hardy [8] and Paul Halmos [7], and watch the NOVA video presentation of Andrew Wiles' proof of Fermat's Last Theorem. Typically, in one class period a panel of research mathematicians answers students' questions about the nature of mathematics and what it means to do mathematics.

We study constructivist learning theory by reading and analyzing articles by Ernst von Glasersfeld [21], Heinrich Bauersfeld [3], and Alan Schoenfeld [17], and we explore teaching practice based on this theory by discussing the work of Deborah Ball [1] and Magdelene Lampert [9]. Two texts that we have used in the course are [4, 12].

Student work in the course includes written reflections on the readings, a research paper, and two curriculum projects. Research papers must be related to the central questions of the course, and supported by scholarly research literature. Students are expected to read, analyze, and understand several articles from peer-reviewed journals. Past paper topics include teaching problem solving, teachers' mathematics beliefs, a comparison of mathematics education in the United States and Singapore, and cooperative learning in the mathematics classroom.

The latter part of the seminar is dedicated to the curriculum projects, which are designed to let students apply and practice what they have learned from the research literature on the theory and practice of learning and teaching mathematics. Working in teams of two or three people, students assess and evaluate a unit from a reform-based middle-school curriculum project. The team's assessment of the unit is to be based on the unit's compatibility with constructivist theory, the NCTM *Standards* and the research literature. Students lead a lesson for the class based on their unit, while classmates evaluate the lesson. Evaluation criteria include the importance and accuracy of the mathematical content, the use of mathematical processes of problem solving, communication, reasoning and proof, and the team's use of their peers' thinking and misconceptions during the presentation. Students later complete a second curriculum project so that instructor and peer criticism from the first lesson can be used to inform the second.

4.7 Discussion

"I can't tell you the extent of how much I learned from you about what mathematics *really* is and how to create an environment where students are able to think and connect mathematically."

— Olivia Schellinger (student), 2008

The preparation of mathematics teachers presents special challenges at the middle grades level. The mathematics content has begun to overlap that of the high school curriculum; thus, a student preparing to teach middle school mathematics might major in mathematics. A mathematics major traditionally requires several abstract as well as applied mathematics courses beyond calculus, and provides an exposure to significant mathematics content. A problem is that those courses are usually taught by mathematicians who may not have much understanding of the K–12 curriculum, pedagogical issues, or the nature of the mathematical knowledge required for teaching. Consequently, the student may find it difficult to adapt his or her knowledge to the needs of the middle grades student.

Our middle school program at UWO was developed for students who plan to teach middle school mathematics, but who might not fare well as traditional mathematics majors or who might not be interested in pursuing such a major. Their level of formal mathematics knowledge entering college is usually significantly weaker than that of a mathematics major, but they may be very bright and they demonstrate an aptitude for mathematical thought processes and an eagerness to understand the subject better. In fact, we look for these characteristics in our elementary education students as they go through their three initial mathematics courses, and we recruit promising students for our middle school program. To ignore the potential in such students would deprive our middle schools of many well qualified and highly motivated mathematics teachers. Furthermore, because our middle school program integrates pedagogy with strong conceptual foundations and addresses the content as it is relevant to the mathematics that our students are preparing to teach, we believe it provides a better preparation for teachers of middle grades mathematics than does the traditional major program.

Between 2005 and 2010, one hundred forty-four students completed the middle school program. Of those, sixty-five (45%) have provided career information to the university, and that data suggests that the graduates of our program fare well: fifty-eight of the sixty-five (89%) are currently employed as teachers.

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I

Programs for Middle School Mathematics Teachers

B. In-service Training Programs

5

The Master of Arts in Middle School Mathematics Program at Salem (MA) State University

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Many educational observers believe that elementary, middle grades, and secondary mathematics teachers need a better understanding of mathematics content. The NCTM, Liping Ma, and many others have used the terms “deep” or “profound” understanding of school mathematics to describe an ideal attribute for a teacher of mathematics [1, 4]. The philosophy behind the Master of Arts in Teaching (M.A.T.) Middle School Mathematics program at Salem State University is based on the idea that middle school mathematics teachers will be more effective in the classroom if they have a deep understanding of mathematics. I will describe the program and the local and national conditions that led to its creation in 2003. It is a small program that seems to be unique in our geographic area for the amount of mathematics it requires.

The M.A.T. program was designed for practicing middle school mathematics teachers who needed a greater understanding of mathematical content. Typically, these teachers were licensed as elementary or middle school generalists and had not taken many mathematics courses as undergraduates. Our M.A.T. program contains more mathematics than other programs for a similar audience. For example the M.Ed. in Middle School Mathematics Education at nearby Lesley University consists solely of education courses [8]. Several pedagogy courses are included in the program and a capstone experience in which teachers implement a project of their own design in their classrooms.

5.1 Background: Teacher Licensure Requirements in Massachusetts

In Massachusetts, passing an exam for obtaining a teacher’s license has been required since 1998 [3]. This exam, the Massachusetts Test of Educator Licensure (MTEL), has a general portion called Communication and Literacy as well as subject- and grade level- specific exams. Passing the MTEL exam is also one of the ways Massachusetts has decided to designate teachers as “highly qualified” in their subject areas, as per the No Child Left Behind Act of 2001 [5].

There are three levels of teacher licensure in the state: preliminary, initial, and professional. A preliminary license requires a bachelor’s degree (with any major) and a passing score on the Communication and Literacy MTEL and a subject area MTEL. Requirements for an initial license include in addition the completion of a Massachusetts Department of Education (MassDOE) approved initial licensure program, which includes pedagogy courses and a supervised student teaching practicum. A professional license requires at least three years of teaching with an initial license and a master’s degree containing at least twelve credits of graduate coursework in the field of licensure. In 2003, when our program was developed, the master's degree required for advancing to professional license was to be a master's having at least half of its credits in the subject area. This was a major impetus for developing our middle school mathematics program.

It is a requirement that teachers progress to professional license. There is a limit of five years on teaching at the lower levels. It is not necessary to receive a preliminary license before receiving an initial license. For example, a teacher who successfully completes student teaching as part of an undergraduate teacher preparation program graduates with a bachelor's degree and an initial teaching license. The preliminary license was created primarily to get career-changers into the classroom quickly.

In Massachusetts, the subject area licenses available are identified by subject and grade level; the license for middle school mathematics teachers is called Mathematics 5–8. One can also be licensed in elementary mathematics (Mathematics 1– 6) or high school mathematics (Mathematics 8–12.). Each license has its own mathematics MTEL. A subject MTEL for a higher grade level may be used to satisfy the lower grade level license requirements, so passing the Mathematics MTEL would satisfy the requirement for Math 5–8 and Math 1–6 as well as for Math 8– 12.

The main difference between the Math 5–8 and Math 8–12 licensure requirements is in the breadth and depth of study of mathematical topics. The most recent MassDOE regulation (from 2007) on subject matter requirements for mathematics licensure lists the following as required knowledge for licensure in Math 5–8:

1. Algebra
2. Euclidean geometry
3. Trigonometry
4. Discrete/finite mathematics
5. Introductory calculus
6. History of mathematics.

The subject matter requirements for licensure in Math 8–12 include the list of topics for Math 5–8 and

1. Abstract algebra
2. Number theory
3. Calculus through differential equations
4. Probability and statistics
5. Non-Euclidean and transformational geometries
6. Applied mathematics or mathematical modeling.

The initial licensure candidate is expected to know the Massachusetts Curriculum Frameworks in mathematics, which draw from the NCTM Standards. Though probability and statistics does not appear on the list of topics that a Math 5–8 teacher should know, it appears on the Middle School Mathematics MTEL [3].

The regulations specify that students in bachelor's degree initial license programs should take at least 36 credit hours of mathematics, which need not be a major, for Math 5–8, or major in mathematics for Math 8–12, but there are no course requirements stated for post-baccalaureate initial licensure programs. Massachusetts depends on the subject MTELs to guarantee that mathematics teachers are competent in the listed subject areas, at least for the preliminary and initial license levels. For example, a person who has a bachelor's degree in, say, English can obtain a preliminary license in Math 5–8 by passing the Communication and Literacy and Middle School Mathematics portions of the MTELs. Since there exist post-baccalaureate initial licensure programs containing no mathematics courses and having only passing the MTEL as a prerequisite, that person could obtain an initial license in Math 5–8 without having had any additional mathematics coursework. Only for the professional license is twelve credits of mathematics required.

5.2 Origins of the Program

The origins of the M.A.T. in Middle School Mathematics lay in a grant proposal that was spurred by the No Child Left Behind Act of 2001. To quote from the grant proposal, "the overall goal of this [program] is to promote a deeper understanding, appreciation and confidence in mathematics for teachers in grades 4 to 8." [6]. The grant, known as Project SALEM, was a partnership with several local school districts. It provided release time for college faculty to design the program and funding for teachers in partner districts to take the new courses. There was also funding for follow-up activities in which the teachers used the mathematics learned in the Salem State courses in their classrooms.

The M.A.T. program, now a permanent offering of Salem State University, was designed to serve teachers teaching middle school mathematics with an initial license who needed a master's degree in their subject area to qualify for professional licensure, as state law required at the time. The law now requires that the master's degree have at least 12 credits of mathematics. In 2003, many of the teachers fitting this description had been trained as elementary or middle school generalists. The programs from which the generalists were graduated would most likely not have required more than a semester or two of mathematics, which would not have prepared them for admission to a program that included traditional graduate mathematics courses, such as the two graduate math programs at Salem State, an M.S. in Mathematics and an M.A.T. in Mathematics.

Nine of the thirteen courses comprising the M.A.T. Middle School Mathematics program are mathematics courses, taught by mathematics faculty. In designing them, we kept the mathematics experience level of our target audience in mind, so the content is really undergraduate-level mathematics. The decisions regarding the makeup of the mathematics courses were influenced by the subject strands in the Massachusetts Curriculum Frameworks and by the MTEL objectives and subject matter requirements for teacher licensure, which are related to the NCATE/NCTM Standards. Some of our course titles came verbatim from those sources. For example, Patterns, Relations and Algebra is identified as a subject subarea of mathematics that the state believes teachers should know, and it is also the title of our first mathematics course. Likewise, Data, Statistics and Probability is a subject subarea listed as an objective on the Middles School Mathematics MTEL and is a title of one of our courses. (A list of the courses and their catalog descriptions appears at the end of the article.) We decided on other courses for a variety of reasons. For example, we included History of Mathematics because we thought it would be valuable for teachers' understanding and perspective and historical topics are mentioned in the state's list of what teachers should know. We decided that it would be beneficial for the teachers to learn to prove theorems, so we included a discrete mathematics course and we include proofs in most of the other courses.

In the process of creating the new courses and putting together the program, we kept in mind the goal of fostering a deep understanding of mathematics. "Deep understanding" is the opposite of a mechanical understanding of mathematics. One who understands a mathematical concept deeply can recognize multiple approaches to its explanation or application, can recognize similarities or differences among various concepts, can draw connections between related concepts, and can successfully communicate the concept to others [1]. We wanted to design courses that are challenging but not beyond the grasp of our audience and include aspects like discovery learning, problem solving, and proofs to help the participants master the mathematics concepts and deeply understand them. Most faculty members who teach the courses include a project to design a unit or lesson plan to help the teachers apply the material to their middle school students. This helps with the understanding of the material.

Colleagues from the School of Education had a hand in designing the three pedagogy courses. One of them, Issues in Mathematics Education, is flexible and allows the instructor to discuss current research trends. The capstone experience, Clinical Experience in Middle School Mathematics Education, involves a research project that the teacher has designed in a prior course, Action Research. The courses have been taught by mathematics education faculty.

Not long after the program was in place in 2004 and our first students enrolled, we began receiving inquiries from people who did not fit the description of our audience. They included people who had never taught before but were interested in teaching middle school mathematics. Many had educational backgrounds in business or the humanities, but had had no pedagogical coursework or student teaching experience. Since it seemed that many of them could benefit from the mathematics coursework, we created a separate track in the M.A.T. program that included a student teaching practicum and led to an initial license, which was approved in December 2008.

5.3 Program Requirements

The M.A.T. degree program is thirty-nine credit hours. We now have two tracks: the original program, hereafter referred to as the professional track, and the initial licensure track. Admission to the professional track requires an initial license for those teaching middle school mathematics, and admission to the initial track requires that the Middle School Mathematics MTEL be passed by the time the first five mathematics courses have been completed. There are no specific requirements for an undergraduate major.

Those entering the professional track must take all nine mathematics, three mathematics education courses, and a three-credit clinical experience, in which the teacher implements an action research project in his or her classroom. The course requirements for the professional track can be found in [9].

There are more pedagogy courses in the initial license track, so as not to increase the credit load, some of the mathematics courses were left out. Enrollees in this track must take the first five listed mathematics courses, choose two of the remaining four as mathematics electives, and then complete a MassDOE-approved kernel of three pedagogy courses, one mathematics methods course, and a student teaching practicum and seminar. The course requirements for the initial license track can be found in [10].

The program began in the spring 2004 and we saw our first two graduates in 2006; altogether, twenty-two teachers have completed our program so far, and there are about twenty-five active students currently enrolled. Teachers from outside the program also take our courses, either as content electives for other programs such as a master's in education, or as professional development required by their school districts.

5.4 The Mathematics Courses

The goal of our middle school mathematics courses is, as we stated in our original Project SALEM grant proposal, "to promote a deeper understanding, appreciation and confidence in mathematics for teachers in grades 5 to 8." [6]. There is no required format for the courses; as with other mathematics courses at our university, the instructor has the freedom to decide what structure will best achieve the goals. The courses tend to have a strong component of group work, with discovery and hands-on activities, although the lecture-and-discussion format is used as well. Class size is currently small, on average about 12 students, but was larger, almost twice as much, when the Project SALEM grant was subsidizing tuitions of those in our partner districts. In small classes, the discussion and group work parts are frequent and lively.

The mathematicians who teach the courses use different materials and have devised experiments and projects to help advance student understanding. Sometimes a textbook is used and other times the instructor uses his or her own notes. In many courses we use graphing calculators and software such as Geometer's Sketchpad and spreadsheets. Here are a few examples of what has been done. Instructors of our Geometry course have used shape blocks to study irregular figures, geoboards to study area, mirrors to study reflection, and have used an experiment involving baseballs, Play-Doh, and grids on transparencies to approximate the surface area of a sphere. In the Data, Statistics, and Probability course, physical probability experiments using dice and coins help teachers explore relative frequency as do simulations using a TI-84 calculator. In the History of Mathematics course, teachers have constructed a set of Napier's rods to explore the lattice method of multiplication.

We try to make clear the connections between the mathematics in our courses to the mathematics encountered in the middle school classroom, which may not always be apparent. One way to do this is to include assignments that tie the subject matter to the corresponding strand in the Massachusetts Curriculum Frameworks. For example, our Linear Systems includes an introduction to matrices and applications such as solving systems of equations and performing geometric transformations. Students often have trouble seeing the relevance of such material. However, setting up and solving systems of two linear equations using multiple methods is a learning standard for grades 7–8, [2, p. 69]. Predicting the result of a geometric transformation on a coordinate plane is also a standard for that grade level [2, p. 70]. Using matrices to achieve those tasks is another method, even though not mentioned in the frameworks. Often the mathematics studied in our courses goes well beyond what would be taught to an eighth grader and no corresponding Frameworks strand exists. Then we emphasize that continuing to do mathematics, whether solving problems or writing proofs, increases understanding of mathematics and the mathematical process and, according to sources such as Liping Ma and the NCTM, makes for a better mathematics teacher.

5.5 What We Have Learned and What the Future Holds

In designing and running the program, we encountered practical and philosophical issues. Practically, we felt we should distinguish the middle school mathematics courses from our traditional graduate mathematics course offerings. We did not want students enrolled in our other two programs, a Master of Science in Mathematics program and a Master of Arts in Teaching Mathematics program, to be able to count the middle school math courses for degree credit. All the courses included the phrase “for Middle School Teachers”, but this would not necessarily prevent the registrar from counting the courses toward a student’s degree. The problem was solved by creating a different prefix, MSM, for the courses in the middle school program (regular graduate mathematics courses have an MAT prefix). Students in the M.S. and our other M.A.T. program are not allowed to take MSM courses for degree credit. Students in the middle school program may take the MAT courses with approval.

Another practical issue that has emerged is that some applicants have a bachelor’s degree in mathematics, making them overqualified mathematically for the program. Even though we are teaching students to think deeply about the elementary mathematical concepts and are helping students apply them to the middle school level, the courses were designed for our original audience of non-math majors. While math majors might be able to derive some value from the courses’ other aspects most of the content is something they have seen before, while it would be brand new to non-math majors. This is a problem if the class is mixed, containing both math and non-math majors. We have yet to resolve the problem completely. If a math major were teaching in a middle school, it would be natural to enroll in our program. Several years ago Massachusetts began to require all undergraduate pre-service teachers to major in an academic subject area. One can no longer major in “middle school education”, for instance. This means that it will be more common for those teaching middle school mathematics to have majored in mathematics as an undergraduate. We will probably need to revise some of our middle school mathematics courses to meet this challenge.

As for the philosophical issues, members of our department were concerned that we would be offering graduate courses that contained undergraduate course content. We decided that what makes the courses graduate level is the approach to the subject matter and its application to the middle school curriculum. When the program was originally conceived, the state requirement for professional licensure was a master’s degree from the department of the subject area of certification. This was the main impetus for housing the program and the mathematics courses in the Mathematics Department instead of the Education Department. We feel the teachers in the program have received a richer mathematical experience having had the courses taught by mathematics faculty.

Program administrators are interested in alumni feedback. In December 2008 we administered a survey to fifty-four teachers who have taken courses in our program. We included those who had completed the program, those enrolled but not yet finished, and those who took some classes but have no plans to enroll. Of the twenty-five respondents, 62.5% reported that they felt “much more confident with the mathematics I teach” and 33.3% felt “somewhat more confident.” As one of our goals in creating the program was to increase teachers’ confidence in mathematics, this was satisfying. Further, 83.3% reported using content or skills they learned in our courses in their classrooms. Five of the respondents had accepted mathematics supervisory jobs, positions which they said they were qualified for as a result of our program. Forty-four percent of the respondents had obtained professional licensure.

In the comments section of the survey, several teachers reported ability to connect higher-level mathematics to the mathematics they taught their students, but a few others felt there should be more focus on explicitly relating the content in the courses to the middle school curriculum. Most said that the courses were difficult for them, but they nonetheless felt they were valuable.

We foresee a number of challenges in the upcoming years. One will be to keep the program populated with students. The Project SALEM grant that subsidized the cost of the courses for teachers in local school districts has now ended, so we will have to work harder to promote the program. The addition of the initial license track will help, as it opens the program to a larger audience (in fact, at present the majority of students in the program are in the initial license track). Nonetheless, the program will likely remain small relative to other education-focused graduate programs here. The administrators in our School of Graduate Studies have sometimes allowed courses to run with as few as seven or eight students, though the official minimum size for a graduate course is twelve students. While getting a low-enrollment course to run usually involves case-by-case justification to the deans (examples: the course is needed by the students for graduation and only runs every other year; the other two graduate math courses running this semester have 18 and 20 students so our average is 15, etc.), there seems to be recognition that the program is well-regarded in the community, is important to it, and should be helped along. Efforts to build up the numbers in individual courses as well as in the program include contacting department heads and mathematics supervisors in local school districts with information about our course offerings each semester, so that they can send interested staff members our way. Offering some courses every other year has helped avoid class cancellations because of under-enrollment.

Another potential challenge has been mentioned: as the effects of the license requirement changes of the past decade begin to be felt, the audience for whom we designed the courses will begin to disappear. Within the next ten years, it is likely that the majority of teachers with an initial license in middle school mathematics will have been mathematics majors in college. We will probably have to revise the mathematics courses to be of more value to the new audience.

The M.A.T. in Middle School Mathematics was created with the goals of increasing the breadth and depth of mathematical understanding of practicing middle school teachers. In response to demand we have created an additional track to train new middle school mathematics teachers. Since the program is well-regarded by school districts in our area, we are confident that we will be able to adjust the program to that community's needs in the future.

5.6 Resources

A description of courses can be found in the Appendix.

Catalog description of the M.A.T. Middle School Mathematics program at Salem State University: `www.salemstate.edu/academics/schools/1229.php`

Good source for introducing teachers to using logic in mathematical proof: Wheeler and Brawner, *Discrete Mathematics for Teachers*, Preliminary Edition, Houghton Mifflin, Boston, 2005.

Excellent historical vignettes, perfect for use in a history of math course as well as other math courses: Berlinghoff, W. and Gouvea, F. *Math Through the Ages: A Gentle History for Teachers and Others*, Expanded edition, Mathematical Association of America, 2004.

Lesson plans, projects, applets. Good for examining in class as introductions to topics or applications, and for making connections to middle school curriculum. `mathforum.org/teachers/middle/`

Collection of relevant links to lessons, projects, math trivia, writing mathematics and even classroom issues. Beware: some sites are better than others. `www.middleschool.net/curlink/math/mthmain.htm`

Website for the NCTM journal *Mathematics Teaching in the Middle School*: `www.nctm.org/resources/middle.aspx`

5.7 Bibliography

[1] Liping Ma, *Knowing and Teaching Elementary Mathematics*, Lawrence Erlbaum Assoc., Mahwah, NJ, 1999.

[2] *Massachusetts Curriculum Frameworks: Mathematics*, Massachusetts Department of Education, November 2000.

[3] Massachusetts Department of Elementary and Secondary Education website: `www.doe.mass.edu/`

[4] *Principles and Standards of School Mathematics*, National Council of Teachers of Mathematics (NCTM), Reston, VA, 2000.

[5] *No Child Left Behind Act of 2001*, Public Law 107 110, One Hundred Seventh Congress of the United States of America, January 8, 2002.

[6] Project SALEM (Salem and Lynn Educating Mathematicians), grant proposal to Massachusetts Mathematics and Science Partnership, PI: Dr. Martha Hunt, October 2003.

[7] Salem State College 2008–2010 Graduate Catalog.

[8] Lesley University M.Ed. program in middle school mathematics: `www.lesley.edu/soe/middleschool/math_initial.html`

[9] Salem State University M.A.T. Middle School Mathematics program (professional track): `www.salemstate.edu/assets/images/ACA_sogs/fs_mat_middleschoolmath.pdf`

[10] Salem State University M.A.T. Middle School Mathematics program (initial track): `www.salemstate.edu/assets/images/ACA_sogs/fs_mat_middleschoolmath_init_lic.pdf`

Appendix

The following are the descriptions of the mathematics courses in the program. All courses are three graduate credits.

MSM 701 Patterns, Relations, and Algebra for Middle School Teachers
Topics include the expression of approximate relationships in data using tables and graphs, linear, polynomial, and exponential relationships, sequences, especially recursive sequences. The course emphasizes multiple approaches to analyzing mathematical relationships (e.g., graphical, tabular, algebraic, numerical, etc.) and will develop a facility with manipulating the mathematical symbolism.

MSM 703 Precalculus for Middle School Teachers Prerequisite: MSM 701
This course is intended to bridge the gap between algebra and calculus. It will develop a firm understanding of the concept of function, how to graphically represent various functions, analyze their behavior and create new functions from old. The course will look closely at various function classes including polynomials, exponential, logarithmic and trigonometric. Functions will be used to model real-life situations.

MSM 705 Geometry and Measurement for Middle School Teachers
A comprehensive coverage of measurement concepts including perimeter, area, surface area, volume, and the Pythagorean theorem. Topics include properties of plane and 3-dimensional geometric figures, the concepts of congruence, similarity, symmetry, transformations and tessellations. An understanding of the nature and techniques of establishing geometric proofs is also central to the course.

MSM 707 Number Systems for Middle School Teachers
This course gives the middle-school teacher a deeper understanding of number systems (integers, rational numbers, real numbers, complex numbers). Topics include divisibility, factorization, Fundamental Theorem of Arithmetic, equivalence relations, congruence, Chinese Remainder Theorem, decimal representation, axioms for number systems, and geometric representation of numbers.

SM 709 Data, Statistics and Probability for Middle School Teachers
Teachers are introduced to methods of graphically displaying, collecting and analyzing data. Techniques involved in computing probability and counting principles will also be introduced. Topics will include measures of central tendency and dispersion, histograms, stem-leaf graph, box plots, binomial probabilities, normally distributed variables, as well as linear and non-linear regression.

MSM 711 Linear Systems for Middle School Teachers Prerequisite: MSM 701
This course gives the middle-school teacher a deeper understanding of systems of linear equations and matrices. Topics include operations on matrices, solving linear systems, inverses and determinants of matrices, and applications of matrices. Particular emphasis will be placed on using matrices in transformational geometry.

MSM 713 Calculus for Middle School Teachers Prerequisite: MSM 703
This course will provide teachers with a conceptual basis for understanding how calculus provides a powerful tool for analyzing change in our world. Topics include limits, slopes and tangent lines, differentiation rules, the chain rule, approximations, Newton's method, extreme values and curve sketching, an introduction to integration with applications to area between curves, the Fundamental Theorems of Integral Calculus and the basic integration techniques.

MSM 715 Discrete Mathematics for Middle School Teachers Prerequisite: MSM 701
This course gives the middle-school teacher a deeper understanding of topics from discrete mathematics taught in middle school. These include combinatorics, graph theory, trees, networks, Pascal's triangle, the binomial theorem, sequences, set theory, and recursion. Students will study logic and methods of proof in order to construct their own proofs. Problem-solving heuristics will also be discussed.

MSM 717 History of Mathematics for Middle School Teachers
This course is a survey of the history of mathematics, with emphasis placed on the development of topics encountered by students in elementary through middle school. Topics include numeration systems of ancient cultures, Euclidean geometry and number theory, origins of algebra, calculating devices throughout history, mathematics of non-western cultures, classical probability and modern topics such as graph theory and fractals.

The following courses are the pedagogy courses in the professional track, which were also designed specifically for this program:

EDG850 Issues in Mathematics Education: Programs and Trends
Students will analyze historical, mathematical and psychological influences in mathematics curricula. Factors that impact mathematics education, such as learning theories, research projects, professional organizations, and international perspectives will be presented and examined.

EDG851 Mathematics for all Learners
Mathematics educators will explore appropriate strategies to use in regular classrooms containing a variety of learners. Strategies for effectively instructing students with learning disabilities, second language learners, and gifted and talented populations will be presented.

EDG852 Action Research in Mathematics Education
This course will examine the quantitative and qualitative techniques needed to design a significant action research project on a current issue in mathematics education. Research design including sampling, design of survey instruments, analyzing data, validity and reliability will be presented. Students will design an action research project to test a hypothesis, which will be carried out in their clinical experience.

EDG992 Clinical Experience in Middle School Mathematics Prerequisite: EDG852
This course will provide an opportunity for a college supervisor, the school system representative and the student to work in concert to provide a full semester experience, the hub of which is the action research project EDG852 Action Research in Mathematics Education.

6

The Math in the Middle Institute: Strengthening Middle Level Teachers' Mathematical and Pedagogical Capacities

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Improving teacher quality is a national need in mathematics education and one many universities and K–12 schools across the country are working in partnership to try to address. This article describes a graduate professional development program at the University of Nebraska-Lincoln (UNL) for certified teachers that leads to a master's degree that is aimed at improving mathematics teaching and learning in the middle grades. An overview of the institutional context for middle level mathematics teacher education in which this professional development program is situated is presented (i.e., state requirements for teachers, university policies, district partnerships). We describe the *Math in the Middle* (M^2) *Institute* [1] and conclude with advice for mathematicians and educators to work in collaboration as they create their program for middle level teachers.

6.1 The Institutional Context

The University of Nebraska-Lincoln, a land grant Research I institution, offers undergraduate majors for future K–12 teachers that lead to a degree from the College of Education and Human Sciences. Students can obtain elementary certification in all subject areas (K–6) or secondary certification in a single subject area (7–12). Certification programs require multiple field experiences, a semester-long student teaching experience, and passing the Praxis Series I: Pre-Professional Skills Test [2]. Elementary teachers must pass the 0011 Elementary Education: Curriculum, Instruction, and Assessment (EECIA) test, a Praxis II series assessment, in order to be considered highly qualified for *No Child Left Behind*.

Elementary education majors are certified as generalists. As part of the (K–6) elementary education program, preservice teachers must take one mathematics methods course (3 semester hours) and four courses (12 semester hours) in mathematics, three of which are specified. One of the mathematics courses is part of what we call The Mathematics Semester, a block of coursework that also includes a math methods course, a bi-weekly field experience in a Lincoln Public School, and a course taught at the elementary school on creating a learner-centered classroom. Requirements

[1] The authors acknowledge the support of the National Science Foundation (EHR-0412502) in doing, studying, and writing about our professional development project. Ideas expressed in this paper are our own and do not reflect the views of the funding agency.

for becoming certified as a secondary mathematics teacher (grades 7–12) include 35 hours of mathematics courses and two math methods courses. The mathematics courses begin with a 3-semester calculus sequence, and conclude with a two-course capstone sequence designed for students interested in becoming secondary mathematics teachers.

6.2 The *Math in the Middle Institute* Partnership

The M^2 proposal was built, in part, on a five-year partnership between Heaton and Lewis that began with another NSF grant, *Math Matters*, which was focused on the mathematics education of elementary teachers. The *Math in the Middle Institute Partnership* is a partnership among mathematicians and mathematics educators at UNL, and mathematics teachers and administrators in the Lincoln Public Schools (LPS), Nebraska's Educational Service Units (ESUs) and many smaller rural school districts. Our website [1] is a source for information about our partnership and the institute that goes beyond what is provided here.

The aim of the M^2 *Institute* is to develop intellectual leaders in middle-level mathematics (grades 5–8) by strengthening the capacities of teachers to improve student achievement in math and to reduce achievement gaps in the mathematical performance of diverse student populations in Nebraska. A closely related goal is to develop in teachers the habits of mind of mathematical thinkers, so that they can then develop them in their students. The grant also funds a research initiative that transforms the M^2 *Institute* into a vehicle for studying educational improvement and innovation. The research agenda has two foci. One is on understanding teachers' capacities to translate the mathematical knowledge and habits of mind acquired through the professional development opportunities of M^2 into changes in classroom practice. The other is on understanding how changes in mathematics teaching practice translate into measurable improvement in student performance. We are particularly interested in how M^2 teachers support one another and other staff in their schools in improving mathematics instruction. For examples of what we are beginning to learn, see Green's dissertation [3] on value-added models to estimate the impact of teachers' participation in M^2 on their students' achievement. Hartman's dissertation [4] on teachers' efforts to learn mathematics in meaningful ways and develop the habits of mind of mathematical thinkers, Smith's dissertation [5] on teachers' efforts to translate learning from the institute into classroom practice, Rolle's dissertation [6] on pedagogical habits of practice, and Pustejovsky, Spillane, Heaton, and Lewis [7] on teacher leadership. Links to these five studies are available at [1]. We have found investigating the nature of teaching, learning, and leadership to be complex; Green's work develops new statistical methodology for value-added models to estimate teacher effects when a school district uses a mixture of norm-referenced and criterion-referenced tests from year to year with different ages of students. Our investigations into teachers' habits of mind have been particularly intriguing. While we focused the M^2 *Institute* on developing teachers' mathematical habits of mind, our research also uncovered regularities and changes in teachers' pedagogical habits practiced within their math classrooms.

Because more than half of Nebraska's population is located in rural areas and in towns of less than 25,000 people, our grant has focused attention on mathematics teachers who teach in rural communities. The priority that *Math in the Middle* gives to rural education permits it to make a unique contribution towards satisfying the needs of students in rural schools and research in mathematics education. As part of our work, in 2004 and in 2006, we hosted a Rural Education Workshop designed to inform the project about the issues facing rural mathematics teachers. Some of our findings have been published [8, 9]. Links are available at [1]. One interesting finding from our study of rural participants within our project is related to the definition of rural—the definition is relative and often reflects where one lives. For instance, someone in Omaha (whose metropolitan area contains around half a million people) sees McCook, Nebraska (population about 8,000) as rural. However, McCook residents see themselves as living in a good-sized city, and see the surrounding small towns (population under 1000), farms, and ranches as the actual rural areas; one would drive for an hour away from McCook before finding a larger town (23,000) and four hours before finding a city with a population over 100,000.

Because our grant seeks to build a university-local school district partnership, teachers who apply to the *Math in the Middle Institute* must teach in a district that agrees to partner with UNL and they must have the support of district administrators. Personnel in the partner districts are involved in the review of applications from their district and make recommendations. Tuition and fees are waived for participants, and the grant provides stipends for teachers when they attend university classes outside their regular school day and travel and subsistence support for teachers who live outside Lincoln. This includes stipends for three weeks in the summer and one weekend day each semester. When classes meet once each semester on a school day, the grant pays for teachers' substitutes.

Elementary	Middle Grades		Secondary	
	not math	math	science	math
15	1	7	1	8

Table 6.1. Certification held by a typical cohort of 32 teachers

Grade 5	Grade 6	Grade 7	Grade 8	7–12 HS	9–12 HS
7	7	5	7	3	3

Table 6.2. Mathematics teaching assignment for a typical cohort of 32 teachers

Data on the credentials and teaching assignments of our first four cohorts of teachers in *Math in the Middle* is given in Tables 6.1 and 6.2, which describe the background of a typical cohort of 32 teachers. A supplement received in Fall 2008 enabled us to extend our partnership and work with two cohorts of teachers from the Omaha Public Schools (the fifth and sixth cohorts overall). These data reflect both the needs of the rural schools with which we have worked and the fact that relatively few teachers in Nebraska have a middle level mathematics endorsement, though nearly half the teachers have an elementary endorsement and over a quarter have a secondary endorsement. Because some schools are 7–12 high schools, some of our participants teach students from grade 7 to grade 12.

To date, 156 teachers from our first five cohorts have earned a master's degree through the program. Very few teachers have dropped out of the program; the program has a 93% retention rate. Because our first cohort graduated in 2006 and because the *Math in the Middle Institute* is a program for current teachers, almost all of our graduates are still teaching or have assumed duties as a math coach or instructional facilitator. A few teachers have decided to pursue a doctorate in education, one graduate is now an assistant principal, and one has retired. We believe that the rest of our graduates are still in a K–12 classroom. This success offers evidence both that the program is accessible to teachers whose educational background leads to elementary certification and useful to teachers who are already certified as a 7–12 secondary mathematics teacher.

6.3 The *Math in the Middle Institute*

6.3.1 Program Goals and Structure

The M^2 Institute is a 12 course (36 hour) graduate program for current teachers designed to offer content-rich courses intended to develop teachers' mathematical knowledge and knowledge of effective classroom pedagogy. As part of the program, teachers conduct an action research project, thereby building their capacities as teachers and preparing them to be leaders among their peers. The institute culminates in one of two degrees: a Master of Arts for Teachers (MAT) with a Specialization in the Teaching of Middle Level Mathematics awarded by the Department of Mathematics in the College of Arts and Science, or a Master of Arts (MA) degree awarded by the Department of Teaching, Learning and Teacher Education in the College of Education and Human Sciences. Participants in the *Math in the Middle Institute* may choose either degree. For full descriptions of the two Master's Degree options, see [1]. Although Nebraska does not require teachers to earn a Master's Degree, in most districts, earning one results in a higher rate of pay. Because all participants in the M^2 *Institute* already hold Nebraska teaching certificates, there is no state or national exam participants must pass as part of their master's programs.

UNL's graduate program in mathematics is focused educating students pursuing a PhD in mathematics, although many of our graduate students earn an MA or MS and some stop after a master's degree. The department of Mathematics had an MAT (Master of Arts for Teachers) degree that required applicants to have a high school teaching certificate and essentially an undergraduate math major. The coursework built on that background and was more appropriate for a high school teacher. Offering the MAT degree with a specialization in the teaching of middle level mathematics was a new development that resulted from our NSF grant. The degree is different from the master's degree that our doctoral students earn. At UNL, students in the Department of Teaching, Learning and Teacher Education (TLTE) may pursue an MA degree in education. For the typical student in TLTE, there would be very little mathematics in an MA program.

Most teachers in Nebraska who teach mathematics hold either a general elementary (K–6) or secondary mathematics (7–12) endorsement; few hold the state's recent 4–9 middle level endorsement. Many of the M^2 *Institute* graduates wanted to have a middle level endorsement on completion of the M^2 *Institute*. However, barriers at the state level prevented most teachers from gaining the certification, since the middle level endorsement requires specialization in two subject areas.

The institute participants go through the 25-month (three summers and the two intervening academic years) program in cohorts. There are two types of courses taken by *Math in the Middle* participants; online courses taken during the school year and on-site courses completed during summer months. The online courses are completed in a standard semester while the on-site courses are completed in one to two weeks' time. The one or two week-long summer institute courses were inspired by the system used by the Vermont Mathematics Initiative (see `www.uvm.edu/~vmi`). Courses meet eight hours a day for five days a week with homework assigned each evening. This approach to instruction respects the many demands on a teacher's time. The academic year courses are best described as "blended distance-education courses." By this, we mean that there is a two-day on-campus component and a distance education component. For the distance education portion of Academic Year courses, we use Blackboard, PC NoteTaker, email, and Adobe Connect communication software.

For the on-campus days of the academic year courses, class meets eight hours each day with a homework assignment overnight. This covers about 40% of the course, making the distance-education portion of the course a reasonable add-on to the teachers' other duties. The order in which our fourth cohort of teachers has gone through the program is given in Table 6.3. Course descriptions are in the Appendix.

Spring	Summer			Fall
	Week 1	Week 2	Week 3	
	MATH 800T	MATH 802T and TEAC 801		MATH 804T
TEAC 800	MATH 805T	MATH 806T	STAT 892	TEAC 888
MATH 807T	MATH 808T	Capstone Course		

Table 6.3. *Math in the Middle Institute* schedule

6.3.2 The Curriculum

The *Principles and Standards* [10], *The Mathematical Education of Teachers* [11], and *Foundations for Success* [12] guide our goals for the pedagogical and mathematical content of the *Math in the Middle Institute*. The twelve courses include seven in the Department of Mathematics, one in the Department of Statistics, three in education offered by TLTE, and a capstone course that can be taken through either Mathematics or TLTE, depending on the teacher's master's program. Table 6.4 provides a list of the M^2 *Institute* courses, in the order they are offered in the institute. For our first course, our program owes a significant debt to Herbert and Kenneth Gross who created Mathematics as a Second Language for the Vermont Mathematics Initiative and wrote a text [13]. A discussion of how we teach this course and a description of two other *Math in the Middle* courses is in an article that appears in this volume [14]. For more information about Mathematics as a Second Language, contact Professor Kenneth Gross [13].

Descriptions of all our courses can be found at [1]; brief descriptions are given in the Appendix. Our website includes a description of three of the courses (MATH 806T, MATH 807T, and MATH 808T), ready for use by mathematics faculty who find them useful. Course materials for other courses are in various stages of development. Please refer to [1] or contact us at `msquared@unl.edu` to learn more about any of our courses.

The curriculum of the *Math in the Middle Institute* includes seven new mathematics courses and a statistics course designed to offer a challenging and coherent mathematics curriculum for middle level teachers. We anticipated that many of the teachers in our program would have the background of someone certified to be an elementary school math teacher and for many, perhaps most, of our teachers it would have been many years since they took a math class; this turned out to be true. Thus, the program starts with a basic course and becomes increasingly more sophisticated. The fact that the secondary certified teachers in the program find our first course, Mathematics as a Second Language, useful is evidence that the profound understanding of fundamental mathematics described by Ma [15] is beneficial to all

UNL Course Number	Course Title
MATH 800T	Mathematics as a Second Language
MATH 802T	Functions, Algebra, and Geometry for Middle Level Teachers
TEAC 801	Curriculum Inquiry
MATH 804T	Experimentation, Conjecture, and Reasoning
TEAC 800	Inquiry into Teaching and Learning
MATH 805T	Discrete Mathematics for Middle Level Teachers
MATH 806T	Number Theory and Cryptology for Middle Level Teachers
STAT 892	Statistics for Middle Level Teachers
TEAC 888	Teacher as Scholarly Practitioner
MATH 807T	Using Mathematics to Understand Our World
MATH 808T	Concepts of Calculus for Middle Level Teachers
TEAC 889 or MATH 896	Integrating the Teaching and Learning of Math (Capstone Course)

Table 6.4. *Math in the Middle Institute* courses

mathematics teachers and that the typical program for secondary teachers assumes (often incorrectly) that mathematics majors have a good understanding of the real number system.

For the education courses, TEAC 800, 801, and 888, we adopted a different approach. Each is a core course in the TLTE Master's program that we have modified to meet the needs of mathematics teachers. They are designed to meet broad TLTE course and MA program goals but taught, when possible, in the context of content related to mathematics education. The capstone course integrates mathematics and pedagogy so teachers can use the mathematics and pedagogy they have learned in the institute in classrooms and help teachers prepare for their roles as leaders.

Using technology is part of many of Math in the Middle courses, online or face-to-face. Each participant receives a TI-84 Plus Silver Edition calculator and uses it, for example, to graph more complex functions (e.g., exponential functions, trigonometric functions, higher degree polynomials), thus emphasizing the idea that a calculator can be a tool in exploring more complicated mathematics than participants might otherwise be able to study. In MATH 804T, teachers use the online assessment software, MapleTA, and in several courses they use Microsoft Excel. We have also introduced all of the participants to `GeoGebra`, an open source software that links dynamic geometry with algebra and calculus.

All M^2 courses have several common features. Homework is assigned, collected, reviewed, and graded (in some fashion) regularly and each course has an end-of-course assignment. Mathematics homework assignments include a variety of problems, ranging from computational to those requiring extensive problem solving, explanation, and mathematical justification. Participants are encouraged to collaborate on assignments in whatever groups they choose but to submit their work individually.

Most M^2 courses divide participants into roughly five subgroups, each assigned to a member of the instructional team. The groups convene daily (during on-site courses) to provide a more intimate atmosphere in which daily homework assignments, questions regarding course content, and other concerns may be discussed. The small groups are important because participants who are hesitant to present their work or ask questions before the entire class are frequently more comfortable doing so in the smaller group.

For the M^2 mathematics courses, the typical End-of-Course Assignment includes: careful explanations for solutions to a set of problems selected by the teacher; reflections about the nature of course learning; and solutions to an "End of Course Problem Set." Because our goal is to help teachers reach a point where they can successfully solve the problems we assign, we permit them to submit solutions, receive feedback, and revise.

Two components of the *Math in the Middle* Master's program have made a contribution to our teachers' development as teacher-leaders in ways that extend beyond assignments in one graduate course and both have been incorporated in the "Master's Exam" requirement for the MA or MAT degree. The first is an action research project and report, and the second is an expository mathematics paper. In the action research projects, teachers identify a problem of practice, read

A mathematical thinker with well-developed habits of mind:

1. Understands which tools are appropriate when solving a problem
2. Thinks flexibly
3. Uses precise mathematical definitions
4. Understands that there exist multiple paths to a solution
5. Is able to make connections between prior knowledge and the problem
6. Knows what information in the problem is crucial
7. Is able to develop strategies to solve a problem
8. Is able to explain solutions to others
9. Knows the effectiveness of algorithms within the context of the problem
10. Is persistent in finding a solution
11. Displays self-efficacy while doing problems
12. Engages in meta-cognition by monitoring and reflecting on the processes of conjecturing, reasoning, proving, and problem solving

Figure 6.1. *Math in the Middle*'s working definition of *Mathematical Habits of Mind*

research articles on the topic, develop an inquiry plan, collect data as they implement their plan, then reflect on their practices as well as their plan as they analyze the results of their project. While teachers regularly try new strategies in their classroom, the systematic collection of data and reflection components tend not to be part of most teachers' normal repertoires.

During the fall of their second year in the M^2 *Institute*, participants take TEAC 888, a course in which they learn about conducting action research in their own classrooms, plan their own action research projects, and submit proposals to do the research to the university's Institutional Review Board. During the following spring, teachers conduct their research, studying a change they chose to make in their teaching. A report on their action research is a degree requirement. Teachers seeking the MA degree are expected to produce a scholarly paper that goes into greater depth with regard to data analysis, discussion of findings, and conclusions.

Teachers seeking the MAT degrees write an expository paper on a mathematics topic assigned by the faculty. The expository papers represent efforts by our teachers to learn new mathematics. Each MAT student receives a different topic, uses available resources (internet, library, and mathematicians) to learn about the topic, and writes an expository paper that is accessible to peers who might want to learn more about it. Our capstone course concludes with two days of oral exams during which participants give a talk about their action research paper (MA) or their mathematics expository paper (MAT). The action research reports and papers and the expository papers are available on our website [1].

6.3.3 Mathematical Habits of Mind

All the mathematics courses have a goal of helping middle level mathematics teachers develop mathematical habits of mind. Mathematical habits of mind (e.g., [16, 17]) represent an enriched view of what it means to do mathematics, based on orientations mathematicians bring to their work [18] and the expectations for K–12 students articulated in dimensions of mathematical proficiency [19] and the NCTM process standards [10]. In our project, we continue to construct and reconstruct our own understanding of the phrase. Figure 6.1 represents the project's current definition, presented as a set of skills and dispositions of a mathematical thinker.

An empirical challenge is to both operationalize and then identify these components in data and do so in multiple contexts of practice from multiple perspectives, including teachers as learners of mathematics working with mathematics instructors and middle school students as learners of mathematics working with middle school teachers. We are also working to understand mathematical pedagogical habits of mind [6], an extension of mathematical habits of mind, as a means of understanding the dispositions teachers may bring to their work of developing these ways of thinking with their middle level students.

6.3.4 Building Teacher Capacity

A key part of our efforts to produce graduates who possess the habits of mind of a mathematical thinker is the use of problems that our teachers call, "Habits of Mind problems." They need not be connected to the mathematics being studied and they are accessible to our participants without demanding a strong mathematics background. Ideally, a problem can be solved in several ways and participants can experiment with data or special cases. Careful reasoning and some creativity in problem solving are required before a thorough solution can be provided. The best problems are also "fun" in some sense of the word. The teachers in our program are fond of the problems and have adapted many of them for use in their classrooms. On [1] are some sample problems that can foster the development of mathematical habits of mind.

The *Math in the Middle Institute* has worked with teachers whose background in mathematics is varied. They share, however, a strong commitment to deepening their knowledge of mathematics and of effective classroom pedagogy, and a willingness to work hard. Our instructional staff has a commitment to providing the support that our teachers need as they progress through our graduate program. The use of Habits of Mind Problems places an emphasis on problem solving, careful reasoning, and the effective communication of mathematical ideas. They help us meet the mathematical needs of our participants by making mathematical content accessible to all participants while offering the best prepared rigorous mathematical challenges.

As participants progress through our courses, we regularly observe them grow in their capacities to engage in the learning of challenging mathematics and in the thoughtful way they approach their own teaching. By the time of our capstone course, the progress can be remarkable, as shown by the action research papers and the expository mathematics papers that the teachers write [1].

Middle school administrators' reactions to M^2 graduates have been uniformly positive. In Lincoln Public Schools, seven M^2 graduates are now mathematics coaches, with duties ranging from coaching third grade mathematics to high school algebra. Lincoln also moved M^2 graduates into leadership roles (such as department chair positions, and membership on curriculum selection committees). In smaller districts, M^2 graduates are called upon by their districts and Educational Service Units to lead mathematics-related professional development for their colleagues.

6.3.5 Student Assessment

In 2011, Nebraska administered a statewide mathematics test for students for the first time. Through 2010, each district was permitted to use any test deemed appropriate. Thus, measuring the impact of *Math in the Middle* on students statewide has been impossible. Students took district tests in varying grade levels at different points during the school year. Most districts had criterion-referenced tests to measure state or district standards. The criterion-referenced tests (CRT) were even more varied than the norm-referenced tests. Because much of Nebraska is rural, the vast majority of the districts have fewer than 50 students per grade level, and many have fewer than 20 which adds to the complexity of estimating teacher effects on student achievement. Across Nebraska, students do quite well on the CRTs averaging 90% proficient or advanced. In many cases, especially in the smaller districts, ceiling effects (100% of students being rated as proficient across multiple years) prevent estimation of value-added teacher effects on student achievement.

Given this assessment situation, we chose to start with what we hoped was a simpler task: analyzing student achievement data from our largest partner school district (at the time). In this district, across grades five through eight, students were given a mix of CRTs and the Metropolitan Achievement Test (MAT) through 2009; the district switched to the Iowa Test of Basic Skills in 2010. Green [3] built on the McCaffrey et al. [20] and Sanders et al. [21] value-added models to allow for assessment to vary from year to year. While the results only showed a very small positive impact of a teacher's participation in *Math in the Middle* on student achievement, several factors could be influencing this. One is that, since value-added models require baseline student scores (to show student growth), only 22 teachers across the first four cohorts were from this district and were linked to student scores in a way for us to establish baseline scores for students. Another factor is that the CRTs had a significant ceiling effect: students who met all criteria in a given year would show zero growth when meeting all criteria for the following year. Lastly, based on our analyses of norm- and criterion-referenced tests used in Nebraska, the overlap between the MAT content and the mathematical habits of mind focused on by M^2 is very weak.

6.4 Conclusion

We recognize that our NSF grant provided us with an opportunity to create a graduate program for middle level mathematics teachers that emphasizes obtaining a deep understanding of mathematical knowledge. In creating it, we focused on helping teachers to acquire mathematical knowledge for teaching and an enriched understanding of effective pedagogy. Because of the support of the NSF, there has been a substantial evaluation of the institute and a research initiative to understand how teachers acquire mathematical knowledge and how that affects the classroom. Of course, a more complete understanding of the long-term impact of teachers' new mathematical capacities on classroom practice is yet to be attained.

At this point, we are eager to share lessons we have learned with faculty on other campuses. Though our program was created for practicing teachers, demand for it has led us to create versions of our courses to offer to undergraduates. One bit of advice we would offer is that teacher education is inherently an interdisciplinary effort and that it is not sufficient just to increase teachers' mathematical knowledge. Thus, we urge mathematicians to seek a partnership with colleagues in mathematics education. By working to create an integrated program of study, we can benefit teachers far better than can be done in isolation. We will willingly share what we have learned with others working to improve the mathematical education of teachers. Information about our courses is available on [1] and we welcome inquires at `msquared@unl.edu`.

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Appendix
Brief Descriptions of Math in the Middle Institute Courses

MATH 800T: Mathematics as a Second Language
This course lays the foundation for developing the "habits of mind of a mathematical thinker," a theme that is further developed in subsequent M2 courses. The approach is to understand arithmetic (number) and (introductory) algebra as a means of communicating mathematical ideas (i.e., as a language). The course stresses a deep understanding of the basic operations of arithmetic, as well as the interconnected nature of arithmetic, algebra, and geometry.

MATH 802T: Functions, Algebra and Geometry for Middle-Level Teachers
This course builds on Mathematics as a Second-Language. A careful study of fractions, ratios, and rational numbers will lay the foundation for a study of algebra. Participants gain a deep understanding of the concept of function and gain a deeper understanding of the algebra and geometry taught in the middle grades. As part of the study of functions, algebra, and geometry, participants study measurement with an emphasis on length, area, and volume. The course also explores ways to increase students' visual literacy as a component of learning, understanding, and communicating ideas in mathematics.

TEAC 801: Curriculum Inquiry
This pedagogical seminar is taught in partnership with a mathematics class and focuses on gaining a deeper understanding of mathematics curriculum development, including historical and contemporary issues that influence curriculum planning and educational change. Participants consider current curricular issues in relation to their mathematics teaching and learning and how the mathematics learned in other M^2 courses transfers into the planned and enacted curriculum of one's own teaching practice.

MATH 804T: Experimentation, Conjecture, and Reasoning
This academic year course is taught using distance learning approaches together with a 2-day, on-site classroom experience. Focus is placed on problem solving, reasoning and proof, and communicating mathematics. With the support of the American Mathematics Competitions (AMC), which is housed in Lincoln, NE, this course utilizes the extensive resources of the AMC to help middle level mathematics teachers develop problem solving skills.

TEAC 800: Inquiry into Learning and Teaching
This academic year course focuses on inquiry into mathematics teaching and learning. Participants are introduced to the field of educational inquiry through a study of various designs and methods of doing educational research. Participants also develop knowledge, skills, and dispositions of educational inquiry through a study of artifacts of mathematics teaching and learning. The course helps participants consider current issues in mathematics education in relation to their teaching and learning of mathematics and what it means to transfer the mathematics learned in other M^2 courses into one's practice as a math teacher. The course also lays the foundation for ongoing work in the area of teacher leadership.

MATH 805T: Discrete Mathematics for Middle-Level Teachers
Discrete mathematics topics introduced in this class include social decision making, vertex-edge graph theory, counting techniques, matrix models, and the mathematics of iteration. The unifying themes for these topics are mathematical modeling, the use of technology, algorithmic thinking, recursive thinking, decision making, and mathematical induction as a way of knowing.

MATH 806T: Number Theory and Cryptology for Middle-Level Teachers
This course focuses on basic number theory results that are needed to understand the number theoretic RSA cryptography algorithm (an encryption algorithm that is used to secure information sent via the internet). As the number theory results are developed, connections to middle level curricula are emphasized and proofs are carefully selected so that those that are included in the course are relevant and accessible to middle level teachers. This portion of the course promotes a deep understanding of the integers and their properties in connection with the operations of multiplication and division. Elementary ciphers (methods for encoding and decoding) are included to introduce the nature of cryptology in preparation for understanding the RSA method. The cryptology related activities are adaptable as enrichment activities for middle level students. The connection of number theory to the RSA encryption algorithm allows the participants to see and understand a relevant, real-world application of mathematics.

STAT 892: Statistics For Middle-Level Teachers
The course offers an introduction to statistics and probability with an emphasis on teaching statistics and probability to middle level students. The course provides the foundation for later study of how data are used in education and for school-based research.

TEAC 888: Teacher as Scholarly Practitioner
This academic year course is taught using distance learning approaches together with a 2-day, on-site classroom experience. It introduces participants to the theory and practice of teacher-led inquiry into effective practice. The course prepares teachers to engage in a school-based action research project that will be conducted during the following spring semester.

MATH 807T: Using Mathematics to Understand our World
This academic year course is taught using distance learning approaches together with a 2-day, on-site classroom experience. It is designed around a series of projects in which students examine the mathematics underlying several socially-relevant questions that arise in a variety of academic disciplines (i.e., real-world problems). Students learn to extract the mathematics out of the problem in order to construct models to describe them. The models are then analyzed using skills developed in this or previous mathematics courses.

MATH 808T: Concepts of Calculus for Middle-Level Teachers
In this summer course, students develop conceptual knowledge of the processes of differentiation and integration along with their applications. The course is designed around a series of explorations (worksheets) through which students are led to discover the main ideas of calculus. Instructors' roles are primarily to answer individuals' questions that

arise in completing the worksheets, facilitate class discussions as the explorations are completed, and summarize the ideas developed in the course as the class progresses through the material.

TEAC 889/MATH 896 (Capstone Course): Integrating the Learning and Teaching of Mathematics
This course is the capstone experience of the M^2 Institute. Considerable time is devoted to discussing how the mathematics learned in M^2 courses can enrich the middle level classroom. The course is an integrated mathematics and pedagogy course whose goal is to enable teachers to be better teachers of mathematics because of the mathematics and pedagogy that they have learned. Concurrently with this course, teachers work on satisfying the master's exam requirements for their master's degree.

7

A Professional Development Program for Middle School Math Teachers in Maryland

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Hood College

7.1 Introduction

Hood College is a small (1400 undergraduates, 1000 part-time graduate students), private liberal arts college in Western Maryland. To respond to the needs of the community, we designed a master's degree in mathematics education for current teachers of mathematics, with a special track for middle school teachers. The program is interdisciplinary. It was designed by a team of faculty from the Departments of Education and Mathematics at Hood, along with consultants from the local public school system. Our goal is to help middle school math teachers who hold a certificate in elementary education become highly qualified teachers (HQTs) under Maryland's interpretation of the No Child Left Behind provisions. There are currently twenty-five students enrolled in the middle school track of our program.

7.2 Background and History

Until recently, Maryland did not offer middle school certification in any subject. Although certification in middle school mathematics is now a possibility [1], most middle school mathematics teachers are certified either in Elementary Education, Grades 1–6 and Middle School or in Secondary Mathematics, Grades 7–12. To be considered a highly qualified middle school math teacher, new teachers (hired after January 8, 2002) must hold a bachelor's degree, hold a valid teaching certificate, and either pass the Praxis II test in middle school math or have completed a major (or equivalent) in mathematics. Experienced teachers (hired before January 8, 2002) need to meet these requirements or earn enough points in a complicated rubric, Maryland's "High, Objective, Uniform State Standard of Evaluation" [2]. As of 2007, approximately 26% of the mathematics courses taught in Maryland schools were taught by non-highly qualified teachers, as defined by the Maryland State Department of Education [3]. Thus, there is a need to help teachers become highly qualified.

Besides the new quality requirements, as a result of recent reform movements in mathematics education and increases in accountability middle school mathematics teachers are being asked to teach more advanced mathematics content (such as what has traditionally been high school algebra and geometry) and to teach it differently than they have. As a result, there is a need for professional development for middle school mathematics teachers who are currently certified in elementary education but are either currently teaching, or would like to teach, middle school mathematics. We designed our program to address this need.

7.3 A Master's Program in Mathematics Education for Middle School Teachers

In the spring of 2008, we began offering a Master of Science in Mathematics Education with a Middle School Track. Students in the program complete eight specified courses – four in mathematics and four in education—and two electives from a list of mathematics, education, and computer science courses, chosen with the help of an academic advisor. We assume that students in the Middle School Track hold a teaching certificate but probably did not take many mathematics courses as undergraduates. Most of our students have taken two semesters of a Mathematics for Elementary Education course sequence in college, plus perhaps a problem-solving or statistics course.

There is no thesis and no comprehensive exam, although students may elect to complete a capstone independent research project. Because the teachers in our program are already certified, no outside examination is required.

Courses are taught in the late afternoon or evening, usually once a week for three hours during the regular school year. We offer at least one required course and one elective every summer.

7.4 Goals

On successful completion of this program, middle school math teachers will have a deep understanding of the foundations of mathematics, including algebra, geometry, data analysis, and mathematical modeling and problem-solving; understand and appreciate mathematics at a level appropriate for their position; and be aware of best practices in mathematics education, including the use of technology for understanding and exploration. Thus upon successful completion of this program, students will be truly highly qualified to teach middle school mathematics in Maryland.

In the program, we build on the NCTM/NCATE standards for initial certification [4] and offer teachers professional development in content and pedagogy in mathematics. Learning objectives for individual courses are included with their syllabi.

7.5 Required Courses

Every student in the program must complete ten courses:

MATH 505 Discrete Mathematics
MATH 500 Statistics
EDUC 595 The Teaching of Statistics & Probability
MATH 501 Explorations in Geometry
EDUC 551 The Teaching of Geometry
MATH 502 Explorations in Algebra
EDUC 552 The Teaching of Algebra
EDUC 596 The Teaching of Mathematical Modeling
Two approved electives from Mathematics, Education, or Computer Science.

The Discrete Mathematics course is typically the first course that a student takes. For many students it is like a jump into an icy lake—terrifying, bracing, dangerous, exhilarating. It forms the foundation for the rest of the program, both in the topics it introduces and in the mathematical habits we hope to develop in our students. Some of the topics may be familiar to them, but most are not. For many students, it is their first encounter with mathematical abstraction. We offer this course every fall and spring and occasionally in the summer, in order to accommodate students who enter the program at different times. It is also required for students in the M.S. in Computer Science program. There are usually 10–15 students in this class.

Students then take the other courses, usually in pairs: a course in mathematics followed by one that explores its teaching. MATH 500 is a basic applied statistics course that is taken by students in several other master's programs (Human Sciences, Management); it is our most popular course and is offered every fall and summer, with enrollments of about 25 students in each class. The other courses in the program are designed for mathematics teachers and are offered on a regular rotation every other year.

After completing the four mathematics and four education courses, students may choose two electives from the following list. This allows us to tailor the program to students' needs. Those who need to shore up their mathematical knowledge or who have a particular interest are encouraged to take more mathematics courses. Those who would like to learn more about using technology can take computer-oriented classes. Students who have a good undergraduate background in mathematics may complete an independent study or action research project. Many of the courses are available because we also offer a high-school track in our program.

MATH 507 Introduction to Graph Theory
MATH 509 Elementary Number Theory
MATH 546 Operations Research
MATH 599: Special Topics in Mathematics (Rotating topics, offered in the summer. Recent topics have included mathematical modeling, calculus, and the history of mathematics.)
MATH/EDUC 575 Independent Study: research on an approved topic
EDUC 597 Action Research Project (year-long course)
COMP SCI 503 Algorithms and Programming I
INFO TECH 512 Elements of Computer Programming.

The mathematics courses are generally taught by full-time mathematics faculty and the education courses by a full-time member of the Department of Education with training in mathematics education. Sometimes we call upon adjunct faculty with specific expertise to teach in the program—a curriculum specialist with the local school system, for instance.

7.6 Algebra and Geometry for Middle School Teachers

Before we began the program, we had much of the framework in place, as we had been offering a master's degree in curriculum and instruction with a concentration in secondary mathematics for some time and we provided mathematics courses to support graduate programs in computer science and other disciplines. But we knew middle school teachers would have special needs, and it was our responsibility to help them fill in the gaps in their mathematics background and prepare them to teach their students in the twenty-first century.

Because the Maryland State Department of Education has developed a set of Core Learning Goals for High School Algebra/Data Analysis and Geometry [5], it is important to help teachers learn more about these content areas. The Maryland High School Assessment (HSA) in Algebra/Data Analysis tests students' knowledge of the Core Learning Goals in this standard. Starting with the graduating class of 2009, students must pass the HSA in order to graduate from high school. Additionally, more and more students are taking high school algebra, geometry, or both while in the 7th or 8th grades. The Maryland State Department of Education has recommended that all students complete the Algebra/Data Analysis course by the end of 8th grade [6]. Thus, more and more middle school mathematics teachers are being called upon to teach what were once high school mathematics courses and, because students must pass the HSA to graduate, there is pressure on middle school mathematics teachers to teach these courses well. In order to prepare them for this, we designed two new courses for middle school teachers:

MATH 501 Explorations in Geometry
MATH 502 Explorations in Algebra.

We found the textbooks from the Connecting Middle School and College Mathematics Project at the University of Missouri to be just what we needed for both courses. The texts, which have the philosophy of the CBMS *Mathematical Education of Teachers* report [7], link mathematics instruction of pre-service or in-service teachers to current middle school mathematics curricula, giving students and teachers the "profound understanding of fundamental mathematics" that gives them the confidence and knowledge to be able to teach well. Each chapter begins with a problem from a middle school textbook and then looks at the mathematics behind it, introducing the students to more advanced ideas and problems. We supplement the texts with examples from other sources, such as the NCTM Navigations Series [8, 9, 10, 11], which show our students how they can use the ideas from the courses in their classrooms, and from the Middle School Mathematics Praxis II exam [12].

As much as possible, we model best practices in teaching the courses, introducing activities and group work for the students, using appropriate technology, having students do research on the history of mathematics, and incorporating student projects and presentations in the class.

The textbook for MATH 501 Explorations in Geometry is *Geometry Connections* by John K. Beem [13]. Students also learn to use The Geometer's Sketchpad [14] with the aid of the excellent guided discovery exercises in *Exploring Geometry with The Geometer's Sketchpad* [15] and *Pythagoras Plugged In: Proofs and Problems for The Geometer's Sketchpad* [16]. We also use Michael Serra's *Patty Paper Geometry* [17] and *Navigating through Geometry in Grades 6-8* [10]. Each student completes a project that explores an advanced geometry topic, chosen in consultation with the instructor. The projects are presented in class, also modeling best practices in teaching.

In MATH 502 Explorations in Algebra, we use the corresponding textbook *Algebra Connections* by Ira J. Papick [18]. We found it helpful to purchase copies of school textbooks from several NCTM-based curricula (Connected Mathematics, Discovering Mathematics) and use them to motivate what we do in class, although there are already ample examples in the *Algebra Connections* textbook. We also use the *Navigating through Algebra in Grades 6-8* resource [8], which contains some nice student activities involving discovering patterns, and an interactive file on the accompanying CD-ROM that illustrates recursive relations.

When students who took this course the first time we offered it were asked to list the Top Ten Things I Learned in This Class and Never Want to Forget (a student's idea), the most popular were:

Modular arithmetic
Coding and decoding
Divisibility rules
The Pythagorean theorem and its relation to algebra
Kaprekar's constant.

Readers may find it odd that we spent time on the Pythagorean theorem in an algebra course, but one of the appealing things about the Papick text is its integration of algebraic and geometric ideas. Kaprekar's constant, a "mysterious" number apparently discovered by the Indian mathematician D. R. Kaprekar in the 1950s, gives students a non trivial example of an algorithm whose outcome can be predicted by algebra. At the end of the course, students' concepts of what "algebra" is had been expanded.

The follow-up education courses in teaching geometry and algebra assume that students have the mathematical knowledge necessary to teach high school geometry and algebra. Therefore, the courses focus on pedagogy. Students examine local, state, national, and international standards and assessments, student misconceptions and difficulties, and effective teaching moves and tools. We read and discuss research articles, cases written by teachers about classroom situations, and we compare and contrast manipulatives and technological aids.

The algebra pair of courses is offered in even-numbered years (the math course in the spring semester, followed by the education course in the fall), and the geometry pair is offered in odd-numbered years. We have offered each course twice in the regular rotation, with enrollments of 20–25 students.

We have found two recent NCTM yearbooks to be perfect texts for the Education courses. In EDUC 551 The Teaching of Geometry, the required text is *Understanding Geometry for a Changing World: The 71st Yearbook of the National Council of Teachers of Mathematics* [19]. Students are also encouraged to purchase either the middle school or high school level NCTM Navigations Series book on geometry [10, 11]. The required texts for EDUC 552 The Teaching of Algebra are *Algebra and Algebraic Thinking in School Mathematics: The 70th Yearbook of the National Council of Teachers of Mathematics* [20] and *Improving Instruction in Algebra: Using Cases to Transform Mathematics Teaching and Learning* [21]. Students are also encouraged to purchase either the middle school or high school level NCTM Navigations Series book on algebra [8, 9]. Course syllabi are available.

7.7 Other Mathematics Education Programs

Hood College also offers (undergraduate) teaching certification in early childhood education, elementary and special education, and secondary mathematics. The early childhood, elementary and special education students complete a two-course sequence in mathematics for elementary education, an applied statistics course, a quantitative literacy course that incorporates Excel spreadsheets, and an elementary mathematics methods course. The secondary education students complete a major in mathematics and a set of state-mandated education requirements. At the graduate level, we grant a master of science degree in Curriculum and Instruction with a concentration in secondary mathematics or elementary mathematics and science, and a master of science degree in Mathematics Education with a high school track.

The master's program for middle school teachers is different from those programs in many ways. It does not offer initial teaching certification. It is meant as professional development for certified teachers, to help them become highly qualified. It focuses on the mathematics that middle school teachers need, and it offers them instruction and support. (Teachers in the high school track concentrate on high school objectives in their education courses and take different math courses—number theory and graph theory, for instance, instead of algebra and geometry.) It balances mathematics content and pedagogy. Half of the courses in the program are mathematics courses, and half are education courses.

7.8 Challenges and Solutions

7.8.1 Scheduling

As we mentioned in Section 7.3, courses during the regular semester are taught once per week in a three-hour marathon. They are hard on the instructor and hard on the student, and we would prefer to teach them in shorter blocks of time. Many of our students drive an hour or more to get to class, so they are understandably reluctant to do that more than once per week. Since they teach all day, it is difficult for them to take more than one course at a time. We have resigned ourselves to teaching courses on this schedule, and we make an effort to break up each class into different segments: lecture, activity, group work, and maybe visiting the computer lab.

7.8.2 Mathematical preparation of students in the program

Though we have admissions requirements for the program (a college degree with a 2.75 undergraduate grade point average, current teaching certification) the background of our students varies wildly, from those who recently graduated from college with a major in mathematics and who have just started teaching middle school math, to those who graduated years ago with a major in elementary education and who are now facing new requirements. We must be flexible to meet the needs of the entire range of students. Some of them may have a required course waived and an elective substituted. For others, the first course they take in the program (usually discrete mathematics) is a terrifying experience, unlike any they have had before. We have a responsibility to help those students find the resources within themselves to be successful, and to develop their skills and confidence to go on to take further mathematics courses. The instructor for the discrete mathematics course is available many hours per week to help students. We have supplemented the office hours with a program that we think is even more helpful. We choose a student who has taken the discrete math course recently and done well in it and hire him or her as a teaching assistant for the next semester. The teaching assistant meets weekly with the instructor and goes over the material that will be covered and the problems that will be assigned, and then holds optional problem sessions for students who feel insecure or lost. The teaching assistant can then give feedback to the instructor about where the students are having trouble. The assistant receives a tuition waiver for one graduate course. This allows us to offer many more hours of help to students from a different person and a different point of view, and helps the teaching assistant as well. We also are in contact with colleagues who offer similar programs and get advice and support from them.

7.9 Assessment, and the Future of the Program

Our program is fairly new, about two years old, and we have been overwhelmed by the interest from teachers in our community and beyond who want and need it. So far two students have graduated from the middle school track, and two more are scheduled to graduate this year. Peter Cincotta, Curriculum Specialist for Secondary Mathematics for Frederick County Public Schools, notes that "Better teachers make better students. Teachers that seek to improve their teaching abilities and their effectiveness with students are teachers that recognize that the art of teaching is not static. The past ten years of research has reinforced the fact that individuals who seek to teach mathematics to middle school students require special skills and training that far exceeds what we thought was needed when these teachers were in middle school. As a district mathematics supervisor, I want the best trained teachers to teach mathematics in our middle schools. The graduate program at Hood College helps me to realize that goal [22]." Students in our program are generally already certified and employed as teachers, and if they take a Praxis exam to become qualified to teach at the middle school level, we do not officially receive their scores. We sometimes get the news from the students themselves (one current student recently emailed us, "I passed the Middle School Math Praxis since being in the program!"), and other anecdotal evidence indicates that students find the program to be helpful. One student with an undergraduate degree in elementary education has recently been hired as a targeted intervention teacher in mathematics at a local elementary school as a result of her coursework in the program. Another is now a math specialist in a middle school.

We actively seek feedback from the students in the program, to make sure that we are offering a program that meets their needs. The Graduate School also administers a survey of current students. In the most recent survey, students indicated that they enrolled in the program to acquire more knowledge (75%), to explore career opportunities (33%), to increase their salary (83%) or for a promotion (42%), and for personal enrichment (58%). All the students who responded to the survey indicated that they felt they were "very likely" to complete the program, taking an average of 3.5 years. In rating the program and curriculum, the respondents' scores were higher than the mean for all graduate programs on every single question. Some comments from that survey:

> I'm seeing math methods described and researched more than I've ever seen in my undergraduate classes or in professional development sessions in [the public school system].
>
> Any and everyone teaching math will be able to use something out of these classes. At first I did this because I had to, now I am doing it because I want to. I am learning so many new methods and techniques of teaching that I want to apply them now, but I've stored them up and can't wait to teach that subject area when it comes back around.

One current student writes, "Since enrolling in the program, I have become more aware of the issues in math education and gained a deeper sense of the meaning of quality mathematics instruction. I have also acquired a collection of rich lessons and instructional strategies to use from collaborating with fellow math teachers in my courses." A recent graduate—one who had an undergraduate degree in mathematics—writes a rather lengthy assessment of her experiences in the program:

> As a result of completing the graduate program, Frederick County Public Schools gave me a raise. It wasn't too much, but it's better than nothing! I am still teaching the same classes that I taught before (Geometry, Algebra I and 8th grade Pre-Algebra), but I feel much more confident in teaching some of the material since we studied it in grad school. Plus I have so many more resources to use in my classes! I use numerous websites that we discussed in class, handouts from class, ideas from other classmates, etc.
>
> I would also like to comment that the classes offered for this grad program fit extremely well in my schedule. It was so convenient to go to Hood after being here at [my middle school]. The time of the classes was perfect too, that there was just enough time after teaching to get to grad class. I loved the option of the summer classes as well—since us teachers have off in the summer, it was wonderful to get a few courses out of the way when I had plenty of free time!
>
> Two more quick things—The first is I felt the size of the grad classes was perfect. Having small classes, you really get to participate a lot during classes and you get to really know your classmates. And second, I feel these classes provided just the right amount of difficulty/challenge. Being a math major in college, these courses at Hood took the math I learned at [my undergraduate college] just a little step further. I honestly loved going to grad classes and building upon the knowledge I already had in the subjects (esp. Algebra and Geometry)!

We have also established a program-level advisory council, consisting of teachers and administrators from nearby school districts, faculty and administrators from local community colleges, and students and faculty from our program. We meet twice a year to discuss how our students perform when they return to their classrooms, and how we can better prepare them to teach mathematics.

In the high school track of our program, students complete a capstone project, one that integrates the mathematical and pedagogical knowledge they have acquired in the program. Our first thought was that middle school teachers probably needed additional coursework more than they needed to undertake a final project, but we are re-examining that. At present, we are assessing students' progress in the program at the individual course level, but requiring a capstone experience could provide a more thorough framework for assessing the program as a whole and offer students a chance to reflect on what they have learned. This is another issue we will take to our advisory council.

We are constantly examining the program and its requirements in light of what middle school teachers need to know and how we can help them achieve their goals. In the meantime, we are confident we are helping teachers and, of course, ultimately helping their middle school students to learn—and perhaps even love—mathematics.

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Appendix
Course Descriptions

MATH 505 Discrete Mathematics
(Either semester, 3 credits) Introduction to the basic mathematical structures and methods used to solve problems that are inherently finite in nature. Topics include logic, Boolean algebra, sets, relations, functions, matrices, induction and elementary recursion, and introductory treatments of combinatorics and graph theory.

MATH 500 Statistics
(Summer and first semester, 3 credits) Basic statistical methods as they apply to data and research in the human sciences and other fields. Topics include frequency distributions and their representations, measures of central tendency and dispersion, elementary probability, statistical sampling theory, testing hypotheses, non-parametric methods, linear regression, correlation, and analysis of variance. Each student may be required to do a statistics project under the guidance of a cooperating faculty member in a discipline such as biology, economics, education, political science, psychology, or sociology.

MATH 501 Explorations in Geometry
(Second semester-odd years, 3 credits) Prerequisite: MATH 505 and a current teaching certificate. This course will examine high school geometry from a more sophisticated point of view, as well as exploring more advanced Euclidean and non-Euclidean geometrics. Topics covered may include analytic geometry, spherical geometry, hyperbolic geometry, fractal geometry, and transformational geometry. Labs in Geometer's Sketchpad will be an integral part of the course.

MATH 502 Explorations in Algebra
(Second semester-even years, 3 credits) Prerequisites: MATH 505 and a current teaching certificate. An examination of basic and advanced algebra concepts for teachers of mathematics. The course includes an introduction to the parts of number theory and modern algebra that underlie the arithmetic and algebra taught in school. The focus is on collaborative learning, communication, and the appropriate use of technology, as well as on a deep understanding of algebraic theory.

MATH 507 Introduction to Graph Theory (Second semester-odd years, 3 credits) Prerequisite: MATH 505 Discrete Mathematics. A rigorous study of the theory of graphs, including simple and directed graphs, circuits, graph algorithms, connectedness, planarity, and coloring problems.

MATH 509 Elementary Number Theory
(First semester-odd years, 3 credits) Prerequisite: MATH 505 Discrete Mathematics. An introduction to the theory of numbers: divisibility, prime numbers, unique factorization, congruences, Euler's phi-function, Fermat's and Wilson's theorems, multiplicative functions, quadratic reciprocity, perfect numbers, and Diophantine equations. Applications include public-key cryptography and integer arithmetic.

MATH 546 Operations Research
(Offered as needed, 3 credits) Prerequisite: An undergraduate degree in mathematics. In-depth study of operations research methods in decision theory, linear programming, distribution models, network models, dynamic programming, game theory, and simulation.

MATH 599A SpTpc:Mathematical Modeling for Teachers
Prerequisite: MATH 505 Discrete Mathematics. An introduction to mathematical models: continuous and discrete, deterministic and stochastic. We will use several kinds of computer software and will focus on active modeling and collaborative learning.

MATH 599B SpTpc:Explorations in Calculus
Prerequisite: MATH 505 Discrete Mathematics. Concepts and applications of calculus that are important in middle school mathematics: sequences and series, functions, rates of change, curve sketching, area. The connection to middle school mathematics is central: students will use middle school math curricula and find the calculus behind the ideas. No previous calculus experience required!

EDUC 551 The Teaching of Geometry
(First semesterodd years, 3 credits)Prerequisites: MATH 501 or equivalent and current teaching certification. This course examines current research and accepted practices in teaching geometry in the secondary school, based on national and state standards. The focus is on problem solving and mathematical reasoning, communication, and integrating geometry with other disciplines. Students will learn to use appropriate instructional materials, including technology, to support the teaching and learning of geometry.

EDUC 552 The Teaching of Algebra
(First semestereven years, 3 credits)Prerequisite: MATH 502 or equivalent and current teaching certification. This course examines current research and accepted practices in teaching algebra in the secondary school based on national and state standards. The focus is on problem solving and mathematical reasoning, communication, and integrating algebra with other disciplines. Students will learn to use appropriate instructional materials, including technology, to support the teaching and learning of algebra.

EDUC 595 The Teaching of Statistics and Probability: Decision Making with Mathematics
(Second semestereven years, 3 credits)Prerequisites: MATH 500 or equivalent and current teaching certification. This course examines current research and accepted practices in teaching statistics and probability in the secondary school based on national and state standards. The focus is on problem solving and mathematical reasoning, communication, and integrating statistics and probability with other disciplines. Students will learn to use appropriate instructional materials, including technology, to support the teaching and learning of statistics and probability.

EDUC 596 The Teaching of Mathematical Modeling: Strategies for Contemporary Problems
(Second semester odd years, 3 credits)Prerequisites: MATH 505 or equivalent and current teaching certification. This course examines current research and accepted practices in teaching mathematical modeling in the secondary school based on national and state standards. The focus is on problem solving through mathematical modeling and mathematical reasoning, communication, and integrating mathematics with other disciplines. Students will learn to use appropriate instructional materials, including technology, to support the teaching and learning of mathematical modeling.

EDUC 597 Action Research/Special Project
(3 credits)Prerequisite: Permission of the program director. Implementation of an action research special project in learning and teaching. Choice is made individually with the course instructor. The action research work is completed in two semesters, beginning in the fall and ending in the spring with a presentation of the findings. It is expected that students will complete a thorough literature review of their topics, clarify a hypothesis about a solution to a learning and teaching classroom problem, collect baseline data related to the problem, design an intervention program, monitor the intervention program being implemented, test the effect of their proposed solution and reflect and generalize about future actions.

EDMA 575 Independent Study:Research
Prerequisites: Six graduate credits in mathematics and six graduate credits in education. Reading and research in mathematics education.

CS 503 Algorithms and Programming I (Either semester, 3 credits)
Prerequisites: Either a minimum grade of B- in MATH 505, concurrent enrollment in MATH 505, or permission of the instructor. Previous experience with a high-level programming language such as Ada, BASIC, C, C++, Fortran or Pascal is recommended.

Introduction to the basic techniques of program development including input, output, assignment, control structures, simple and aggregate data types and subprograms. All phases of the course will focus on problem-solving strategies, modular design, and debugging techniques. Students will learn a high-level programming language, which will be used to implement programming concepts and do programming assignments.

IT 512 Elements of Computer Programming (First semester, 3 credits)
This course provides students with an introduction to programming concepts and techniques used in problem solving. Students will study general programming concepts and a modern programming language that illustrates them. Students will design, implement, and test programs to solve problems primarily in IT, business and science. Students will develop the ability to plan and develop programs logically, and learn to write, test, and debug programs. Topics include I/O, expressions, types, variables, branching, loops, web programming, program planning, and simple multimedia programming. Students will apply their knowledge through programming projects.

Syllabi for the required courses are available.

II

Courses for Middle School Mathematics Teachers

A. Discrete Mathematics

8

Discrete Mathematics: A Course in Problem Solving for 21st Century Middle School Teachers

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8.1 Introduction

This article describes how discrete mathematics topics have been incorporated into courses for middle school teachers at five universities using materials supported by the NSF-funded project, Discrete Mathematics for Prospective K–8 Teachers (DUE-0443317). It discusses our discrete mathematics materials and the philosophy behind their development and describes instances of implementation from each of five field-testing classrooms. Authors include the PI, co-PI, and four project participants.[1]

8.2 Background

In the past twenty years, Rosenstein (mathematician) and DeBellis (mathematics educator) have collaborated with mathematicians, mathematics educators, and expert classroom teachers to design and develop discrete mathematics programs and materials for K–12 teachers. We have collaborated on three NSF-funded professional development projects (TPE-895-5176, TPE-915-5231, ESI-945-4406) and one NSF-funded curriculum development project (DUE-0443317) and we have produced workshop materials for K–8 teachers through the professional development project (Leadership Program in Discrete Mathematics) and upper-level undergraduate course materials for prospective K–8 teachers through the curriculum development project (Discrete Mathematics for Prospective K–8 Teachers). The Leadership Program was funded through Rutgers University and the curriculum development project through Shodor, a

[1]This material is based upon work supported by the National Science Foundation under grant No. DUE-0443317. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

non-profit organization located in Durham, NC. In this article, we discuss the history of our work, general principles for teaching an activities-based discrete mathematics course in college, and offer examples of courses at five institutions.

The Rutgers workshop materials and activities focus on developing the mathematical needs of K–8 teachers and can be offered in a three-week professional development format. The materials provide teachers with an opportunity to have introductions to graph theory, combinatorics, iteration and recursion (including fractal geometry), and other topics (including voting, codes, fair division, and sorting). DeBellis and Rosenstein also developed college course materials and activities, titled: *Making Math Engaging: Discrete Mathematics for K–8 Teachers*. These educational materials are designed to help prospective and practicing teachers develop their mathematical thinking and reasoning skills in college undergraduate or graduate courses. They provide teachers, particularly middle school teachers, with an opportunity to learn new mathematical content using an activities-based approach and provide a context within which they can implement these concepts in their classrooms.

In addition, DeBellis and Rosenstein with colleagues Eric Hart (of American University in Dubai) and Margaret Kenney (of Boston College) wrote a set of discrete mathematics recommendations and classroom lessons for K–12 teachers. That product [1, 2], a component of NCTM's Navigations series, provides educational materials for school teachers to use in their classrooms.

In [3], the National Council of Teachers of Mathematics outlined three important areas of discrete mathematics—combinatorics, iteration and recursion, and vertex-edge graphs—that "should be an integral part of the school mathematics curriculum. . . . Combinatorics is the mathematics of systematic counting. Iteration and recursion are used to model sequential, step-by-step change. Vertex-edge graphs are used to model and solve problems involving paths, networks, and relationships among a finite number of objects" [3, p. 31]. However, K–12 teachers (and college faculty) have been left to interpret what discrete mathematics topics are of interest and what topics should be taught at each grade level. An early article that helped to fill the gap is the chapter of the New Jersey Mathematics Curriculum Framework that addressed discrete mathematics [4, Chapter 14]; it is available at [5]. The reality is that discrete mathematics topics have been taught across all K–12 levels without systematic or coherent implementation.

One important contribution of the discrete mathematics Navigation books is a framework of recommendations for how these three areas of discrete mathematics should span the K–12 grade levels. For example, for graph theory, it is recommended that "all students [in middle grades] should:

- Represent concrete and abstract situations by using vertex-edge graphs and represent vertex-edge graphs with adjacency matrices
- Describe and apply properties of graphs, such as vertex degrees, edge weights, directed edges, and isomorphism (whether two graphs are the "same");
- Use graphs to solve problems related to paths, circuits, and networks in real-world and abstract settings, including explicit use of Euler paths, Hamilton paths, minimum spanning trees, and shortest paths;
- Understand and apply vertex coloring to solve problems related to avoiding conflicts;
- Use algorithmic thinking to solve problems related to vertex-edge graphs;
- Use vertex-edge graphs to solve optimization problems." [1, p. 10]

Similar recommendations are offered for combinatorics and iteration and recursion [1, pp. 6–13]. Thus, courses in middle school teacher preparation programs (and graduate programs) should include discrete mathematics topics so that graduates will be prepared to introduce them to their students.

In the *Common Core State Standards* [6], topics in discrete mathematics are not explicitly mentioned. The intention of the standards are content topics for students to be college and career-ready and many writers do not yet see discrete mathematics topics as essential. They do mention discrete mathematics [6, p. 57] among a list of advanced mathematics courses appropriate for high school (along with courses such as calculus, and advanced statistics). As such, discrete mathematics is appropriate for the preparation of middle school mathematics teachers but its strength lies in it being contemporary, accessible mathematics where students learn and experience the behaviors of mathematical practice [6, pp. 6–7]. It is through attracting, exciting, and engaging students with mathematics that they regard as relevant that we begin to see success in problem-solving.

8.3 The Course Materials: *Making Math Engaging* Curriculum

The design of our materials and courses embodies the philosophy that prospective and practicing teachers need to build their level of mathematical knowledge and “teachers should develop the habits of mind of a mathematical thinker and demonstrate flexible, interactive styles of teaching” [7, p. 8]. We believe that prospective mathematics teachers should, in their undergraduate training, experience mathematics courses where they learn new mathematical content taught in a pedagogical style that they will be expected to emulate. Our materials focus on developing problem-solving skills and habits of mind that engage and invite the prospective K–8 teacher to think mathematically—initially, without realizing it!

The materials, *Making Math Engaging: Discrete Mathematics for K–8 Teachers* (hereafter, *Making Math Engaging*), consist of a textbook that focuses on the learning of the mathematical content, an activity book that helps frame good problem solving practices, and a classroom guide, that includes sample K–8 classroom lessons. The materials focus on graph theory topics, including those recommended by NCTM, and are written for K–8 teachers. The materials model the perspective that mathematical learning and thinking comes through engagement with the topic—in practice that means that students should read a bit, then figure things out a bit, then read some more, reflect, and figure things out some more. By reading the text and doing activities, students initially imitate mathematical behavior and then, over the course of a semester, they begin to develop their own habits of mind as they achieve success in solving hard problems.

We have field-tested materials at thirteen universities and colleges located in seven states with 317 undergraduate students and 30 graduate students from Spring 2005 through Fall 2007. While we field-tested across a wide variety of discrete mathematics courses, we learned that the materials are most valuable to college faculty looking to teach a mathematics course to middle school teachers (either an upper-level undergraduate course or a lower-level graduate course) or a mathematics course to elementary school teachers working on a mathematics specialist degree (as upper-level graduate course).

Currently, the discrete mathematics topics presented in the materials include graph coloring, properties of graphs, isomorphic graphs, regular and irregular graphs, trees, bipartite graphs, planar graphs, and applications of graphs. Through the content and the 62 carefully sequenced activities, students springboard into other mathematical conversations such as: even and odd numbers, counting problems, methods of proof, constructing examples and counterexamples, and problems in mathematics such as the Four Color Conjecture which became the Four Color Theorem in 1976 and the Traveling Salesman Problem for which no efficient algorithm is known.

The curriculum topics continue to evolve. We are including systematic listing and counting topics, and iteration and recursion into the activities so that the *Making Math Engaging* materials will parallel topics found in the Navigation series. Some courses use the *Making Math Engaging* materials as the only required text, while others require them along with the student’s choice of one of the Discrete Mathematics Navigation Series texts (either K–5 or 6–12). We will also be developing classroom activities for college classrooms in an Instructor’s Manual. For more information regarding the curriculum, including sample materials and a table of contents, see `www.shodor.org/discretemath/` or contact project PI Valerie DeBellis; contact information can be found on the website.

8.4 Examples of Course Content

Many colleges have different goals for creating a semester-long discrete mathematics course for K–12 teachers, with each college desiring a different list of discrete mathematics topics to be taught. Creating one set of materials that would satisfy all is unrealistic. However, most college faculty want a course that would build students’ problem solving and reasoning abilities while studying new mathematics. The outcome is topics in graph theory with activities that work similarly across colleges. Here are examples of activities:

The opening activity in the discrete math course is the U.S. map coloring problem. At the beginning of the class, students are grouped around maps of the continental United States where the interiors of the states are white, and each group is trying to determine (using chips of different colors) the minimum number of colors that must be used to color each state so that bordering states have different colors (so that you can recognize the borders). This is an important part of the course because it is an approachable and engaging activity that has endless variations and extensions. Students learn from the outset that they will often be challenged by questions such as: Why? How do you know? and What happens if we change the problem slightly? These are questions that the students are generally not accustomed to answering, and they set the tone of investigation, conjecturing, experimentation, and mathematical justification that will be a focus in the rest of the course.

The U.S. map coloring problem gives students a new appreciation for mathematical modeling and they learn structures of the model that will force (or not force) the use of a new color. As the problem illustrates, the graph theory content of the discrete mathematics course provides the engaging context and the real-world connections that help students relate to the mathematical focus of the course: argumentation and justification.

A second type of activity that provides students with a mathematical opportunity they seldom encounter in other courses concerns graph counting and classifications. For example, consider the problem: Construct all different (non-isomorphic) connected graphs with exactly five vertices, two of which have degree 3 and three of which have degree 2. (The degree of a vertex is the number of edges that emerge from it.) Such problems are appealing to the students because they present an immediate and identifiable task: to create graphs that meet the conditions. As is the case in the map problem, difficulties emerge as the task unfolds. When there are many graphs that meet the conditions, how do students organize their constructions to account for them all? How do they know that they have constructed all possibilities and that none of the graphs are the same (isomorphic)? What appears on the surface to the students as a simple problem involving listing solutions becomes a challenging problem in mathematical justification. Because the solutions can be complex and the justifications intricate and subtle, there are opportunities for students to hone their mathematical communication skills. They do not have to convince their professor–they must make arguments that satisfy their peers, often a more difficult task.

The link between "reasoning and justifying" builds an environment of sense-making, which ultimately leads to formal proof. Many teachers, whether prospective or practicing, have trouble providing complete proofs and trouble seeing why they need to do it. For example, when students try to find all possibilities in a situation, it is not always easy to see how to make an organized list and then explain to others why the list includes all possibilities and does not include duplicates.

For the bipartite graph:

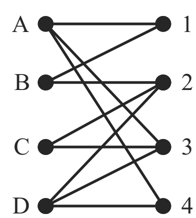

Figure 8.1. How many perfect matchings are possible in this graph?

Ask the question, "How many perfect matchings are possible in this graph?" Students typically work to find all perfect matchings in a bipartite graph with equal-sized parts and labeled vertices by listing as many matchings as they can find. (A *perfect matching* is a set of vertex-disjoint edges that contains all the vertices.) They conclude that they have all possibilities when they can't find another one and the items in their lists may not be distinct. The prospective or practicing teacher becomes aware of the need to have a system for listing the matchings so that all are convinced that each matching is listed exactly once. A problem that arises is how to keep track of choices.

For the graph in Figure 8.1, a student might start by noticing that A could match with 1, 3, or 4. The choices may or may not lead to a perfect matching as other vertices are considered and other choices made. The problem solvers may be overwhelmed with reasoning through all the possible edges to consider as matches and use a random strategy, hoping they will hit upon a solution. If the activity is allowed to finish in this direction, then, in the end, no one knows for sure if all possible choices have been made.

A way to help teachers organize their choices and keep track of their reasoning is to use a tree diagram. Each time the problem solver makes a choice, it is recorded by inserting another branch into the diagram. Using the graph from Figure 8.1, we might start the diagram with the three ways of matching the vertex labeled A. It can be matched either with 1, 3 or 4.

The first three branches of this tree represent the ways we can match vertex A. Now assume A is matched with 1. What might happen from there? B must be matched with 2, but C could be matched with either 3 or 4. If C is matched with 3, then D can be matched with 4 and we have found a perfect match. However, if C is matched with

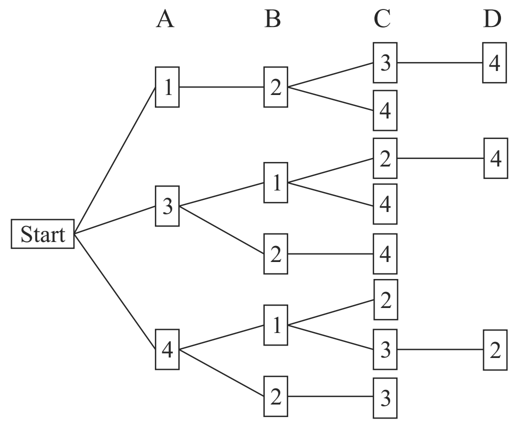

Figure 8.2. Tree Diagram for finding all perfect matchings for the graph in Figure 8.1.

4, then no perfect match can be found. Each branch is followed as choices are made and branches terminate when no choice remains. In the end, the matchings are maximal paths in the tree. Counting all the paths counts the number of matchings! This provides a visual way to understand that we have made all possible choices, and uses one of the basic structures studied in graph theory. It is interesting to discuss the differences and similarities in the trees that may arise using this strategy. Problems like this, provided students persevere, are understandable, make sense, and build confidence in their ability to reason mathematically.

Another example, found in *Making Math Engaging* [8], is John Conway's Game of Sprouts. The simplest version, considered in the book, is played as follows: The starting board consists of two vertices with no edges. Two players take turns making moves, as illustrated in Figure 8.3, where a move consists of connecting two vertices, not necessarily distinct, with an edge, and then putting a vertex on it, splitting it into two edges. There are two conditions: edges may not cross and vertices may not have degree greater than three. The last person who can make a move wins. These conditions force every game to last at most five moves.

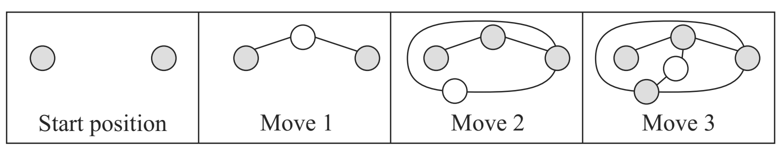

Figure 8.3. Three possible moves in the Game of Sprouts

The text asks students to decide who wins the game if Player I's first move consists of an edge drawn between the two vertices. Students had to answer this question so that they were certain that their answer was correct and could communicate this to the instructor. In short, they had to give a proof. Here are the steps by which the instructor led them to success:

- The class generated a list of all possible responses that Player II could give to Player I's opening move. When the list was complete, each response was placed on blue Post-It note that were distributed to groups of students, one Post-It per group.
- Each group was asked to generate lists of all possible responses of Player I to Player II's move, and each was placed on a yellow Post-It note.
- Each student received one or two of the yellow Post-It notes, and on green Post-It notes generated all possible responses of Player II to Player I's move.
- A game tree was constructed on the board, with one "Start" vertex, one vertex for Player I's initial move, then the blue Post-It notes for the next level of the game tree, yellow for the second level and green for the third. For each of the green Post-It notes, it was not difficult to decide who would win the game.

Thus, the class was able to work through a hard problem that few students could have solved individually. They built and analyzed a tree diagram of all possible moves following the first move. They persevered, argued, organized their data, convinced themselves of the winner, and acted as a community of mathematicians. The visual model helped students understand the problem and see how different thinking schemes can produce the same answer.

Our last content example is “human graph” activities. We call them “human graphs” because the vertices of the graphs are the students. The edges are strands of yarn held by two students. We ask a group of students to form a graph of a certain type using themselves as vertices and the strands of yarn as edges.

For example, the students can be divided into groups of five, with each group being asked to form a graph that has three vertices of degree 2 and two vertices of degree 3. Then two groups can be asked to determine whether the graphs they have formed are the same or different. This activity can serve as an initial step in the class’s effort to find all connected graphs with 5 vertices that have three vertices of degree 2 and two of degree 3, a problem discussed earlier.

We also ask each group of five students to form a graph that has two vertices of degree 2 and three of degree 3. Although this task seems similar to the previous one, it is impossible to form one. The activity leads to a discussion of properties of odd and even numbers (e.g., the product of two odd numbers is odd) and to the conclusion that a graph cannot have an odd number of vertices of odd degree.

Here is a human graphs activity for 10–20 students. We give each student one strand and ask her or him to hold one end and give the other end to another student who is not already holding two strands. We caution students not to create cycles, so if Tanya is connected to Cathy, who is connected to Bonnie, who is connected back to Tanya, we ask them to break one of the connections and to exchange some strands. (We also try to do that if we see four students who form a cycle, but it may be hard to spot groups that have formed larger cycles.) We now ask the students to untangle themselves. While doing so they are allowed to let go of a strand momentarily. What happens is that, despite our instructions not to form cycles, they have formed a large cycle (or, sometimes, two or more smaller cycles). This is surprising, and leads to a discussion of why this happens. In fact, any graph in which every vertex has degree 2 (as those they constructed) must either be a cycle or consist of two or more cycles. The students can then determine all possible outcomes that could arise with twelve students (for example) if we were able at the outset to eliminate all groups that had cycles of 3 or 4. There could be a cycle with 12 vertices, or one cycle of 7 and another of 5, or two cycles of 6—three possibilities altogether. If we weren’t careful to eliminate the cycles of 3 or 4, there could be other possibilities.

We also use human graphs to demonstrate problems involving graph coloring. For example, we construct the graph in Figure 8.4 on the floor, using paper plates for the vertices and masking tape for the edges, and ask fifteen students to stand on the fifteen vertices.

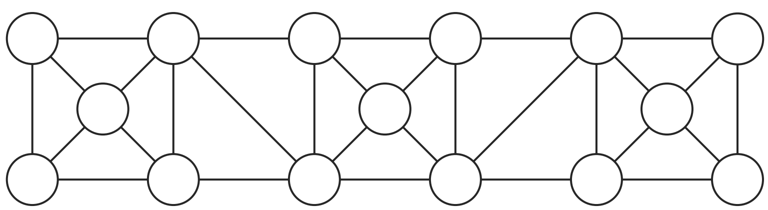

Figure 8.4. Floor Plan for the Human Graph Activity

The students are asked to say hello to each of their neighbors and to hold up their fingers to indicate how many neighbors they have so that the instructor can verify that all students know who their neighbors are. Each student gets three sheets of paper—red, white, and blue—and, when given a signal to start, is asked to color his or her vertex by holding up one of the three sheets of paper. If two neighboring students are holding up sheets of the same color, one of them must change colors. The goal is to color the graph so that each student is holding up a color and no two neighboring students are holding up the same color.

What often happens with the graph is that the students on the left color their part of the graph properly and the students on the right color their part of the graph properly, but the two colorings don’t fit together in the middle. Completing the task then becomes a good exercise in communication and cooperation. After they have completed the task, another group of students is asked to do the same activity . . . but this time without any talking.

For another graph coloring activity, one that can be challenging, we arrange 10–15 students in a circle, give each a strand of yarn, and have them find another student to hold its other end (but not one standing next to them). Students place their strands, in position, on the floor and we tell them that strands on the floor can not cross, so they may have to remove or rearrange some strands. (In fact, they will have to remove at least three strands.) The resulting graph is outerplanar, though we do not mention this. Next we give another strand to each student and ask that they create an edge between them and the person standing to their right. The students place these strands, in position, on the floor so that the graph is visible. Each student is now connected to the two students who are standing next to him or her in the circle and to zero or more other students. Each student has three sheets of paper—red, white, and blue—and must color his or her vertex with one of the colors by holding it up. Is it possible to color the graph using three colors? The surprising answer is yes, although for some configurations, it may be difficult.

Human graphs provide a way for tactile or kinesthetic learners to optimize their mathematical understanding. They learn best through experience or physical movement. They like to experience the world and act out events; human graphs provide a way for them to understand the mathematics content.

8.5 Course Formats of Established Discrete Mathematics Courses for Middle School Teachers

These course format examples will show how different types of universities with different teacher training programs implement the Making Math Engaging curriculum. One example is a capstone 300-level math course in an undergraduate middle school endorsement program (Northeastern Illinois University - NEIU); one shows the benefits of a cross-listed math course where undergraduate and graduate students share the same learning experience (University of Illinois at Chicago - UIC); one is an introductory math course in an undergraduate middle school endorsement program (Rutgers University - RU); one shows how a graduate discrete mathematics course is used to help middle school teachers who are elementary certified become highly qualified (The College of New Jersey - TCNJ); and the last example shows how discrete mathematics is implemented in an education course (East Carolina University - ECU). The courses are offered in programs that certify or endorse students. Certification programs build experiences toward a teaching licensure, endorsement programs give experiences toward providing a specialized statement appearing on a teaching certificate. Endorsements may identify a subject, such as “mathematics,” or a target population, such as “middle school,” that a certificate holder is authorized to teach.

8.5.1 Northeastern Illinois University (Tanya Cofer)

NEIU is a commuter college with approximately 12,000 students on the north side of Chicago. It is a federally-designated Hispanic-serving institution with an historical focus on teacher education. One of the newest programs is a middle school mathematics and science endorsement program called Concepts in Integrated Math and Science with Pedagogy. It immerses prospective teachers in integrated mathematics and science content and provides them with connections to middle school teaching. Students are recruited from the ethnically diverse communities served by NEIU and partnered community colleges, and most graduates are committed to working in the Chicago public school system.

NEIU’s Concepts in Discrete Mathematics for Middle School Teachers course (MATH 381) is a capstone 300-level course in the endorsement program. It has been offered every spring semester since 2006. Before enrolling in it, students generally complete all five required integrated mathematics and science course pairings, which are taught jointly by mathematics and science faculty. The pairs are: geometry and physics, numbers and chemistry, algebra and biology, statistics and environmental science, and a capstone calculus and ecology pair. The discrete mathematics course is a stand-alone course designed around the *Making Math Engaging* materials. It has two foci: making real-world connections and emphasizing mathematical argumentation and justification.

Following the *Making Math Engaging* curriculum, there is emphasis on active learning and group problem-solving. A short, interactive lecture is provided, and then students work in groups of three or four on interactive activities. On completion, representatives are asked to come to the board to present their solutions and to justify their conclusions mathematically. A comfortable atmosphere is cultivated in which students accept being challenged mathematically by their peers and their instructor. In this way, students construct their own content knowledge, learn to communicate mathematically, develop skill in constructing mathematical arguments, learn to critique the arguments of others, and learn to appreciate and interpret varying approaches to solving a problem. There is no pressure to cover a set amount of material: in a capstone course, there is the freedom to spend as much time as necessary on fruitful and challenging problems and arguments. It is a course on developing mathematical habits of mind and on learning that true mathematical understanding starts (not ends) when a task is completed.

The graph classification problems are appealing because they focus on constructive arguments with concrete mathematical objects (vertex-edge graphs). The program students see that the material is easily translated to the middle school level. Class and discussions with students often revolve around adaptations of material to the middle-level. The program's focus on modern and reform approaches to curricula help the students understand that, for them and their future students, both mathematical content and process are important. In the first three years that the course was part of the middle school endorsement program, at least six program students have used materials adapted from it as part of lessons designed for middle school students with similar trends continuing beyond these initial years.

The concepts and activities presented in the *Making Math Engaging* curriculum provide numerous opportunities for prospective middle school teachers to view mathematics in places and contexts that are new, unexpected, and appealing. They learn that mathematics is not only about numbers and quantities, but also about reasoning, relationships, and justification. There is mathematics in games (Sprouts, for example) and in map coloring and scheduling. This new way of seeing mathematics can affect the mathematical practice of middle school teachers for the better. Student reviews of the course have been encouraging. One student stated that "the reasoning and logical thinking is extremely helpful" while another commented that "this course was incredibly useful and should be taught for all math teachers. Concepts here are relevant."

Over the 2009–2010 academic year, NEIU's middle school mathematics and science endorsement program was expanded to incorporate practicing middle school teachers who wish to return to seek endorsements. The expansion provides us with an opportunity to adapt the discrete mathematics course (and other program courses) to appeal to and suit a new audience. We will also try to find ways to encourage prospective and practicing teachers to work together to broaden and deepen their content experiences. As we expand and adapt the course, we will continue to build a library of resources.

Resources from NEIU's discrete math course can be found at:

`www.neiu.edu/~tcofer/MATH381/NEIU_DM_Resources.html`

General information on NEIU's Middle School Math and Science Endorsement program can be found at:

`www.neiu.edu/~middle/`

The Chicago Public Schools site is:

`cmsi.cps.k12.il.us/ViewNewsDetails.aspx?pid=899&id=12255`

8.5.2 University of Illinois at Chicago (Bonnie Saunders)

UIC is a major research university with 25,000 students in Chicago. The UIC Mathematics, Statistics, and Computer Science (MSCS) department is ranked in the American Mathematical Society's Group I. Within MSCS the Office of Mathematics Education (OME) offers five degree programs and Illinois teaching certification to prepare elementary, middle, and high school mathematics teachers.

On a semi-regular basis MSCS offers the course, MTHT 491 Topics in Teaching Elementary/Junior High School Mathematics: Discrete Mathematics. It may be used to meet requirements for several programs offered at UIC or by the Illinois State Board of Education and has been offered since 2006.

Discrete Mathematics is an elective course in the Middle School Endorsement program and the Masters in Elementary Teaching program at UIC. It is cross-listed so that undergraduate students who are elementary education majors with a concentration in mathematics and practicing elementary teachers can enroll. Some graduate students who are not yet teaching will enroll in the course either for progress towards an endorsement or for fulfilling requirements for their degree. Most, but not all, of the teachers work in the Chicago Public Schools (CPS).

The course was first offered fall semester 2006; teachers were given support from CPS so there was a full class of twenty-six practicing teachers and four undergraduate students. The second time it was offered there was not the same level of support; three teachers, six undergraduate students, and one graduate student enrolled. Fall semester 2008 there were thirteen teachers, one undergraduate student, and one graduate student. Undergraduate students are more likely to be able to take the course if it is offered in the spring due to the scheduling of required courses. Many of the students in the first course were enrolled or subsequently enrolled in our MST program and are now graduated. Many are teacher leaders in the Chicago Public Schools.

The class meets once a week for 15 weeks. Class meetings are three hours in the evening to accommodate practicing teachers' schedules. We use the Making Math Engaging curriculum and in a typical class, we work through the activity book with students discussing and presenting their solutions first at their table and then to the entire group. Direct instruction is kept to a minimum. This, in part, is made possible because the text is written so that practicing and prospective middle school teachers can read and understand it. There are regularly scheduled quizzes and exams, and required homework is assigned each week.

Students in the middle grades programs at UIC find discrete mathematics fun and challenging in a way that makes them better mathematicians. As one teacher commented, "The idea of struggling with a problem in class has encouraged me to have my students struggle more with problems." and another commented, "This course has changed my old perspective about math as a dry subject' to math, the rainbow subject.' Each new lesson gravitated me toward it and perked and alerted my brains and mind."

Information about the Mathematics Education Programs in MSCS at UIC can be found at

`www.math.uic.edu/mathed/`.

Bonnie Saunders' webpage is

`www.math.uic.edu/~saunders/`.

8.5.3 Rutgers University (Joseph G. Rosenstein)

RU is the state university of New Jersey and is the state's premier public research university. Rutgers' three campuses, in Newark, New Brunswick, and Camden have over 52,000 students; its main campus in New Brunswick has 36,000 students, including over 28,000 undergraduates.

Until recently discrete mathematics was included in New Jersey's mathematics standards K–12. Graphs and combinatorics were included in Standard 4 (Data Analysis, Probability, and Discrete Mathematics) and recursion was included in Standard 3 (Patterns and Algebra). This Curriculum Framework in Mathematics, developed in response to an earlier version of the standards, includes a chapter of classroom activities and is available at `dimacs.rutgers.edu/nj_math_coalition/framework.html`.

In New Jersey, teacher licensure for middle school can occur in two ways. One can be certified to teach K–12 in a subject (usually by completing a Bachelor's degree in the subject and passing the subject's Praxis II test), or someone who has K–5 elementary certification can obtain a middle school endorsement by completing 15 credits in the subject and passing the subject's Praxis II exam. The latter approach was adopted in 2004 as a response to the No Child Left Behind (NCLB) act. Prior to 2004, elementary certification in New Jersey was K–8.

At Rutgers, members from the Department of Mathematics have developed three new courses for prospective K–8 teachers in response to the new middle school endorsement in mathematics—one on numbers and algebra, one on geometry, and one on discrete mathematics and problem solving. All three have a strong emphasis on reasoning and communication. They may also be taken by prospective teachers who need to take a mathematics course to complete their general requirements. A student must pass an algebra placement test in order to enroll in any of them. In addition, students who seek the middle school mathematics endorsement will typically also take precalculus, calculus, or mathematics for the liberal arts to complete the required five math courses.

The discrete mathematics course is given in the spring semester (beginning in 2007) and meets twice a week for 80 minute sessions. It attracted nine undergraduate students in 2007, seventeen in 2008, and twenty in 2009. The course is titled as Problem-Solving and Reasoning with Discrete Mathematics and it is described as "an activity-based course where you will be working in groups to investigate problems that are interesting and engaging and that can be used in K–8 classrooms." Students are informed that "at the elementary and middle school levels, discrete math is a good arena for discussing problem-solving and reasoning, which play an important role in national and New Jersey math standards."

Since the course is given in the mathematics department, we focus on content but students learn through activities and discovery. Each class session includes an activity that the students complete, usually in pairs or larger groups, and then discuss with the entire class. Because these methods appeal to the learning styles of prospective middle school teachers, they gain a deeper understanding of the content. Students are given a weekly homework assignment that is reviewed by the instructor, two exams during the semester, and a final exam at the end of the semester. Students are given reading assignments and are expected to come to class prepared to answer questions based on what they have read. The text for the course is *Making Math Engaging*, and most of the topics in these materials are discussed over the course of the semester.

Joe Rosenstein's webpage is

`dimacs.rutgers.edu/~joer/joer.html`.

8.5.4 The College of New Jersey (Cathy Liebars)

TCNJ, formerly Trenton State College, is a highly selective, public liberal arts institution with approximately 6000 undergraduate students and 1000 graduate students. TCNJ's campus is located approximately halfway between Philadelphia and New York in suburban Ewing Township where it began as Trenton Normal School.

After No Child Left Behind was passed, there were many middle school mathematics teachers in local school districts who were elementary certified and not highly qualified according to the new law. Through grant funding, a series of graduate mathematics courses for middle school teachers was developed at TCNJ and taught on site after school in Trenton. The courses were designed to help middle school teachers become highly qualified. Most elementary certified teachers are not familiar with discrete mathematics. Some do not even know what it is, so one of the courses that was developed is Math 597, Discrete Math for Middle School Teachers.

The course was first taught in the fall of 2006 with twelve middle school and high school teachers from the Trenton School District and again in fall of 2009 in the Ewing School District with six teachers enrolled. The primary focus of the course was on content, but it included methods of teaching the content using physical materials, models, technology, and middle school curricula. *Making Math Engaging* was used as the primary textbook for the course. Assessment consisted of homework assignments, three short tests, a mid-term project, and a final project that required the teachers to redesign two lesson plans using what they learned in the course and teach at least one of the lessons with a written reflection to follow up. Approximately two-thirds of the semester was spent on vertex-edge graphs using the *Making Math Engaging* text. Topics included coloring maps and graphs, the four color theorem, regular and irregular graphs, trees and tree diagrams, bipartite graphs, planar graphs, and applications of graphs. Some of the text was skipped because it was too difficult for the audience. The remainder of the semester was spent on logic and sets, combinatorics, and recursion, using *Discrete Mathematics for Teachers* by Ed Wheeler and James Brawner as a resource [9].

After the middle school endorsement went into effect in New Jersey, more and more elementary education majors at TCNJ began to realize that they can become more marketable if they obtain a middle school endorsement, particularly in areas of shortage such as mathematics and science. So there was a need to develop courses designed specifically for this audience that would prepare them for teaching mathematics at the middle school level. In response, undergraduate versions of the graduate level mathematics courses for middle school teachers were created. They were the same as the graduate versions, although the students were assessed differently; instead of completing a final project, the undergraduate students took a final exam.

MAT 117, Discrete Math for Middle School Teachers, was taught at TCNJ in the spring of 2008 with fifteen students and in the spring of 2010 with seventeen students. The students in the class were mostly Elementary Education majors with a few Special Education and Deaf/Hard of Hearing Education majors. Essentially the same topics were covered as in the graduate course with a few topics added, such as factor graphs and perfect matchings. Some topics, such as proof by contradiction and graph constructions, were discussed in more depth than in the graduate course.

For the mid-term project in both courses, students were given a choice of three projects that each involved a different aspect of vertex-edge graphs. For example, one of the project choices was

> You have decided that you want to teach your students about isomorphic graphs and have decided to use The Geometer's Sketchpad (GSP) to do it. Create two sets of two isomorphic graphs on GSP that do not look like they are isomorphic. For each set of graphs, use GSP to move around the vertices to make the images identical. (Label each of the vertices of the isomorphic graphs so that you can tell the same vertices are connected to same other vertices.) Hand in a printout of prior and post movement for each set of graphs and a set of directions or worksheet that your students would do to complete the activity.

Another project involved an application of graph coloring to grouping students in their class and a third involved a traveling salesperson. Students were evaluated on their mathematical accuracy, organization and clarity, and presentation of the project.

Syllabi for the graduate courses can be found at

centerforstem.pages.tcnj.edu/tcnj-collaboration/courses/.

8.5.5 East Carolina University (Margaret Wirth)

ECU is a constituent institution of the University of North Carolina, with an enrollment of nearly 26,000 students, located in eastern North Carolina. It is the third largest university and the largest teacher training institution in the state system.

In 1994, ECU initiated the NSF-funded MIDDLE MATH Project (DUE-94455152). This undergraduate faculty enhancement effort involved 60 mathematicians and mathematics educators in an effort to define the content, instructional methods, and tools needed to develop effective programs. ECU's discrete mathematics course was developed during the 1996–1999 academic semesters. At that time, it was housed in the mathematics department. Since 2004 it has been taught and redesigned by the Department of Mathematics and Science Education (MATE), which is now the department of Mathematics, Science, and Instructional Technology Education (MSITE).

The mathematics concentration at ECU is designed to prepare teachers of middle grades mathematics, but it serves other populations, including the group of elementary education majors who concentrate in mathematics. Middle grades education majors have eight 3-semester hour courses required for their mathematics concentration. They have been designed for middle grades teachers. Two courses are now taught by math faculty (MATH 1065 College Algebra, and MATH 2129 Elements of Calculus) and six courses are taught by mathematics education faculty (MATE 1267 Functional Relationships, MATE 2067 Data and Probability Explorations, MATE 3067 Algebra and Number Foundations, MATE 3167 Geometry and Measurement, MATE 3267 Concepts in Discrete Mathematics, and MATE 3367 Mathematical Modeling).

The discrete mathematics course, MATE 3267, is taught once a year and taken during the spring semester of a student's junior year. It is one of two capstone courses in problem solving and alternates with the MATE 2700 discrete mathematics course for secondary mathematics majors. Although designed for middle school prospective teachers, the MATE 3267 course enrollment often includes a mix of elementary and secondary mathematics prospective teachers. Enrollment has increased over the years and varies from fourteen to twenty students per class each spring. Currently, the course is being taught in both an online format and a traditional classroom format.

MATE 3267 includes a variety of discrete mathematics topics to satisfy state content requirements. *Making Math Engaging* curriculum is used to cover problem solving, graph theory, and mathematical induction. Teacher supplements are used to discuss social choice, dynamical systems, logic, and sets. Since 2004 the course syllabus has included a unit plan component that requires students to collaborate in groups of four to five to create a set of electronic lesson plans including worksheets, projects, assessments, and PowerPoint presentations for a discrete mathematics topic as a culmination activity in lieu of a final exam. The students choose a topic from an instructor-prepared list that includes topics that have not been taught during the semester.

The students are assigned groups based on their choices and begin working on their unit plans during the last week in March. By the end of March the instructor, using activities and problem solving techniques suggested in the class text has taught a unit on graph theory and introduced probability and social choice, integrating matrices in the lessons. The students are to model these class lessons in the creation of their units. The unit plans are due in electronic form and in hard copy on the last class day of the semester. The students follow up with group presentations of their unit plans to the class during the final exam period.

The creation of the discrete unit plans serves several purposes. The topics taught in the discrete course are a first-time experience for the undergraduates. Creating lessons from unfamiliar content material requires groups to do research on the topic and follow up with discussion about it before beginning their unit construction. Determining the sequence of learning objectives, organizing the content for optimal student understanding, creating engaging lessons, and then providing assessments to determine if the students learned anything, all while teaching yourself the topic is a worthwhile learning experience for future teachers. Collaboration is emphasized and supervised by devoting several regular classroom sessions to group work and unit discussions with the instructor. Although the unit plan assignment is structured, the actual creation of problems, projects, and assessments for the unit is open and allows for creative problem solving and discovery.

The resulting unit plans vary in sophistication and reflect the levels of the undergraduates' teaching expertise, communication skills, mathematical knowledge, and instructional abilities. Some are tool sets and do not reflect a high level of problem solving, while others are engaging and involve real problem solving strategies. The students know the units are a first step in preparing discrete mathematics lessons for their classrooms. These are not meant to be a final plan for any topic, but rather one constantly under construction as the students evolve in their teaching careers. The compendium of units is copied electronically and distributed to each student at the exam day group presentations. At least four unit plans per semester are created, which means that students receive at least seven discrete mathematics unit plans (three from the class lectures, four from their unit plans) ready to use when they are teaching in the field.

The success of the units can be measured by the responses of the undergraduate students who are proud of their products and eager to teach their new discrete mathematics units to middle grades students. Through word of mouth and professional presentations, requests for copies of the discrete mathematics units from practicing teachers in the field has been an added and unexpected by-product of the course unit assignment. The experienced teachers understand the time and effort necessary to create effective lesson plans and value the units for their own classroom instruction.

Copies of the course syllabus, course calendar, unit plan, and sample units are available at `www.personal.ecu.edu/wirthm`.

8.6 Conclusion

Discrete mathematics is an important course to offer prospective and practicing middle school teachers. The content (such as vertex-edge graphs, combinatorics, and iteration and recursion) is important to their digitally-native students, who are beginning to think about possible career paths, and it can help facilitate a community of mathematicians among teachers. The core of the subject can be an exploration of the heart and soul of mathematics: reasoning and justification. Prospective and practicing middle school teachers will begin to reason, argue, communicate, write, and create representations to say mathematically what they mean as they engage in problems that are easy to understand but difficult to solve. The students can work on challenging problems with colleagues, sometimes for long periods of time, and experience mathematics as an endeavor rather than an algorithm.

Vertex-edge graphs build bridges for middle school teachers, connecting abstract and difficult mathematical ideas such as graph isomorphism and inductive arguments to approachable mathematical tasks that have concrete representations. The courses in which the *Making Math Engaging* materials have been field tested serve diverse middle school teacher populations. Some students take it as an introductory mathematics course; some undergraduates take it as a capstone experience. Still others take the course as practicing teachers for professional development. Whatever department offered it, the essence of the material remains. The tasks and concepts are approachable for all prospective and practicing teachers. The level of the student and the expectations of the instructor set the level of the course. In programs that emphasize connections with the sciences, the modeling and applied aspects of the content can be emphasized. In an education course, students can spend time working on activities themselves and then adapting their ideas into lesson plans suitable for their classrooms.

The curriculum puts engaging concepts in the classroom and empowers the students and instructors together to create and adapt their courses. It is this adaptability that makes the material an integral part of the NCTM standards for school mathematics. For middle school teachers and students, discrete mathematics is a great equalizer. It is a subject that requires little prerequisite knowledge (unlike calculus), yet challenges students to make rigorous mathematical arguments and to explore patterns and connections. Such a course embodies 21st century mathematical thinking and problem-solving.

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9

Guided Discovery in a Discrete Mathematics Course for Middle School Teachers

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We describe a discrete mathematics course for in-service middle school teachers taught exclusively using guided discovery. The article describes the structure and facilitation of the course itself and the set of notes used as a textbook. We discuss the benefits of using guided discovery with middle school teachers and student reaction to the experience.

9.1 Introduction

The Master in Teaching Middle School Mathematics (MSM) program at Salem State University is designed for middle school teachers who already have an initial license to teach grades 5–8 math in Massachusetts (or equivalent certification) and who wish to be eligible for professional licensure with certification in middle school mathematics. The program was established in the spring of 2004 under a Title II-B: Massachusetts Mathematics and Science partnership grant.

We discuss the use of guided discovery in the MSM course in discrete mathematics, one of nine mathematics courses required for the degree. The courses provide the students with a deeper understanding of the mathematical content in or related to the middle school curriculum. In each course there is a emphasis on problem solving as well as some discussion of logical reasoning and proof writing. Most students in the MSM program have not majored in mathematics and their background in formal mathematics is limited.

In the spring of 2008 the MSM discrete mathematics course was taught by Reva Kasman. In the class were twelve middle school teachers and one high school teacher. Only one middle school teacher had been an undergraduate math major and the high school teacher had minored in math. She chose a guided discovery approach for the course, with students using course notes written by Ken Bogart [2] and adapted by Mary Flahive [3] as a textbook. We describe the logistics of the class and reflect on the experience of teaching the course in this manner.

9.2 Course Materials

9.2.1 Adapting the Notes for Middle School Teachers

Ken Bogart's original notes were designed for undergraduate math majors. Although using an upper-level textbook with middle school teachers can be difficult, the problem solving format of the notes allowed an easier adaptation than more formal presentations.

[1] This material is based upon work supported by the National Science Foundation under Grant No. 0410641.

For an example of how the notes can be effectively used at a variety of levels, we consider one of the first problem sequences. Students are presented with the problem:

> The coach of the team in Problem 3 knows of an ice cream shop along the way where she plans to stop to buy each team member a triple-decker cone. The store offers 12 different flavors of ice cream, and triple-decker cones are made only in homemade waffle cones. (Repeated flavors are allowed; a triple-decker with three scoops of the same flavor is possible. Be sure to count Strawberry, Vanilla, Chocolate as different from Chocolate, Vanilla, Strawberry, etc.)
>
> (a) How many possible triple-decker cones will be available to the team members?
>
> (b) How many triple-deckers have three different kinds of ice cream?

Students do not have any formulas or suggested approaches at this point, and they are likely to experiment and use their intuition. The next problem in the sequence introduces the definition of a function and asks several questions, such as:

> How many functions are there from the 3-element set $\{1, 2, 3\}$ to the 2-element set $\{a, b\}$?

The sets are small so it is possible to list all functions, and the instructor can suggest the usefulness of arrow diagrams or input-output tables (the latter are usually familiar to the teachers) as alternatives to the formal notation of the text. Since students are apt to count each row of a table (see Figure 9.1) as one function, this provides an opportunity for clarifying that each table represents a single function.

x	$f(x)$
1	a
2	b
3	a

or

input	output
1	b
2	b
3	b

Figure 9.1. Functions represented by input-output tables

The notes next direct students to compare the number of functions from a 3-element set to a 2-element set with those from a 2-element set to a 3-element set. The instructor can recommend more experimentation until students are able to identify a general pattern and handle a subsequent problem for which an exhaustive list would be overwhelming:

> How many functions are there from any 3-element set to any 12-element set?

With that result in hand, the original counting problem reappears, now in the context of functions:

> Redo [the ice cream problem] by constructing a function from the 3-element set of positions in the triple-decker to the 12-element set of flavors. Give an explicit verbal description of your function.

At this early point in the notes students are told that the inputs should be the three cone positions and the outputs should be the twelve flavors, so this is a natural place for a discussion of why the reverse choice does not work: "What happens if our inputs are the flavors and our outputs are the cone positions? Would our definition of function work if we had a cone with vanilla on both the top scoop and the bottom scoop?"

While discrete mathematics is often not an official part of the middle school curriculum, this problem sequence on functions correlates well with the recommendation in the 8th grade section of the Common Core State Standards for Mathematics (see page 52):

> Students grasp the concept of a function as a rule that assigns to each input exactly one output.

Developing student understanding of functions is a continuing theme of the notes, and this discussion foreshadows later problems where students are expected to have sufficiently refined their knowledge of the concept of function so they can define the domain and co-domain as well as the relation.

In their algebra standards for grades 6–8 the National Council of Teachers of Mathematics recommends that students be able to "represent, analyze, and generalize a variety of patterns with tables, graphs, words, and, when possible, symbolic rules". Nearly all problems in the notes expect students to identify or apply patterns, and to model information in useful and appropriate ways.

The Massachusetts Curriculum Frameworks for Mathematics recommends that students in grades 5–6 "use tree diagrams and other models (e.g., lists and tables) to represent possible or actual outcomes of trials [and] analyze the outcomes". Trees appear explicitly in chapter 4 on graph theory, but tree diagrams also arise naturally in the solutions to counting problems such as

> One of the schools sending its team to the tournament has to travel some distance, and so the school is making sandwiches for team members to eat along the way. There are three choices for the kind of bread and five choices for the kind of filling. How many different kinds of sandwiches are available?

The most valuable lesson for middle school teachers using the notes is not the content but the experience of learning in this manner. Common Core State Standards (CCSS) have recently been adopted by Massachusetts and many other states. These include (page 53) eight recommended Standards for Mathematical Practice, described as "varieties of expertise that mathematics educators at all levels should seek to develop in their students." The first three are:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.

To develop these skills in their students, teachers should become model practitioners themselves. The characteristics of a proficient learner recommended by the CCSS become second nature to the teachers in the class as they puzzle through problems without examples to mimic, actively discuss mathematics with their peers, and share their ideas clearly in writing.

9.2.2 Structure of the Notes

Bogart's method of guided group discovery [4] built on earlier work in pedagogy, including Neil Davidson on small group discovery [5]; Ed Dubinsky, et al., on the genetic decomposition of mathematical knowledge [1, 6, 7]; and Alan Schoenfeld on problem solving [11]. Unlike many classes in which information is conveyed primarily in lectures, students in any group discovery class play a principal role in the acquisition of content knowledge and are actively involved in their own learning both inside and outside the classroom. The course notes consist almost exclusively of problem sequences, with a small amount of connective exposition. There are no worked examples in the text and no solutions are provided at the back of the book. The notes were constructed to contain enough problems for students to discover for themselves key concepts while not containing so many that students would be overwhelmed by the volume of new ideas. Each sequence begins with low-level questions that involve concrete experimentation. The difficulty and the level of abstraction gradually increase within the problem sequences. The discovery-based approach easily allows an instructor to choose how far to go in each unit. The early problem sequences emphasize the importance of understanding small examples and this is a continuing theme throughout the notes. As the book proceeds, students are explicitly encouraged to try multiple methods of finding solutions to problems, as well as to make connections with prior information.

The problems allow for some flexibility in the level of justification required for a proper solution. Although rigorous formal proofs were not required in our class, the use of detailed contextual explanations was encouraged in both verbal and written work. As an example, for a question on using a function to describe the distribution of twelve tee-shirts to nine students, the statement "The function is surjective because every element of the co-domain has a preimage" would not be considered sufficient justification, while the explanation "The function is surjective because the co-domain is the 9 students, the domain is the 12 tee-shirts, and we know that every student gets a tee-shirt" would be satisfactory.

The table of contents for the notes is given in the appendix on p. 95. Our class covered most of the combinatorial principles in chapter 1, including the sum, product, bijection, pigeonhole principles and Ramsey numbers. Students also worked through the sections from chapter 3 on equivalence relations, the binomial theorem, Pascal's triangle, and Catalan numbers; chapter 4 on graph theory and some optimization algorithms; and the additional material on functions and digraphs in the first appendix of review material.

Chapter 2 of the notes covers formal mathematical induction. From her previous experience using the notes with in-service high school teachers, the instructor thought students tended to concentrate on the standard template for these proofs and as a result missed the inductive nature of the arguments. To avoid this with the middle school teachers, chapter 2 was replaced with supplementary materials that focused on applied recursion problems and concrete inductive reasoning. (Section 9.5.2 contains a further discussion of these materials.)

9.3 Course Structure

The class met weekly in 140-minute sessions for a 15-week semester. The learning process for the students consisted of several components: at-home preparatory work, in-class small-group discussions, in-class presentation of solutions, and subsequent demonstration of mastery through homework and exams. The final project was an instructional module at the level of their home classroom.

9.3.1 Preparatory Work

Students had their introduction to each new topic outside of class. A week before each class period students were assigned a selection of approximately six to ten (usually consecutive) problems from the notes to work on at home. This developed essential concepts and results. The work began by investigating concrete examples or applying formal definitions. Subsequent problems often directed students to conjecture more general results and to support their conjectures with evidence and proof. We have compiled a list of the weekly homework assignments.

Students were expected to work on the problems on their own and to bring their results to class, regardless of their state of completion or correctness. The experience of reading and interpreting unfamiliar mathematical text was new to the students, as was the absence of worked examples and a solution appendix in the text. Students were encouraged to keep records of their experimentation, questions, and conjectures, which provided the basis for in-class exploration with their group.

9.3.2 Small-Group Discussions

Due to the advanced preparation, students came to class already engaged in the learning process and generally eager to meet with their peers to compare answers and to solve any challenging problems that they had been unable to solve independently. When the class period began, students were broken into groups of three to four students, sometimes assigned by the instructor and sometimes self-selected. The assignment of groups was designed to ensure that students worked with everyone in the class at some point during the semester and to prevent students from consistently relying on one of their stronger classmates for direction.

The small groups engaged in mathematical dialogue for approximately 80 minutes—they discussed solutions, puzzled through examples, and developed strategies for any problems that were still unsolved. During this time the instructor circulated and spent several minutes with each group to listen, ask questions, and draw out ideas, always remaining with a group only long enough to ensure that they were able to continue having a mathematically fruitful discussion. Although the instructor did not confirm the correctness of solutions, she occasionally suggested examples for the students to investigate or posed related questions that were likely to provide insight if students were on a misguided path or felt unable to move forward.

9.3.3 Presentation of Solutions

After a short break, the remainder of class time (about 50 minutes) was devoted to presentations of the solutions from the groups. Because there was not enough time to go through every solution, the students and instructor would jointly compile a list of problems that seemed most important to discuss. Students were naturally interested in seeing the solutions to the more challenging problems, but the instructor made sure that the problems would contain all the essential ideas of the unit. Having observed the groups in the first part of class, the instructor also tried to include problems where students had particularly innovative or diverse approaches.

Each student was expected to present at least two solutions at the board during the semester. Of the twelve students who completed the class, eight presented between four and seven problems, one gave three presentations, and only three students satisfied just the minimum requirement of two problems.

Presentations were generally done voluntarily, but as the semester progressed students would supportively prompt shyer members to present the group's efforts. On occasion, the instructor encouraged particular students to share an alternative solution or to show an idea that other groups had overlooked. Students were supportive of each other during presentations, helping each other by catching errors and filling in the necessary details. However, debate was not uncommon when multiple approaches to a problem were presented and students had strong feelings about which technique they preferred.

Normally the instructor observed presentations from a seat at the back of the room and allowed the students to direct the discussion. On the rare occasions when the instructor chose to spend a few minutes at the board she generally shared an extra example to clarify a concept or reminded students of an earlier problem that was connected to something they had just discovered. The instructor did not want to establish a pattern of providing the last word on matters as that would inhibit the spirit of independence and sense of empowerment that the students were developing.

9.3.4 Homework Sets

As mentioned earlier in section 9.3.1, students were assigned a selection of problems from the notes every week to prepare for the next class meeting. These problems were usually well-mined after the in-class discussion and often full solutions had been presented to the whole class. Furthermore, the linear nature of the notes precluded saving problems for the graded homework sets.

Thus, the instructor created supplementary homework assignments with exercises modeled on the problem sequences from the notes. To avoid confusion, "problems" refer to the numbered questions in the notes, while "exercises" refer to the questions on the supplemental homework assignments. A homework set was due approximately every 3–4 weeks, providing a chance to accumulate sufficient familiarity with a topic before applying it to the homework exercises.

Working the exercises gave students an opportunity to demonstrate their individual mastery of the material. Another critical goal of the homework sets was to require students to explain their solutions carefully from beginning to end with clearly written justifications. Although rigorous formalism was not required, students were expected to write detailed and logical explanations and to use proper notation and terminology. Moreover, homework sets were the most significant source of individual feedback and evaluation from the instructor, since the instructor's role during class was limited to providing general responses to group concerns.

9.3.5 Exams and Review Days

Students demonstrated their mastery of the material on two in-class exams—a midterm and a comprehensive final. As with the homework sets, the exams included computational and short-answer explanation questions and required students to apply what they had studied to new situations.

With a guided discovery textbook, students tend to lose sight of how much they have actually learned. So, the class periods before the midterm and final exams were used as opportunities to take stock of accumulated knowledge. The preparatory work for them consisted of listing and defining key principles and terms (both formally and in the students' own words) and illustrating these with examples. Students were asked to identify problems in the notes that exemplified different concepts, to create their own problems that would use the same techniques, and to determine where they still had any confusion. The review periods began as usual with small-group discussion, during which many of their questions were answered by their peers. The instructor addressed any remaining concerns at the board during the final hour of class rather than having student presentations.

9.3.6 Instructional Module Project

For a final project, students developed an instructional module designed at the level of their home classroom. As explained in the instructions, students were expected to choose a topic in discrete mathematics not explicitly covered in the course. The topics chosen included voting theory, apportionment, probability, expected value, the golden ratio, Fibonacci patterns, and graph coloring.

Each project had both a written component and a 15-minute oral presentation to the class. The written part consisted of an explanation of the mathematical content of the chosen topic, lesson plans, worksheets, and general guidance for a teacher. The students incorporated many hands-on investigations and experimental classroom activities into their lesson plans, although they were under no obligation to use a discovery-based approach. Two groups (the ones working on the golden ratio and the Fibonacci numbers) used their activities in one partner's classroom before presenting their projects to our class.

Students could work individually or in pairs for this project. To simplify scheduling the in-class presentations and to encourage collaboration, the instructor preferred that students work with a partner. Ten of the twelve students worked in pairs. However, insisting that everyone do the project with a partner might pose an undue burden since most of our students work full-time during the day.

9.3.7 Grading Scheme

While participating in group discussion is essential for student success in guided discovery, it is vital that students demonstrate an independent mastery of mathematical content. Because of this, most of the assessment for the course was based on individual work. Percentages for the course grade were allocated as follows:

10%: overall participation
35%: written solutions to homework sets
15%: instructional module project
20%: in-class midterm exam
20%: in-class final exam.

Information on the assessment of the instructional module project can be found on the last page of the instructions.

Students' overall participation grades were based on their preparation of assigned problems, the quality of their involvement in the small groups, their contributions to whole-class discussions, and their presentations of solutions. The instructor assessed participation weekly and assigned each student a score out of 5 possible points. At the start of term students received a rubric distinguishing the levels of 0-5 points. Most students earned the full five points, though early in the semester a few students received four points due to their reluctance to talk in their groups. As students became used to the class and their peers, this ceased to be a problem and usually everyone fully participated.

9.4 Establishing the Class Format

The students were naturally apprehensive about the prospect of a group discovery course, and at the start of the term the class had several concerns. Many students believed that their ability to do (and teach) mathematics had come from studying worked examples and then applying similar techniques to similar problems. Consequently, the lack of worked examples and solutions could seem like an insurmountable obstacle to their learning. Moreover, the (intentional) sparseness of formal exposition in the course notes led some students to wonder if they were being asked to teach themselves the course material without adequate tools. For some students this initial anxiety was exacerbated by the fact that they were unfamiliar with the professor, since they did not yet have an established trusting relationship with the person who was ostensibly there to guide and evaluate their work. From previous experience teaching a course in this style, the instructor knew that it was especially important to address these issues at the beginning of the term and to include a discussion of the pedagogical reasons for choosing a guided discovery approach. Students were frequently reassured early in the semester that they were not isolated in the learning process and that their grades would be based on clearly defined assessments.

To help the students adjust to the textbook and the class format, a typical class period was modeled on the very first day, with the obvious exception that the students had not worked on problems in advance. After an initial discussion of what guided discovery means and what students could expect in the class, small groups were formed and students worked together on the first six problems in the notes. Later in the period, volunteers presented solutions at the board and the class ended with the assignment of problems to be attempted at home for the next class meeting.

While most of the beginning problems are concrete counting questions (for example, determining the number of possible sandwiches that can be made with three bread choices and five fillings), one lesson that should emerge from the sequence of problems is the utility of working with ordered pairs. During our first class, one student was especially intrigued by this concept and proposed a conjecture about counting ordered triples. Anxiously, one of his group-mates responded that the book didn't ask about that and she didn't like going beyond what was asked. This provided the instructor with an opportunity to encourage and discuss the nature of the course, including the expectation that students should freely investigate generalizations to gain a deeper understanding of the material.

Students arrived for the second class with some level of distress because for some problems they had only random ideas and scratch work, and they were also unable to determine whether their work was correct or even on the right track. They were reassured that this was to be expected and that the group discussion was intended as a time to work through these challenges. After some preliminary confirmation within their groups of the answers to the more routine problems, students spent the bulk of class time on the problems that had caused difficulty.

In subsequent weeks, groups were almost invariably able to solve all the problems together, even ones that had seemed inaccessible to them at home. Students soon became accustomed to the format of the class and comfortably engaged in dynamic mathematical discussions.

9.5 Concrete Contexts for Abstract Notions

Although the notes cover a great deal of abstract mathematics, most topics are introduced with or applied to concrete contexts. With this approach, the material is conducive to experimentation with models and examples, and technical jargon is minimized. The following three examples show how deep mathematical concepts emerged from simple models or representations.

9.5.1 Functions

In chapter 1 of the class notes, problem 7 points out that "the idea of a function is ubiquitous in mathematics." While this is certainly a true statement, in many courses functions are seen only as algebraic expressions in which to substitute values. Also, in many examples the domains and co-domains are tacitly assumed to be subsets of the real numbers. For example, a standard textbook question about the domain of a function might be "Find the domain of $f(x) = \frac{6x}{x-4}$," and the expected answer is "$x \neq 4$."

As earlier illustrated in section 9.2.1, using functions in combinatorial arguments requires students to have a highly conceptual understanding of function. For example, problem 18 in chapter 1 asks "In how many ways can you pass out nine different candies to three children?" and then encourages students to use techniques about counting functions in their solutions. In solving this problem students must decide whether the domain will be the set of children or the set of candies, requiring knowing and applying two restrictions from the definition of a function—that a single input can be associated with only one output, and that every input must have an output. Although in practice we may "give candies *to* the children" (in some sense, assigning candies to a child), we cannot use a child's name as the input for our function, since the output might be multiple candies or none at all. Rather, to create a function we need to input a candy and assign the candy's owner as the output.

This interpretation of the problem also provides a concrete context in which to discuss more abstract concepts such as injectivity and surjectivity. Unlike the horizontal line test, which in a student's mind may or may not be related to the definition of one-to-one function, here a student can clearly explain that the function is not injective because we are forced to give at least two candies to a single child, and that whether a particular function is onto is based on whether each child receives some candy.

9.5.2 Recursion

We have mentioned in section 9.2 that chapter 2 of the course notes was replaced with supplementary materials on recursion based on Fibonacci puzzles compiled by Ron Knott. In keeping with the spirit of the notes, the unit focused on first recognizing the recursive nature of a problem and then justifying a recursive formula. The patterns were based on physical situations rather than abstract sequences and many of the arrangements involved Fibonacci relationships, which most students already knew.

Consider the following homework exercise from this unit:

> Mrs. Hilbert is letting her students play with Cuisenaire rods. For those unfamiliar with these manipulatives, they are colored rods of different lengths—for example, all rods of length 2 are red and all rods of length 5 are yellow. You should know two things about Mrs. Hilbert's set of Cuisenaire rods. First, she lost all her

rods of length 1 in an unfortunate papier mâché incident last year. Second, she has the Special Extended Set that contains rods of every positive integer length (starting at 2, of course). The order of the rods matters as long as the pieces are different colors (so 2-3 is different from 3-2, but there is only one way to count 2-2).

1. Let R_n represent the number of different ways that students can lay out the colored rods to make a row of length n. Find R_2, R_3, R_4, R_5, R_6, and R_7 explicitly. (Remember that we have no rods of length 1.)
2. Find a recursive formula that expresses R_n in terms of previous terms in the sequence.
3. Justify your formula. Here is a suggestion: divide your configurations of length n into two categories—those that end with a rod of length 2 on the right, and those that don't.

Students drew arrangements of rods to determine R_n for several values of n, and they used the illustrations to obtain and justify their recursive formulas. Each configuration of length n can be associated either with a configuration of length $n-2$ (by removing a final 2-rod) or of length $n-1$ (by shaving 1 unit off the final k-rod where $k>2$). In this manner students were able to explicitly connect each term in the Fibonacci recursive formula $R_n = R_{n-2} + R_{n-1}$ with the arrangements of Cuisenaire rods. Moreover, such an argument relies at least implicitly on the use of the bijection principle learned in chapter 1 of the notes.

9.5.3 Ramsey numbers

The Ramsey number $R(m,n)$ is the smallest number R such that within any set of R people there is either a subset of (at least) m mutual acquaintances or a subset of (at least) n mutual strangers. In terms of graph theory, $R(m,n)$ is the smallest number R such that an edge coloring of a complete graph K_R (the complete graph on R vertices) with two colors (say red and green) will contain either a red K_m or a green K_n. This topic appears in chapter 1 of the course notes as an optional application of the generalized pigeonhole principle (GPP), which states

> If a set containing more than kn elements is partitioned into n blocks, then at least one block has at least $k+1$ elements.

With two variables, k and n, this statement seems fairly abstract. However, consider the following concrete question posed to the students by the instructor as a supplement to similar problems in the notes:

> When Bob arrives at the party there are five people already there. He looks around and mentally notes who he knows, and who he doesn't know. What does the generalized pigeonhole principle tell you about the size of the two categories?

Bob is dividing people into two categories, so $n=2$ in the GPP. Since the set has five people in it and we must have $5>kn$ to apply the principle, we can also choose $k=2$. Now the GPP tells us that at least one category has at least $k+1=3$ elements; that is, Bob knows at least three people at the party, or there are at least three people that Bob doesn't know.

It is perhaps surprising to cover Ramsey numbers with middle school teachers. Though the middle school curriculum is unlikely to include such an advanced topic, we found the experience of working with Ramsey numbers to be valuable for middle school teachers in several respects. Our students found it constructive (though challenging) to write clear formulations of Ramsey numbers in terms of strangers and acquaintances before abstracting to graphs. The homework built on this by asking students to write similar reformulations in other contexts, such as in the following exercise:

> An outdoor park has six marked entrances. Each pair of entrances is joined by a trail. On any given day, each trail is designated "Bikes Only" or "Pedestrians Only". Explain why every day there will either be a triangular path for bikers or for pedestrians.

Once they had been introduced to the purely graphical definition, students were able to independently experiment with edge colorings of graphs. Students independently made conjectures, such as that $R(m,2)=m$ for any $m \geq 2$, and supported their conjectures with simple but logically valid arguments. The topic also naturally led to discussions about the mathematical notions of existence and nonexistence, and the distinction between necessary and sufficient conditions. For example, the fact that $R(4,3) \neq 8$ can be proved by coloring a K_8 in such a way that no red K_4 or K_3 is present, but showing that $R(3,3)=6$ cannot be done by coloring a K_6 to include either a red or green K_3.

9.6 The Student Experience

One of the primary benefits of using guided discovery is the opportunity for students to act as practicing mathematicians in much the same way that chemistry students perform and interpret scientific experiments. Students become independent problem solvers, build intuition, and develop some level of comfort with unrehearsed mathematical dialogue. The experience is valuable to middle school teachers in both obvious and more subtle ways.

The organization of the textbook compelled students to construct key ideas as they naturally arose in the problems. This fostered the students' ability to work confidently with new concepts and unfamiliar ideas in their own teaching, which is essential when they are required to adopt new curriculum materials or standards. Because there were no answers or worked examples in the text, students had to develop and listen to their own intuition. It also reinforced the message that there are numerous approaches to solving a problem, which affected how students interacted with their peers. In particular, they learned that it is important to follow carefully and understand unexpected solutions that do not resemble their own. For middle school teachers, this is particularly significant in terms of their interactions with their students in the classroom, and when grading written work.

Students developed written communication skills primarily through the supplemental homework sets. At first some students tended to write just enough for the instructor to know they had the basic idea of a solution, assuming she could fill in the omitted details. Possibly the supportive classroom environment encouraged this misconception because students did not always challenge each other's explanations during class presentations if everyone understood the main idea. This may have been translated into the message that it is acceptable to neglect precise reasoning in written work if the reader gets the big picture. In any event, the instructor provided substantial feedback on written work throughout the semester, and by the final homework set the quality of submitted work showed a marked improvement. While it is not always clear how to interpret another instructor's grades, we nevertheless share the fact that on the first homework set grades ranged from 54% to 100% with an average score of 84%, while the third and final homework set produced a grade range of 91% to 100% with an average score of 96%. In large part the increase in scores was due to an improvement of the students' abilities to present their solutions in a clear manner with appropriate supporting examples and narrative text.

As the term progressed, students also became increasingly able to recognize the importance of using correct terminology and the value of choosing appropriate notation. For example, initially a common practice among the students was to represent everything by numerals. With experience they came to see the disadvantages of naming seven children $1, 2, \ldots, 7$ if the problem required each child to choose from among five types of fruit that they named $1, 2, \ldots, 5$. With instructor guidance the students became more inclined to use clarifying and contextual notation, such as denoting the children by $c_1, c_2, \ldots, c_7$ and the fruit by $f_1, f_2, \ldots, f_5$. As they grew more comfortable doing this in their own work, they were less likely to feel overwhelmed when they encountered formal mathematical notation in the text.

Although students in a group discovery class don't receive content knowledge from their instructor in traditional ways, the instructor plays a vital role in the classroom and imparts many lessons about what it means to do and to teach mathematics. When the instructor encourages students to explain what they have tried or where their group is having trouble with a problem, the students see that careful and patient listening to others can lead to progress, sometimes in unexpected ways. When a group is stuck, the instructor asks appropriate questions to help students learn to make use of knowledge that they already possess. By offering good examples for impromptu investigation, the instructor impresses upon the students the advantage of checking small or familiar cases when trying to comprehend general principles, and the importance of looking for counterexamples when testing a new conjecture. We have found that over time students instinctively recreate this process in the absence of the instructor and begin to develop additional strategies for overcoming challenges.

Because all students were expected to present work at the board, there was an appreciation that everyone had something to contribute and the "class stars" did not have a monopoly on correct solutions. Students also developed a willingness to share ideas before they had been polished and an acceptance of mistakes as an inevitable part of the learning process.

Late in the semester, one student remarked that while she had worked harder in this course than in any of her previous mathematics courses, she was surprised at how much less preparation she had needed for the midterm exam. There was agreement from other students that the ideas from the course were understood and deeply ingrained in their minds as a result of the guided discovery approach.

Not all lessons are easily absorbed. While student anxiety about guided discovery became almost negligible as the course progressed and everyone appeared to greatly enjoy their time in class, some students repeatedly expressed concern that they still couldn't solve all the weekly problems by themselves. Although they knew that by the end of class they would understand the material, it was difficult to convince some students that group discussion is an integral part of the learning process and not a remedial measure. Many have an established belief that being an expert means having the ability to solve problems correctly the first time. With these students there remained some mild frustration that all their hard work did not seem to pay off in this (unrealistic) way.

That said, the overall reaction to the experience was highly positive. Students commented on the gratification that came from finding a solution after investing considerable time and effort, and that formulas made more sense when they were discovered rather than provided by the text. One student noted that his intuition had been strengthened by the investigatory process, especially by finding natural situations where exceptions and subtleties arose.

9.7 Accessing Resources

We are eager to share our course materials with other instructors who are considering teaching a similar course. On the site supplemental materials designed by Reva Kasman are posted, including homework sets, worksheets, a rubric for grading participation, and instructions for the module project assignment. In addition, the current edition of the adapted notes that formed the course text and an instructor handbook for the notes are also there. The topics covered in the handbook include: how to use the notes, how to use class time constructively, possible grading schemes, and ways to motivate students to do more. There are also comments from previous instructors on specific problem sequences. In addition to general advice, the handbook contains a chapter-by-chapter review of the content of the notes and separate chapter summaries for possible distribution.

9.8 Concluding Remarks

The decision to use guided discovery with middle school teachers can be daunting at first, and this is exacerbated by the students' initial anxiety and reservations. However, we have found that despite their early fears middle school teachers are ideal participants in this method. They typically have an appreciation for the notion of playing with concrete examples before abstracting to general principles, and they are articulate about their own experiences of the learning process and how it could be mirrored in their classrooms. To promote acclimation and acceptance of the non-traditional approach, we recommend early discussion of the pedagogical attributes of the method, including its advantages in a course for teachers. We also believe that to be successful the instructor must be committed to using guided discovery consistently for the whole course, lest the students decide that they can wait out the guided discovery portions of the class until the instructor presents the material more directly. Our experience of watching students enthusiastically engage in mathematical dialogue with their peers has been extremely rewarding, and the ultimate level of student satisfaction with their own understanding was similarly high. We have found the challenges of teaching in this style to be well worth the effort, and we encourage other instructors to adopt this method in their courses for middle school teachers.

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Appendix
Table of Contents for the Course Notes

COURSE NOTES

II

Courses for Middle School Mathematics Teachers

B. Geometry

10

Geometry for Middle Grades Teachers at Northern Kentucky University

Bethany Noblitt
Northern Kentucky University

At Northern Kentucky University (NKU), the course Geometry for Middle Grades Teachers is required of all pre-service middle grades mathematics teachers and of pre-service elementary school teachers who have an emphasis in mathematics. This student-driven, inquiry-based and activity-oriented course has been successful in its goal to have students be able to solve problems in geometry, reason mathematically, and to present clear and concise explanations of their work. The problem-solving philosophy and implementation of the course have contributed to its success.

10.1 History and Philosophy of the Course

This course was taught for several semesters using the text, *Geometry for Teachers*, by C. Patrick Collier [1]. Topics covered included two-dimensional geometry, three-dimensional geometry, isometries, congruence, similarity, and measurement. The topics covered were appropriate for the course, and the philosophy used by the text's author agreed with the philosophy of the course. In the preface to the third edition of the text, Dr. Collier says, "Rather than plan a class by asking myself, 'What shall I tell them today?' I ask myself, 'What shall I have them do today?'" [1] The instructors of this course adopted this philosophy as well. Unfortunately, the text is no longer in print which prompted the instructors to develop course notes based on it. The content of the course notes is a subset of the units of study in Collier's text. There is no required textbook for the course, and instructors use course notes as the primary resource for the class. Dr. Collier generously gave permission to use and adapt his materials and exercises for use in the Northern Kentucky University Geometry for Middle Grades Teachers course. The course notes have grown and developed over the semesters, but the general philosophy, spirit, and content of the course has stayed in line with Dr. Collier's text.

The philosophy of the course is complex and is based on Collier's philosophy of student engagement. First, the course is primarily a geometric problem-solving course in which the students explore geometrical concepts using hands-on activities, constructions, manipulatives, and technology. By encouraging multiple approaches to problems and the use of multiple representations, the course facilitates the formation of connections among areas of mathematics and between mathematics and real world problems. Second, students work cooperatively on solving problems. We believe that written, oral, and visual modes of communication are essential in understanding and doing mathematics. The discussion and sharing of concepts, strategies, and solutions by the students are integral to the course.

10.2 Prerequisites

Middle grades mathematics education students at Northern Kentucky University are required to complete 21 hours in mathematics coursework. The courses include: Mathematics for Elementary and Middle Grades Teachers I and II, Geometry for Middle Grades Teachers, Introduction to Probability, Introduction to Statistical Methods, and two courses from a list of electives that include college algebra, finite mathematics, discrete mathematics, pre-calculus, and calculus.

Mathematics for Elementary and Middle Grades Teachers I and II are prerequisites for the Geometry for Middle Grades Teachers course. Mathematics for Elementary and Middle Grades Teachers includes elements of problem solving, number sense and numeration, number systems, number theory, patterns, and functions. The content of Mathematics for Elementary and Middle Grades Teachers II course includes elements of algebra and geometry.

The goal of Mathematics for Elementary and Middle Grades Teachers I and II is to introduce students to mathematical concepts important to understandings elementary mathematics and to develop skills in problem solving. The courses prepare the students for the problem-solving view of the Geometry for Middle Grades Teachers course. The definition of problem solving used in these courses and in the Geometry for Middle Grades Teachers course that used by the National Council of Teachers of Mathematics (NCTM): That "problem solving means engaging in a task for which the solutions method is not known in advance" [6].

Pre-service middle grades teachers and pre-service elementary grades teachers with an emphasis in mathematics take the Geometry for Middle Grades Teachers course. Typical enrollment for the class is between twenty and thirty students with less than half being pre-service elementary grades teachers. Within the past ten years, NKU has gone from offering one section to four sections of the course per academic year.

10.3 Learning Objectives

The problem-solving philosophy of the course is reflected in the learning objectives for the course. The objectives for the course include:

- Students should be able to problem-solve in geometry.
- Students should be able to use dynamic software to aid in problem solving.
- Students should use mathematical modeling to solve geometric problems. The term "mathematical modeling" refers to the use of mathematics to aid in the solving of real-world problems.
- Students should make conjectures based on their observations of geometric phenomena.
- Students should be able to present clear and concise explanations of their work.
- Students should be able to reason mathematically.

They are met through the exploration and investigation of geometric ideas and concepts such as the classification and description of plane figures and figures in space, congruence, similarity and symmetry of plane figures and figures in space, and isometries.

10.4 A Typical Class Meeting

The course is a three-credit course that typically meets for 75 minutes two days a week. There is no required textbook for the students in this course but the course notes, including exercises, are made available on the course Blackboard website. Students are required to print and bring them to each class meeting. The notes were adapted from [1].

A typical class meeting begins with an introduction to the topic of the day. For example, the instructor might ask a question designed to draw on the students' previous knowledge of the topic, which may come from prerequisite coursework or from reading the course notes. Or, the instructor might assign an in-class activity designed to motivate the day's discussion. This introduction tends to be brief, as most of the class period is reserved for students to work on activities and problems together in small groups. Sometimes, all students are working on the same activity or problem, but more often the small groups have been assigned different problems. Once the groups have had enough time to make progress on or complete their problems, they present their problem to their classmates. The class is encouraged to ask the group questions about the problem and the group is expected to give accurate answers. However, if a group does not know the answer to a question, that is okay, too. Sometimes the class asks questions that the group has not thought about and the exchange of ideas is often the beginning of a rich discussion. If the presentations of problems take longer than one class period, the presentations continue at the beginning of the next class, after which the process begins again with the instructor introducing some new ideas or topics on which the students are expected to reflect, investigate, and share.

Without the students' active participation in solving and presenting problems, the class would not be what it is. Discussion and sharing of concepts and strategies and presentation of strategies are integral to the course.

10.5 Content

The content of the course has six units: topology, shapes in the plane, shapes in space, motions in the plane, motions in space, and similarity and measurement. They provide the students with experiences in both two and three dimensions. Their content is described below, with more details later.

Topology Students are introduced to topology. Curves and networks are included, along with classic topology problems.

Shapes in the Plane This unit addresses two-dimensional figures, some of their properties, and relationships between them.

Shapes in Space Students move from two-dimensional to three-dimensional figures, exploring properties and relationships.

Motions in the Plane The idea of congruence is discussed in terms of isometries and transformations.

Motions in Space This unit explores symmetry of three-dimensional figures.

Similarity and Measurement Similarity of two-dimensional (and three-dimensional) figures and how areas (and volumes) of similar figures are related comprise the main content.

10.5.1 Topology

This is the first time many of the students in the class have heard the term topology. Much of the introductory discussion is about figures being topologically equivalent. The topic of topological equivalence is covered in the Collier text [1, p. 2934] and in the course notes. Class discussion of topological equivalence revolves around ”stretching,” ”squeezing,” and ”twisting” one figure until it looks like another, with the instructor showing examples. A key element in this discussion is that no tearing is allowed. The figures used in this discussion include two- and three-dimensional ones. Students explore the classic coffee mug and donut example in Figure 10.1 as well as others developed by instructors.

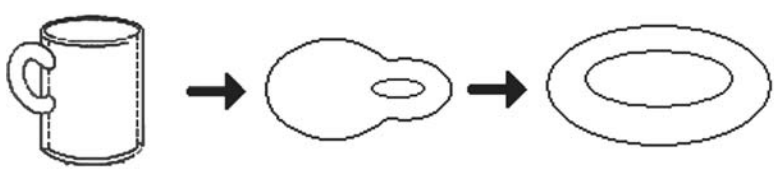


Figure 10.1. Picture can be found at [2].

This classic example draws the intended response from students. They make comments such as, “What?!? A coffee mug and a donut are not equivalent.” A good discussion about the difference between equivalence and topological equivalence follows comments like this.

When students begin to understand the meaning of topological equivalence, they are asked questions like,

Which letters of the alphabet are topologically equivalent?

Students feel comfortable with these questions because the letters of the alphabet are familiar. Because they are confronted with the shapes of the letters, they can concentrate their thinking on what it means to be topologically equivalent and not on trying to determine what the figures are. Once the students have done problems exploring topologically equivalent letters of the alphabet, they are ready to consider topological equivalence of figures with which they are less familiar, answering questions like,

Consider the two objects in Figure 2. Are they topologically equivalent?

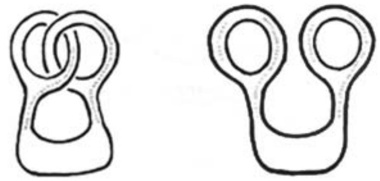


Figure 10.2.

Students respond to the question by discussing what it would take to get one figure to become the other figure. This is where the discussion regarding tearing really comes in to play. Once students realize that they would have to tear one of the rings on the first figure and then glue it back together, they are able to communicate their understanding that the figures are not topologically equivalent.

10.5.2 Shapes in the Plane

The unit begins with a discussion of polygons. Because this topic is one with which the students should have experience, the instructor provides them with problems that question their understanding of polygons. For example, students are asked to find a formula for the number of diagonals of an n-gon. To successfully solve the problem, the students must understand exactly what a diagonal is, and often the fact that a diagonal does not have to lie in the interior of the polygon goes against their intuition. Students also make connections between polygons and topology. For example, the topological equivalence of polygons (and other figures) to circles is explored.

Polyominoes are introduced in this unit. A polyomino is the polygon formed by the outline, or border, of the figure obtained when congruent squares are joined. The squares are joined so that the outline, or border, forms a polygon, which is defined as a simple closed polygonal curve, and whenever two squares have a point in common, their intersection is an edge or a vertex. Polyominoes are classified according to the number of squares needed to construct them.

In Figure 10.3 are examples of polyominoes. The bold outline indicates the polyomino, while the dashed lines indicate the squares that are joined to make it.

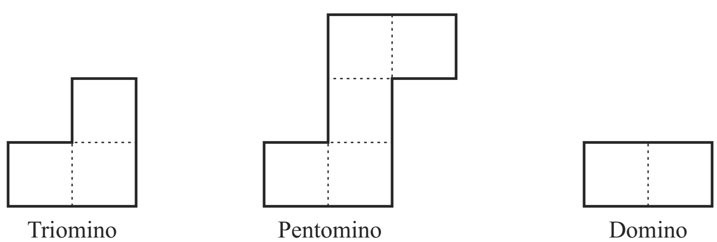


Figure 10.3.

Polyominoes can be constructed with any number of squares. A polyomino constructed with n squares is referred to as an n-omino. Students consider n-ominoes where n is at most seven.

In Figure 10.4 the shapes are not polyominoes because there are two squares whose intersection is not an edge.

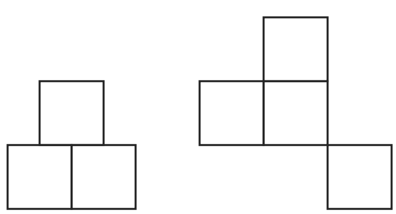


Figure 10.4.

While typically polyominoes are new to student, they find the definition easy to understand. Using polyominoes to dig deeper into a familiar concept like congruence can prompt interesting discussions. For example, students are asked to build the noncongruent tetrominoes. One of the first questions is "Are two polyominoes congruent if you can flip one to get the other?" Discussing the answer gets at that deeper understanding of congruence. The discussion of congruence becomes even richer when they are asked to find the building number an n-omino, that is the number of noncongruent $(n + 1)$-ominoes that can be made by attaching a square along an edge of an existing square to the n-omino. By examining polyominoes for their building numbers and comparing building numbers of noncongruent n-ominoes, students are exploring the ideas of congruence of plane figures and symmetry in a new setting.

It this unit The Geometer's Sketchpad (GSP) [9] is used for the first time. There are exercises in which students explore geometric concepts related to plane figures using GSP. For example, from the course notes, the students are asked to

> Draw three different quadrilaterals, making at least one of them concave. For each quadrilateral, connect the successive midpoints of each side, forming a new quadrilateral. Make a conjecture about the new quadrilateral formed. Can you prove your conjecture?

By using GSP, students do not have to draw a diagram. After a short lesson on using GSP, most students are ready to work independently on a problem such as the midpoint quadrilateral problem. In it, students typically conjecture, correctly, that the midpoint quadrilateral is a parallelogram. GSP allows the students to be more confident in their conjectures. What they have more trouble with is deciding what they need to do to justify their conjecture. Though proofs are discussed in class, they are not a focus of the course. The instructors have chosen to focus more on problem-solving and reasoning than on formal proofs. Often when a proof is desired, the instructor will lead the class in a discussion that will result in the proof. The class discussions, along with the use of GSP, prepare students for writing better proofs. For the midpoint quadrilateral problem, the students may find slopes using the Measurement tool in GSP to verify that the opposite sides of the midpoint quadrilateral are parallel. Then, by dragging vertices of the quadrilateral, students can see that the midpoint quadrilateral remains a parallelogram for any quadrilateral they create.

Tessellations are introduced in this unit. Though this is a topic with which students have experience from a prerequisite course, it is made new by making connections between regular and semiregular tessellations (a familiar topic), tessellations using polyominoes (a new topic), and dual tessellations (a new topic). The dual of a tessellation is tessellation formed from it by connecting the centers of polygons that share a common side. Upon investigation, students are surprised to learn that the dual tessellations of the regular tessellations are regular tessellations. Figure 10.5 illustrates that the dual of the tessellation of regular triangles is the tessellation of regular hexagons and the dual of the tessellation of regular hexagons is the tessellation of regular triangles. The tessellation of squares is self-dual.

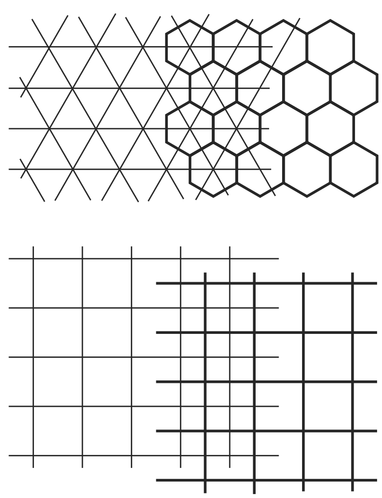


Figure 10.5. Picture can be found at [5].

The concept of duality in two dimensions can be expanded to duality in three dimensions. Once students understand duality in two dimensions they are able to make the connections necessary to understand duality in polyhedra in three dimensions. The concept of duality in three dimensions is discussed in the next unit, Shapes in Space.

10.5.3 Shapes in Space

The class begins this unit with a discussion on polyhedra. The discussion of dual tessellations in the previous unit leads to an understanding of duals of polyhedra. In three dimensions, the concept analogous to dual tessellations is dual polyhedra. The website of the National Library of Virtual Manipulatives [7] has an applet where students can visualize the platonic solids and their duals, as well as other applets that bring hard-to-visualize concepts to life. A goal of the course is to improve students' visualization skills. The National Library of Virtual Manipulatives is a tool we use to help students meet the goal.

Included in the discussion of polyhedra are networks and nets. Students are expected to gain an understanding of multiple representations of geometric concepts and networks and nets are two-dimensional ways of representing three-dimensional figures. They help students to improve their visualization of the figures and how they can be represented in the plane.

Prisms and pyramids are polyhedra with which students are familiar. The course looks more deeply at the definitions of these polyhedra and different patterns that can be found within these figures. Here is a problem taken from [1] in which the students are asked to investigate the relationship of the edges of a polyhedron.

> Let A, B, C be vertices on the base of a triangular prism. Let D, E, F be the other vertices of the prism so that AD, BE, and CF are edges. Verify that edge AB has one edge parallel to it, four edges that intersect it, and three edges that are skew to it. Does a non-base edge have the same pattern of edges related to it? Investigate the same relationship for prisms that have regular polygons having four, five, six, and eight sides as their bases. That is, choose an edge of the base of one of the prisms and tell how many edges are parallel to that edge, how many edges are in lines that intersect the edge and how many edges are skew to that edge. Describe any patterns that you observe.

Out of the three types, parallel, intersecting, and skew, parallel edges tend to be the most difficult for students to determine. Some students will see only the parallel edges that lie in a horizontal or vertical plane. They will not visualize a non-horizontal or non-vertical plane, potentially missing a pair of parallel edges. Because the different planes can be hard to visualize, at times the skew edges can be hard for them to identify as well. Once the different patterns are determined, students must describe these that occur and they are asked to communicate and generalize them. By investigating, communicating, and generalizing the patterns and relationships, students gain a deeper understanding of what a prism is and how its components are related.

The students' visualization and representation skills are challenged when the discussion of polycubes begins. A polycube is a three-dimension analogue to the polyomino: a collection of congruent cubes placed face-to-face, forming a polyhedron. Here is a duocube: In class, students work with polycubes in different ways. First, isometric dot paper

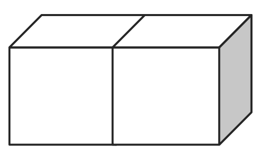


Figure 10.6.

is used to draw polycubes from different viewpoints. This exercise in visualization and representation is difficult for many students. Often students will say, "I just can't draw," or "I am not an artist." For those students, instructors may recommend the use of online resources designed to help students create isometric representations of polycubes. These online resources include applets found at [3]. Using the website, students can represent the T-shaped quadracube drawn in Figure 10.7 from three different viewpoints.

Some of the discussions regarding polycubes are like polyomino discussions. Students consider the building number of polycubes. The building number of an n-cube is the number of noncongruent $(n + 1)$-cubes that can be formed by adding a cube to the n-cube. This prompts investigations of congruent polycubes, which are not trivial. Whether or not one allows polycubes that are reflections of each other in space to be considered congruent will affect the building numbers of polycubes. So, when discussing building number of polycubes, students gain an appreciation of the importance of definition. Here students are first introduced to the notion of reflection in space, a topic explored in greater detail in the unit Motions in Space. Because we have no way to make two space figures coincide, reflections in

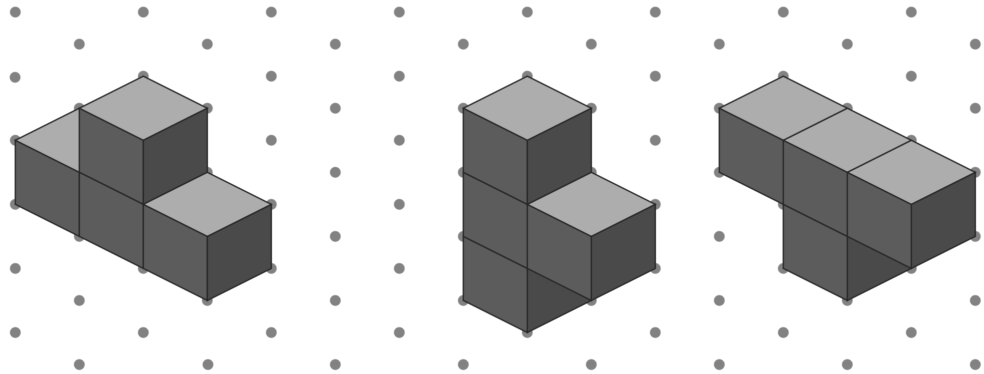
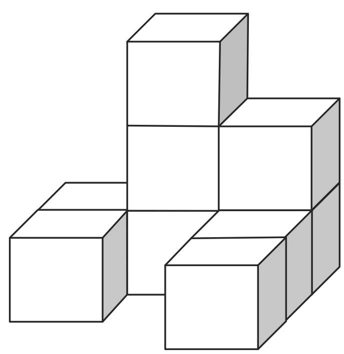


Figure 10.7.

space can be difficult for students to visualize, but using models of polycubes made of multilink cubes help students improve their visualization of congruent polycubes. Students need experiences with determining congruences using the physical polycubes to be able to visualize congruence without them.

Stack maps are another way of representing a three-dimensional polycube in the plane. A stack map is a map that indicates the number of cubes stacked vertically in a polycube and can be thought of as an aerial view of a polycube obtained by considering its layers. In Figure 10.8 is an example of a polycube and its stack map.



1	3	2
1	0	1
0	0	1

Figure 10.8.

Students like to work with stack maps because they them as easier two-dimensional representations of polycubes. However, students must understand that not all polycubes may be represented by a stack map. For a polycube to have a stack map, it must be able to lie in a position in which each cube is supported by a cube below it. There may not be any unsupported cubes in a polycube represented by a stack map. Students usually understand this requirement and its necessity. They will add to the class discussion by explaining that if unsupported cubes were allowed, they would not be able to interpret the meaning of a "1" in a stack map. They ask themselves the question, "Would that '1' mean one cube in the bottom layer, or in one of the other layers of the polycube?"

The congruence of stack maps and the congruence of polycubes is the topic of much rich discussion in the class. What does it mean for two stack maps to be congruent? Can a polycube have two (or more) noncongruent stack maps? Do congruent stack maps imply congruent polycubes? Do congruent polycubes imply congruent stack maps? These are all questions that elicit thoughtful responses that require investigation on the part of the students. A nice discussion of polycubes and the geometry of polycubes can be found in [1].

Another problem adapted from [1, p. 98] is given in which students must demonstrate an understanding of congruent stack maps and congruent polycubes.

> Make up your own congruence problem by doing the following. Put nine numerals in a 3-by-3 square array to make a stack map. Draw a picture of the polycube represented by the stack map. Then arrange the nine numerals to make another map that is congruent to the first. Then draw a picture of the polycube represented by the second map. Try to draw one of the polycubes from a different point of view. Exchange pictures of the two congruent polycubes with a classmate and ask them to verify that the figures are congruent. You do the same with the polycubes that they created.

In this problem, students work with their classmates to come to a better understanding of the topic. The problem asks students to draw a picture of the polycube. Rather than drawing free-hand, students often request to use isometric dot paper. Though they find isometric drawings difficult at first, once they gain experience with them, they usually prefer them to free-hand drawings. Problems that encourage collaboration and communication are vital to ensure the deep level of understanding that is the goal of the class.

10.5.4 Motions in the Plane

This unit begins with a discussion on congruence of plane figures and isometries. Initially discussions revolve around the idea of congruence correspondences. A congruence correspondence is the statement of congruence that exists between two congruent figures. For example, in Figure 10.9 there is a congruence correspondence of $ABCDE \cong VWXYZ$. The particular order in which the vertices are written in the congruence correspondence indicates that $\overline{AB} \cong \overline{VW}$, $\overline{BC} \cong \overline{WX}$, $\angle X \cong \angle C$, and so on.

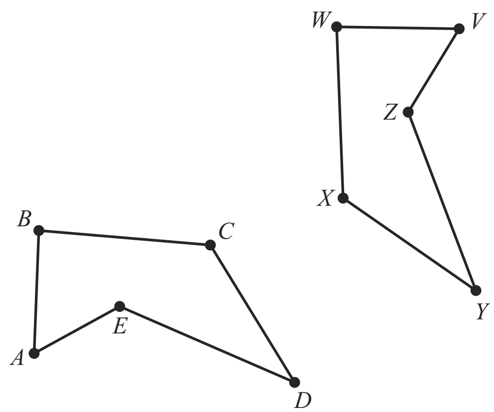


Figure 10.9.

In this unit, three isometries (reflections, rotations and translations) are discussed, stressing terminology and notation. A goal of the unit is to continue to develop students' spatial sense. Many tools are used: miras, tracing paper, geoboards, and GSP. Fixed points and lines of transformations, and motion orientation help students develop their spatial sense. Understanding of isometries is enhanced by recognizing them as functions and writing them as such. A functional expression for reflections is given by $M_{AB}(P)$ and interpreted as the reflection of P across line $\overline{AB}$. A functional expression for rotations is given by $R_{Q,ABC}(P)$ and interpreted as the rotation of P about point Q by angle ABC. A functional expression for rotations may also be given by $R_{Q,\frac{a}{b}}(P)$, which is interpreted as the rotation of P about point Q, by the fraction $\frac{a}{b}$ of a complete revolution. A functional expression for translations is given by $T_{AB}(P)$ and interpreted as the translation of P by directed line segment $\vec{AB}$.

In this unit, geoboards are used in exercises as well as in class activities. Here is an example of an exercise that uses a geoboard and requires student understanding of the isometries covered and the functional aspect of the isometries.

> Make a drawing of a five by five rectangular geoboard. Name the points on the geoboard A through Y by going across from left to right starting at the top. Do the following:
>
> 1. Write three correct functional expressions containing M_{CM}.
> 2. Write three correct functional expressions of the form $R_{\#\#}(\#) =$ # for each of the rotations $R_{Q,AFG}$, $R_{L,\frac{1}{4}}$, $R_{N,GMQ}$ by filling in points, angles, preimages, and images where appropriate.
> 3. Write three correct functional expressions of the form $T_{\#\#}(\#) =$ # for each of the translations T_{KL}, T_{MQ}, T_{JM}, T_{AV} by filling in points, angles, preimages and images where appropriate [1, p. 121].

Student difficulties with the exercises mostly consist of not understanding the functional notation and how to express their work using it. Also, they express concern when the transformation that they choose yields an image that does not coincide with a point on the geoboard. In this case, telling them that they need to select their pre-image points appropriately usually will help them get on the right track.

This unit continues with students exploring the ideas of reflection congruence, rotation congruence, and translation congruence and their relationship with that of congruence in general. Students determine congruence correspondences relating congruent figures and decide which type(s) of congruence exist. Other investigations require students to find an example of two figures that are congruent but not reflection, rotation, or translation congruent. This leads to a discussion of the composition of transformations, specifically, of glide reflections. An interesting problem posed is to determine the parameters (the line of reflection and the translation vector) for a glide reflection. To do this, students must have a good understanding of glide reflections.

10.5.5 Motions in Space

In this unit, many of the ideas discussed in the Motions in the Plane unit are discussed in three dimensions. The unit begins with a discussion of the idea of congruence in three dimensions and how congruent figures can be reflected or rotated in space to coincide with each other. The idea of coinciding three-dimensional figures is hard for students to visualize. Relating it back to two dimensions leads to a consideration of translations, reflections, and rotations. [1, pp. 167–174] uses polycubes to discuss reflections and rotations in the plane (Collier). For example, rotating the polycube in Figure 10.10 one-quarter turn about a vertical axis (clockwise, looking down from above), gives the polycube in Figure 10.11:

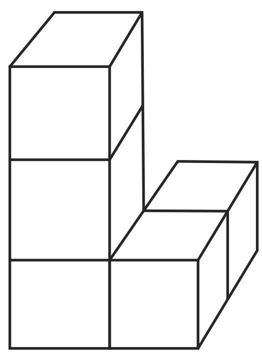


Figure 10.10.

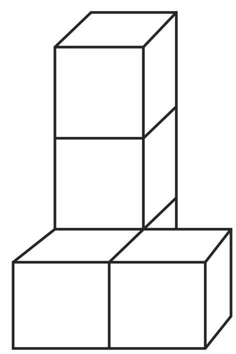


Figure 10.11.

A plane figure might be moved to coincide with another plane figure by reflecting it in a line. Reflections in space can be especially difficult for students to visualize, because space figures are reflected in a plane, and unfortunately this cannot be accomplished physically. Consider a pentacube P and a plane. Figure 10.12 illustrates the pentacube P reflected in the plane.

After rotations and reflections in space are discussed, the ideas of rotation symmetry and reflection symmetry for three-dimensional figures are introduced. After discussing these topics for polycubes, the students are asked to relate their findings to other three-dimensional figures such as prisms and pyramids. Usually students find the concept of symmetry in polycubes more difficult than the concept of symmetry in polyhedra with which they are familiar such as prisms and pyramids. Here is an example of an exercise:

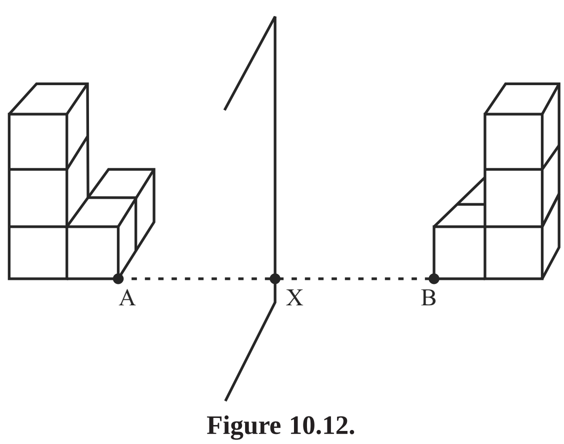


Figure 10.12.

Consider a right prism. Investigate the relationship between the number of edges in its base and the number of its reflection symmetries. Begin with a right prism whose bases are regular triangles, proceed to a right prism (not a cube) whose bases are squares, and continue.

Consider the prisms described above. Investigate their rotation symmetries. That is, try to find a relation between the number of edges in the prism's base and the number of rotation symmetries the prism has. You might want to find the number of axes of rotation symmetry as well as the number of rotation symmetries.

Consider a right pyramid whose base is a regular polygon. As described above, relate the number of reflection symmetries to the number of edges in its base.

Consider the pyramids described above. Investigate the relationship between the number of edges in the base of such pyramids and the number of their rotation symmetries [1, p. 175].

Typically, students can identify the rotation and reflection symmetries present in right prisms and right pyramids. From there, most students complete the exercise successfully.

10.5.6 Similarity and Measurement

This is the final unit of the course. It begins with a discussion of what it means for two figures to be similar, and, more specifically, centrally similar. Two plane figures are said to be *centrally similar* if one figure, say figure B, is obtained from figure A by enlarging or shrinking figure A by a scale factor about a center O. Figure 10.13 shows an example of two figures that are centrally similar, centered at O, with a scale factor of 2.

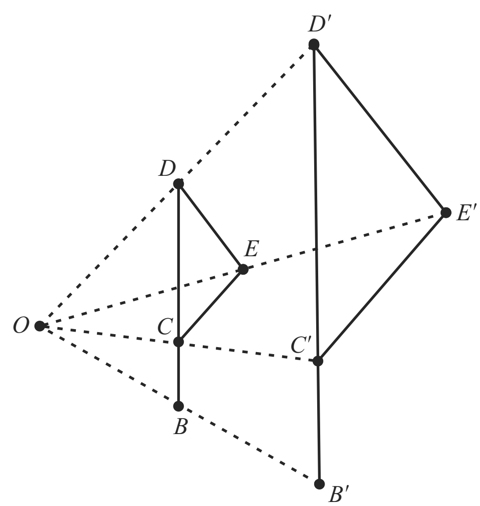


Figure 10.13.

In Figure 13,

$$\frac{OD'}{OD} = \frac{OE'}{OE} = \frac{OC'}{OC} = \frac{OB'}{OB} = 2$$

because the scale factor is 2. Also,

$$\frac{D'E'}{DE} = \frac{C'E'}{CE} = \frac{C'D'}{CD} = \frac{C'B'}{CB} = 2.$$

The notation for the central similarity function is $C_{O,2}(BCDE) = B'C'D'E'$. In general, when $C_{Q,k}(A) = A'$, then A is on ray QA' and the length of se QA' is k times the length of QA.

Most of this unit centers on a discussion relating the area of similar two-dimensional figures and the volume of similar three-dimensional figures. Polyominoes are used in the discussion of area and two-dimensional figures. Polycubes are used in the discussion of volume and three-dimensional figures. Students are asked to do the following problems.

Draw two different hexominoes. Find the perimeter and area of each. Then draw two more polyominoes so that each is similar to one of the hexominoes with a scale factor of 2. Find the perimeter and area of the

> larger polyominoes. Discuss what would happen if the scale factor is 3. Discuss what would happen if the scale factor was changed to 4. Explain, in general, the relation of perimeter and area of similar polyominoes.
>
> Construct two different hexacubes so that at least one is not a prism. Find the surface area and the volume of each. Then construct two more polycubes so that each is similar to one of the hexacubes with a scale factor of 2. Find the surface area and the volume of the larger polycubes. Discuss what would happen if the scale factor is 3. Discuss what would happen if the scale factor is 4. Explain, in general, the relation of surface area and volume of similar polycubes.

By working on the problems, students investigate and conjecture relationships themselves, without the instructor telling them what they are. Determining the generalization is often difficult for students. Typically, they are able to find the examples requested, but then sometimes they are unable extend to what would happen in general.

10.6 Assessment

Assessment in the course typically consists of assignments, projects, quizzes, and exams. These forms of assessment may vary according to the instructor. In the next section, typical assignments and projects are described. This section provides examples of items that have appeared on quizzes and exams, so as to provide a sense of the types of problems that are given to students. When writing problems for quizzes and exams, instructors attempt to maintain the same sense of problem-solving by posing non-routine problems. The following problem has appeared on an exam for the course.

The two rectangles below are congruent.

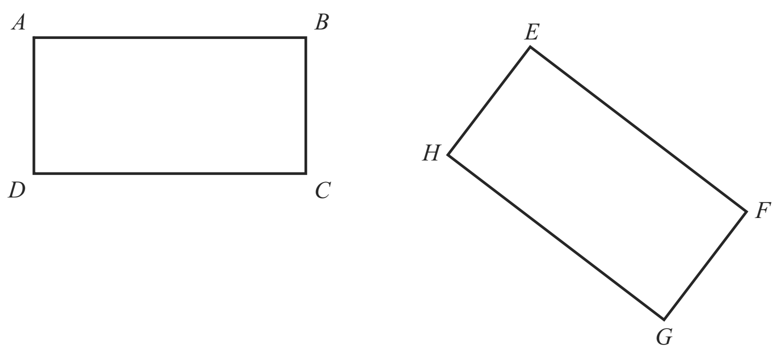


Figure 10.14.

a. Find all congruence correspondences between the two rectangles.

b. Are the figures translation, rotation, or reflection congruent? If so, which? Thoroughly describe the translation, rotation, or reflection correspondence in each case, providing all appropriate parameters.

For this item, students need to understand that certain congruence correspondences will indicate a reflection congruence, rotation congruence, or translation congruence. Students also must be able to determine which type of congruence is indicated by a particular congruence correspondence and figure out its parameters. For rotation correspondence (if there is one) they must determine the center of rotation and the angle of rotation. For reflection correspondence (if there is one), they must determine the line of reflection. For translation correspondence (if there is one), they must determine the translation vector.

Another item that has appeared on an exam is:

> Determine the number of diagonals of a prism whose base is a regular n-gon. Determine the number of diagonals of a pyramid whose base is a regular n-gon.

This item does not tell students how to proceed to determine the rule for the number of diagonals. Students respond well to this item, as they have participated in problem-solving throughout the course. The problem requires students to collect data, examine it for patterns, generalize their results, and it requires students to recognize that this is what they need to do to solve the problem.

In general, instructors try to provide as little information to students as possible on an assessment item. Instructors don't want to "give it away." They want students to be able to find a correct answer, and they want them to determine what to do to get it.

10.7 Assignments and Projects

Assignments often require students to use GSP. One assignment asks students to investigate the Pythagorean Theorem in ways they have not before. The Pythagorean Theorem states that for every right triangle, the square of the longest side equals the sum of the squares of the other two sides. This assignment asks the students to use GSP to explore this kind of comparison in acute triangles and obtuse triangles. The students answer the questions: How does the square of the longest sides compare to the sum of the squares of the other two sides? How do they compare in obtuse triangles? How do they compare in acute triangles? In answering, students discover for themselves that if the square of the longest side is more than the sum of the squares of the other two sides, then the triangle is an obtuse triangle. Similarly, they will see that the sum of the squares of the two smaller sides is greater than the square of the longest side, then the triangle is an acute triangle.

After students complete the investigation of the Pythagorean Theorem, they are asked whether the Pythagorean relationship holds if figures other than squares are attached to the sides of the right triangle. They are to use GSP by first constructing a right triangle. Then, students construct three similar figures (other than squares) such that one side of each figure is also a side of the right triangle. Students determine the areas of the similar figures to determine if the area of the largest polygon is equal to the sum of the areas of the smaller polygons. This assignment is an investigation into familiar geometric topics, in which students discover mathematical truths that are new to them. Students' reactions to the assignment vary. The assignment is open-ended, which sometimes can cause anxiety. Without a step-by-step procedure to follow, some students are at a loss for how to proceed. This assignment is typically given at the beginning of the semester when the course and its philosophy is new to the students. But once students get used to expecting that they will explore problems on their own, they tend to rise to the occasion and appreciate this type of learning.

Another exploratory assignment is one in which students are asked to investigate transformation composition using GSP. In this assignment, they are not given a particular figure to transform, nor are they given any of the parameters of any transformation. They are to make a conjecture about the single transformation (if it exists) that is equivalent to a given composition of transformations. For example, what single transformation is equivalent to a reflection followed by a reflection? As we know, the answer is dependent on the relationship of the lines of reflection. However, the students are not told this. They are prompted with questions that should be considered when making their conjectures. So, for a reflection followed by a reflection, students are prompted with the questions: Does it matter if the lines are parallel or intersect? What if you changed the order in which you did the reflections? Students are assessed on the conjectures made and the extent to which they justify their conjectures with mathematical evidence based on their exploration using GSP.

Long-term projects are also assigned in this course. They are designed around a topic in which the instructor would like the students to delve more deeply. One example is an investigation of fractals, a mathematical topic with which these particular students may have little experience. In the project, students will learn about fractals by working through a unit on fractals designed for middle grades students. This project was adapted from [4]. Because most of the students in the Geometry for Middle Grades course have not previously learned about fractals, they work through the unit just as the middle grades students would, learning how to make some well-known fractals such as the Sierpinski triangle and the Koch snowflake. They also explore fractal properties such as self-similarity, fractional dimension, and formation by iteration. Students take the unit further by discussing the mathematics they learned and by reflecting on the unit as a future teacher, discussing its potential use in the classroom along with adaptations they might make. The project allows students to investigate and learn about a new mathematical topic as well as to reflect on a unit developed around it. That this project involves a real-life unit for middle school students is highly motivating for students, as they see it as information that is directly applicable in their future classrooms.

10.8 Conclusion

Students in the course are challenged in ways in which they may have never been challenged before. They are held more responsible for their own learning by the expectation that they participate in investigative problems and present their findings to their classmates. Presentations have the benefit of motivating the students to do well because their classmates are counting on them to solve problems and to present them clearly. The assignments and projects challenge the students by requiring them to spend significant amounts of time on a single problem and to reflect on it as they solve it. Students appreciate the depth at which they are expected to understand the content, once they realize that it is the depth that makes their understanding so meaningful.

The instructors of the Geometry for Middle Grades Teachers would like to thank Dr. C. Patrick Collier for granting permission for the use of the materials in his *Geometry for Teachers*.

10.9 Bibliography

[1] C. Patrick Collier, *Geometry for Teachers*, Waveland Press, Inc., Prospect Heights, Illinois, 1994.

[2] "Elementary Introduction to Topology," accessed on July 22 2012, `en.wikipedia.org/wiki/Topology#Elementary_introduction`

[3] "National Council of Teachers of Mathematics Illuminations Site," accessed on July 22 2012, `illuminations.nctm.org/`

[4] "Fractals for Elementary and Middle School Teachers," accessed on July 22 2012, `math.rice.edu/~lanius/frac/`

[5] "Mathworld Dual Tessellation," accessed on July 22 2012, `mathworld.wolfram.com/DualTessellation.html`

[6] National Council of Teachers of Mathematics, *Principles and Standards for School Mathematics*, National Council of Teachers of Mathematics, Reston, VA, 2000.

[7] "National Library of Virtual Manipulatives," accessed on July 22 2012, `nlvm.usu.edu/en/nav/vlibrary.html`

[8] "Picture of Handcuffs," accessed on July 22 2012, `britton.disted.camosun.bc.ca/jbhandcuff.htm`

[9] "Geometer's Sketchpand," accessed on July 22 2012, `www.keycurriculum.com/products/sketchpad`

Appendix A
Additional Activities

1. (Topology) Which of the plane figures below, if any, are topologically equivalent to each other?

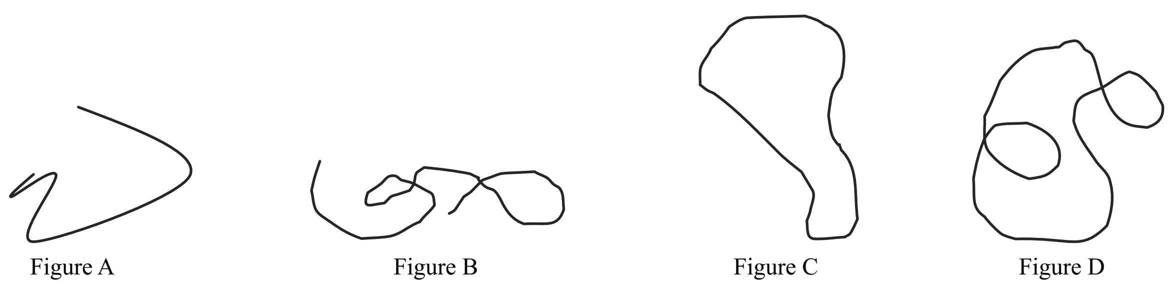


Figure 10.15.

2. (Topology) The Handcuffs Puzzle: For this puzzle, use two participants and two four-foot long pieces of rope. Tie a loop in each end of a piece of rope so it can be worn as handcuffs. Give each person one rope that they will wear as handcuffs. Before putting the handcuffs on, each participant loops their "handcuffs" around each other so they are tied together as shown in Figure 10.16.

   The participants are to get themselves apart without removing the handcuffs or cutting through the rope.

3. (Shapes in the Plane) Can two congruent figures have more than one congruence correspondence? Discuss your answer using two congruent regular triangles and two congruent squares as examples.

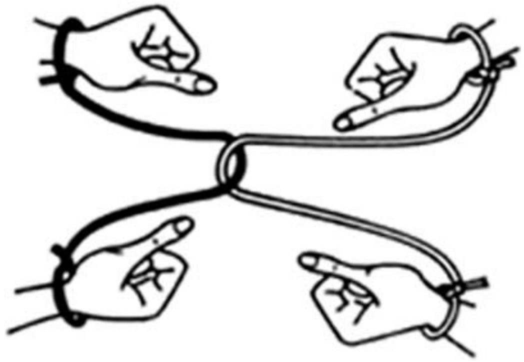


Figure 10.16. Picture can be found at [8].

4. (Shapes in the Plane) Find the tetromino that has the least building number. Find the tetromino that has the greatest building number. Find the pentomino that has the least building number. Find the pentomino that has the greatest building number. Find the hexomino that has the least building number. Find the hexomino that has the greatest building number. Explain how you might predict whether a n-omino has a greater or lesser building number than another n-omino. [1, p. 60]

5. (Shapes in Space) Draw the pentacubes that are also prisms. Draw the pentacubes that are not prisms. How many distinct pentacubes exist? Determine which pentacubes cannot be represented by stack maps, which can be represented by stack maps in exactly one way, and which can be represented by stack maps in more than one way. [1, p. 101]

6. (Motions in the Plane) Find the parameters for the given preimage and image under a glide reflection (Figure 10.17). That is, find the line of reflection and the translation vector.

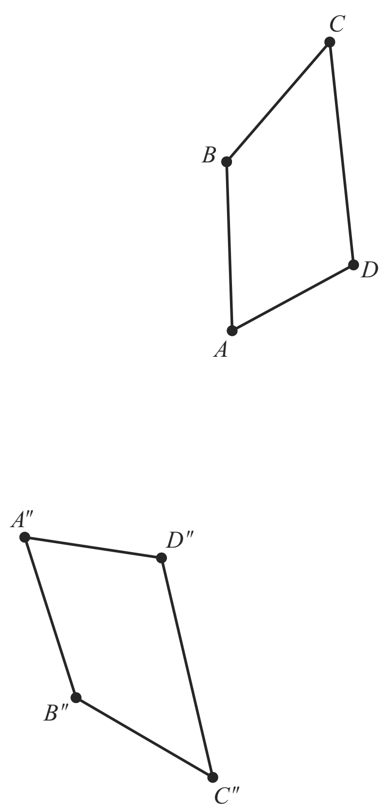


Figure 10.17.

7. (Motions in Space) Locate a solid with the following symmetries:
   (a) One axis of order 4.
   (b) Four axes of order 3.
   (c) One axis of infinite order.
   (d) An infinite number of axes of infinite order.
8. (Motions in Space) Sketch a tricube that is a rectangular prism. Then locate all of its planes of symmetry. Which planes cut through faces? How do they cut the faces? Which planes cut through edges? How do they cut the edges? Tell the total number of reflection symmetries. Describe how the planes of symmetry are related to one another. [1, p. 170]
9. (Similarity and Measurement) Verify the area relation for similar plane figures by making up three examples on square dot paper. That is, construct three pairs of similar geoboard polygons (polygons drawn on dot paper). Find the area of each polygonal region. Show how the ratio of the areas of the regions compares with the scale factor. Avoid using squares or right triangles.

Appendix B
Syllabus

MAT 240-001 — Geometry for Middle Grades Teachers

3 credit hours

Prerequisites MAT 141 with a grade of C or better. MAT 240 is open only to students majoring in elementary or middle grades education. If you have any doubt about your readiness for this course, you should discuss this with me as soon as possible.

Additional Information The General Math Department Syllabus can be found at `www.nku.edu/~math/pages/about/GeneralSyllabus.php`.

Text Although there is no required text for this course, notes and exercises will be posted on Blackboard.

Highly Recommended Software Geometer's Sketchpad Software 4.0 Win/Mac. Berkeley, CA: Key Curriculum Press. Since there is no text required for this course, it is the hope of the instructor that students will purchase the Geometer's Sketchpad Software. You can purchase it at `www.keymath.com/x18112.xml` among other places. The cost is about $40.

Material to be covered See the Course Assessment Objectives found on the last page of this document. The extent to which you meet the course assessment objectives will be measured by items on exams and projects.

Foci and Goals of the Course The content of the course explores geometrical ideas using constructions, manipulatives, and technology. The overall goal of this course is to extend the concepts learned in MAT 141, especially geometry, patterns and reasoning. This is mainly a problem-solving course with measurement ideas appearing throughout. Written, oral, and visual modes of communication are essential in understanding and doing mathematics. The discussion and sharing of concepts, strategies, and solutions by you are integral to this course. You will work both individually and cooperatively in problem solving activities. Manipulatives, computers, and calculators will be used when appropriate to help you with your mathematical conceptual development. Viewing problem solving with multiple approaches and multiple representations can facilitate the making of connections among areas of mathematics, between mathematics and other disciplines, and between mathematics and real-world problems. Engaging in inquiry activities, you will employ your reasoning skills and will link concrete and abstract mathematical concepts. This is a mathematics course, not a methods course. The purpose of the course is to introduce you to some important concepts as a basis for the mathematics taught at the middle school level. We will often use manipulatives common to the middle-grades classroom for the purpose of seeing how they might be useful in learning particular concepts.

Grading Your grade will be based on the following:

5 Assignments 20 points each =	100 points
5 Quizzes 20 points each =	100 points
1 Project 50 points =	50 points
2 Exams 100 points each =	200 points
1 Final Exam 100 points =	100 points

There are 550 points possible. Your grade will be determined by the total number of points you earn.

495–550 points	A
440–494 points	B
385–439 points	C
330–384 points	D
329 points or below	F

Class Attendance and Participation Attendance will be taken. If you cannot attend a class meeting, it is your responsibility to notify me and make arrangements to discuss the outcomes of the class discussion and assignments. You are expected to check Blackboard regularly for announcements, class notes, exercises, and assignments.

Assignments To receive credit for assignments, they must be complete and submitted on the due date. Only in extreme circumstances, subject to the approval of the instructor, will late work be accepted, or make up exams given.

Tools You will find several manipulatives useful in this course. Tracing paper and dot paper (both with square and triangular designs) are used liberally. Dot paper is posted on Blackboard. At times you may also need a straightedge, compass, and protractor. In addition, we will also use The Geometer's Sketchpad software. This software is available in labs on campus. It is highly recommended that you purchase the student version of the Geometer's Sketchpad software. You will also need a calculator and access to the Internet.

Exams and Quizzes Quizzes will precede Exams by about one week and will give you an idea of what to expect on the Exams. There will be two Exams in addition to the Final Exam. Exams after the first will be cumulative. Content for the Exams will come from your in-class notes, class notes on Blackboard, exercises provided on Blackboard, computer activities, class discussions, and assignments.

Student Honor Code "The work you do in this course is subject to the Student Honor Code. The Honor Code is a commitment to the highest degree of ethical integrity in academic conduct, a commitment that, individually and collectively, the students of Northern Kentucky University will not lie, cheat, or plagiarize to gain an academic advantage over fellow students or avoid academic requirements." "You are bound by the Student Honor Code neither to give nor receive any unauthorized aid on examinations [quizzes or any graded assignment]."

Students with Disabilities "Students with disabilities who require accommodations (academic adjustments, auxiliary aids or services) for this course must register with the Disability Services Office. Please contact the Disability Service Office immediately in the University Center, Suite 320 or call (859)572-6373 for more information. Verification of your disability is required in the Disability Services Office for you to receive reasonable academic accommodations."

This schedule is tentative and is subject to change by the instructor.

MAT 240 Course Assessment Objectives

Objectives related to specific topics and procedures

1. Students should be able to classify and describe plane figures including polygons and polyominoes.
2. Students should be able to determine, describe, and classify tessellations.
3. Students should be able to determine and describe the congruence of two geometric objects in a plane.

4. Students should be able to determine and justify similarity of plane figures.
5. Students should be able to describe isometries in the plane and determine the image of a figure under an isometry.
6. Students should be able to determine the symmetry of plane figures.
7. Students should be able to classify and describe space figures, including polyhedra and polycubes.
8. Students should be able to determine and describe the congruence of two geometric figures in space.
9. Students should be able to describe isometries in space and determine the image of a figure in space under an isometry.
10. Students should be able to determine the symmetry of space figures.
11. Students should be able to determine and justify similar figures in space.
12. Students should be able to determine and justify the relation of areas (and perimeters) of similar figures in a plane.
13. Students should be able to determine and justify the relation of volumes (and surface areas) of similar figures in space.

Objectives related to general course outcomes

1. Students should be able to solve problems in geometry.
2. Students should be able to use dynamic software to aid in problem solving.
3. Students should use mathematical modeling to solve geometric problems.
4. Students should make conjectures based on their observations of geometric phenomena.
5. Students should be able to present clear and concise explanations of their work.
6. Students should be able to reason mathematically.

II

Courses for Middle School Mathematics Teachers

C. Number Theory and Abstract Algebra

11

A Number and Algebra Course for Middle School Math Teachers

Michael Mays and David Miller
West Virginia University

11.1 Introduction

Extended learning courses in Number and Algebra for in-service Middle School Math Teachers have been taught at West Virginia University since 2002. The format of the courses has remained constant: a two credit hour mathematics course and a one credit hour corequisite Curriculum and Instruction course, part I of both Math and C&I taught in the fall semester and part II in the spring semester. Splitting the three hour content into a mathematics portion and a C&I portion is done to make clear the applicability of the mathematics. The courses were conceived as part of an initiative for in-service middle school mathematics teachers in the NSF funded statewide professional development initiative called MERIT (Mathematics Education Reform Initiative for Teachers). More recently the courses have been offered as part of the Southern Regional Education Board (SREB) Making Middle Grades Work program, which is independent of the MERIT initiative, but has a similar philosophy and aims to meet similar goals of increasing capacity and teacher depth of knowledge at the middle school level. The courses are appropriate for in-service and pre-service middle school math teachers, but so far have proved to be of greatest use for in-service teachers.

The objectives of the courses are to increase knowledge and competence for middle school mathematics teachers in content and pedagogy related to the teaching and learning of number and algebra. For the two credit hour mathematics courses, the content portion means

- Improve understanding of basic concepts and skills in the area of number and algebra.
- View number and algebra from an advanced perspective.

The pedagogy portion in the curriculum and instruction course considers classroom relevance

- Relate the advanced mathematical topics to topics taught in the middle school classroom.
- Examine current research in teaching and learning mathematics.
- Explore model middle school mathematics curricula such as Connected Mathematics.

Every week applications are developed and explored to demonstrate the utility of the mathematics. Technology including graphing calculators, Java applets, and other Web resources are used to investigate number and algebra concepts and to model and solve real world problems. Some of the applets are described in section 11.5. The applets are available for download and may be incorporated into other courses.

11.2 History and Background

In 2000 West Virginia received an NSF grant for a five year program to improve mathematics teaching in middle schools, called Project MERIT. It was a collaboration of the Department of Education, the State College and University Systems of West Virginia, the West Virginia Council of Teachers of Mathematics, and the West Virginia Mathematics and Science Coalition. As part of the professional development, Marshall University and West Virginia University developed courses adressing content strands in middle school mathematics: Number and Algebra, Geometry, Discrete Math/Data Analysis, and Functions and Change. Topics were incorporated in the courses putting mathematical ideas in the context of grade level content and subsequent need. Thus the course in Functions and Change included ideas setting the stage for calculus. Each strand was covered in six credit hours of course work: a two credit hour Mathematics course and a corequisite one credit hour Curriculum and Instruction course offered in the fall semester, and a sequel to them in the spring semester. Completion of the set of courses was worth twenty four credit hours of mathematics and mathematics education, suitable for

- Endorsement in Middle School Mathematics for those who have an elementary education certificate and pass the West Virginia Middle School Mathematics certification exam.
- Application toward a Master's Degree in Secondary Education with an emphasis in middle school mathematics for in-service teachers.
- Preparation for National Board Certification or highly qualified teacher status as required by No Child Left Behind (NCLB).

A cooperative agreement allowed course credit hours to be accepted by West Virginia University and Marshall University. Marshall University was responsible for the courses in Geometry and Discrete Math/Data analysis, and West Virginia University for the courses in Number and Algebra and Functions and Change. Subsequent course development has been supported by the Benedum Foundation, and for the WVU courses by the Southern Regional Education Board, which made them part of their Making Middle Grades Work improvement plan. Dr. Robert Mayes, currently Professor in the Department of Teaching and Learning at Georgia Southern University, was influential in envisioning the initial course design and in obtaining Benedum and SREB support for subsequent course development.

Of all the courses developed for MERIT and for SREB, the Number and Algebra courses have been offered the most often. Table 11.1 shows the schedule and enrollments. We also include retention data. All the students who finished the courses earned a grade of C or better.

All the students who have taken the course the last three years have been in-service teachers. In the first three years the enrollment consisted entirely of West Virginia residents, so we were able to have a launch and a class session in conjunction with the annual spring meeting of the West Virginia Council of Teachers of Mathematics (WVCTM). In 2008 and 2009 the enrollment consisted entirely of teachers in the Louisville, Kentucky area, and we were able to organize face-to-face meetings at the University of Louisville. The Louisville classes were funded by a "GEAR UP Kentucky" grant that provided for course assistants who met regularly with participating teachers, audited the on-line class sessions, and proctored examinations.

Term	Course	Beginning Enrollment	Ending Enrollment
Spring 2002	I	7	6
Fall 2002	I	5	4
Spring 2003	II	5	4
Fall 2003	I	8	6
Fall 2004	I	2	2
Spring 2005	II	4	4
Fall 2005	I	6	5
Spring 2006	II	4	4
Fall 2006	I	13	12
Spring 2007	II	8	7
Spring 2008	I	10	10
Spring 2009	I	19	18

Table 11.1. Number and Algebra enrollment history

The courses serve the immediate needs of in-service teachers, and the topics have the most obvious relevance to the middle school curriculum. They are appropriate for pre-service teachers, especially those in the five year education program at WVU, which awards an undergraduate degree from the Eberly College of Arts and Sciences in the teaching areas of emphasis and a graduate degree in Education from the College of Human Resources and Education. (Secondary education students can complete a major in mathematics. Elementary and middle school education students usually complete a major in multidisciplinary studies that can include enough mathematics courses to constitute a minor.) The set of mathematical topics to be covered as listed in the syllabus is ambitious but achievable over the academic year in six credit hours:

- Set theory (including the well-ordering principle)
- Number systems and binary operations (whole numbers, integers, rational numbers, real numbers)
- Abstract algebra (rings, integral domains, fields, polynomial rings, order, density, and completeness properties)
- Number theory (divisibility, the division algorithm, The Fundamental Theorem of Arithmetic, distribution of primes, sequences and inductive reasoning, recursive reasoning, and recurrence relations)
- Modeling (change - linear and curvilinear, finite differences, exponents and polynomial expressions, lines of best fit, function as process and object, and the method of least squares).

The goal in emphasizing the algebraic underpinnings of arithmetic is to show the intrinsic differences among number systems, in terms of what properties and operations are included, and to remind the students of the role of deductive reasoning in developing mathematical ideas. The topics mirror the content competencies for number and algebra described in [3]

- Understand and use field axioms on the set of real numbers
- Understand decimal representation and other base representations
- Use prime factorization and relate it to algebra
- Make conjectures about prime and composite numbers and provide justifications to prove or disprove them
- Understand the mathematics behind algorithms, including alternate algorithms and samples of student generated algorithms
- Represent arithmetic operations using multiple models and manipulatives
- Read, write, compare, order, represent, estimate and compute with numbers in a variety of forms: integers, rational numbers, decimals, percents, square roots, perfect squares, irrational numbers, complex numbers, and numbers written in scientific and exponential notation
- Understand algebra as a language to represent or describe scientific or mathematical situations.

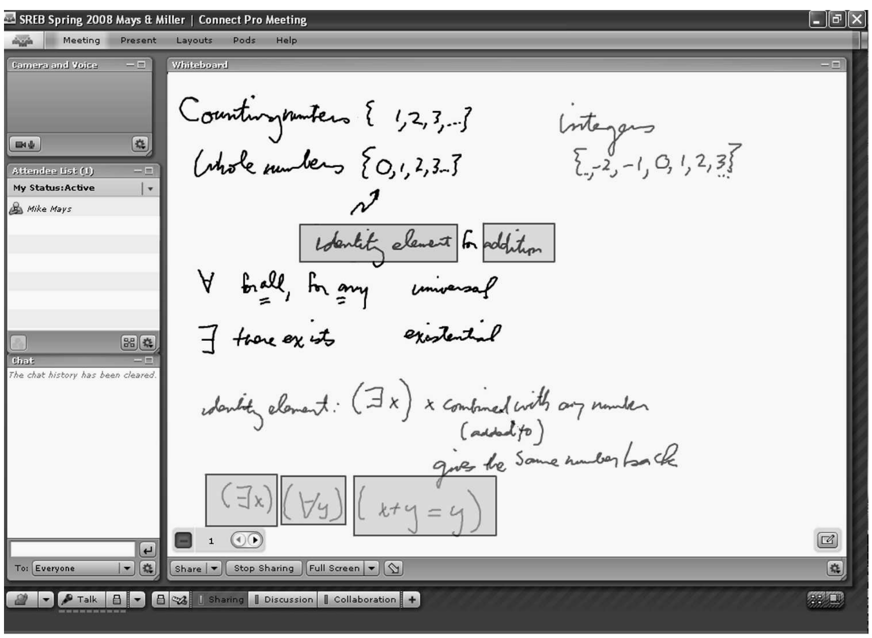
The document [3], with competencies separated into content competencies, process competencies, and instructional competencies, foreshadows the organization of the Common Core State Standards that are being widely promoted today.

11.3 Detailed Course Outline

The courses operate under what we call the ACT paradigm: a three-step cycle to introduce and develop a new topic. First an Active Assignment is made to give students a few days to read and prepare for the interactive online session. It is available by Friday afternoon for a Monday night class, posted on the course website. First there is a reading assignment, either from the textbook (typical for the Math course) or a research article (more likely for the C&I courses), along with Focus questions for the reading that will stress key concepts and skills to be mastered and Focus problems assigned to assess understanding and skill. A quiz based on the reading assignment is posted online using WebCT. It must be completed by the following Friday, so that students can use as resource material the readings, and the development of the ideas in the class session, and the postings by students on the discussion board. The discussion board is available on the WebCT course website from the time the Active Assignment is posted. It is an asynchronous component of the course (so it is available at any time, regardless of whether other students are participating at that moment or not). Building on the Active Assignment, the Conceptual Class session discusses topics from the reading, relates the mathematics to situations in the middle school classroom, and gives insights into the Focus questions that the students discuss on the discussion board. This is where the in-service teachers often have anecdotes to share about their classroom experiences. They have the option of sharing by speaking or by writing comments in a chat window. The last session on a topic is a chance to Talk Together in a less formal online office hours format, where ideas from the discussion board can be explored and students can request elaboration. The Conceptual Class session is recorded so it is optionally synchronous (so students are encouraged to participate while the session is being recorded, but if they cannot the recording is available for later reference), but the Talk Together class is a mandatory synchronous session (so students are all to be logged on and participating while the session is going on), to provide a sense of community and pace. Our experience is that in-service teachers need mandatory scheduled activities to keep the course at a high enough level of priority among their other obligations. Otherwise they fall behind, and despite their best efforts have trouble catching up again.

As a year long course in Number and Algebra, the material is divided so that the first semester focuses on Number (structure of number systems grounded in abstract algebra) and the second semester on Algebra (variables linked to properties of polynomial rings, polynomial modeling built on finite differences, and least squares). In both semesters, the mathematics is presented at the level of an undergraduate mathematics modern algebra course in which the focus is on the structure of number systems, but the examples and activities are relevant to a middle school mathematics class. Each semester is split into six units, consisting of a mathematics week devoted to a single topic followed by a curriculum and instruction week devoted to the relevance of the topic to the middle school classroom.

Unit 1 is on sets and whole numbers. The mathematics class models operations of arithmetic using set theory (addition compared with set union, multiplication with Cartesian products) and the C&I class explores Venn diagrams and logic problems connected with them. A screen shot from the Adobe Connect session for Unit 1 is shown in Figure 11.1. As in all units, there is an analysis of the mathematical content of a lesson chosen from Connected Mathematics [4]. According to the overview available online, the Connected Mathematics Project (CM) was funded



In an Adobe Connect session there are things going on in the pods on the left, a webcam image of the speaker on the top, a list of participants in the middle, and chat conversations at the bottom. All on-line sessions from the most recent offering of the first semester of the Number and Algebra course, along with a syllabus, are available at `www.math.wvu.edu/~mays/524/Streaming.html`

Figure 11.1. Freehand screen snapshot from discussion of number sets.

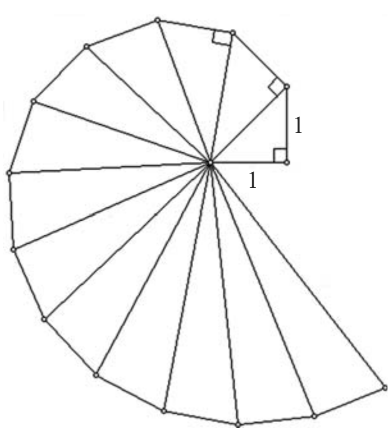
by NSF between 1991 and 1997 to develop a mathematics curriculum for grades 6–8. It is published as a series of independent, but conceptually related, units designed to help students develop understanding of important concepts, skills, procedures, and ways of thinking and reasoning in number, geometry, measurement, algebra, probability, and statistics. The lesson from Connected Mathematics used in Unit 1 is “The Product Game,” from the CM 6th grade unit Prime Time.

Unit 2 expands the number set from the whole numbers to the set of integers. From the point of view of modern algebra we explore the consequences of having additive inverses and an additive identity. We develop laws of exponents by extending the idea of whole number exponents as repeated multiplication to the case of negative exponents. The reading from Connected Mathematics is Investigation 3 on Subtracting Integers from the 7th grade unit, Accentuate the Negative. An excerpt from “Developing Mathematical Power in Whole Number Operations,” in [6] provides a basis for online discussion.

The next unit incorporates order: ordered integral domains are the setting for discussing solutions to inequalities. We discuss the Euclidean Algorithm here, and primes and factorization in elementary number theory. A mathematics education research article discussed is [14], looking for misunderstandings in the concept of prime numbers between pre-service elementary school teachers and middle school students. The Connected Mathematics Investigation 5 on Factorizations from the Prime Time unit (6th grade) is the lesson analyzed.

Since proportional reasoning is such an important topic in middle school mathematics, we devote two course units to rational numbers and their properties and study arithmetic operations on rational numbers. The rational numbers provide the first example of a field, so we revisit closure, inverses, and the distributive law in the context of the field axioms. Field properties for finite sets leads us back to topics in number theory, with an analysis of Cayley tables for addition and multiplication in modular arithmetic for prime moduli. We see when general linear equations have solutions as in Chapter 4 of [12]. Sense-making of operations includes common sense interpretations of division and multiplication of rational numbers. The Connected Mathematics units Bits and Pieces I (BP-I) and Bits and Pieces II (BP-II) (6th grade) provide class activities for discussion in BP-I Investigation 2: Comparing Fractions and BP-II Investigation 7: Dividing Fractions. Research articles discussed include The Distributive Property Undersold by Jane Watson [13] and Fractions and Multiplicative Reasoning by Patrick Thompson and Luis Saldanha from [11].

The last unit in the first semester is about the systems of real numbers and complex numbers. Real numbers are introduced with a discussion of completeness. Dedekind cuts (defining real numbers as upper/lower bounds of bounded sets of rational numbers, which can be described as cuts separating the real line into two pieces) are introduced in a discussion on the historical evolution of real numbers, but the most immediate link to middle school mathematics for real numbers is decimal representations, including the representation of rational numbers as periodic or terminating decimals. Complex numbers make the most sense in the language of closure, this time algebraic closure, so that we can be sure polynomials of any degree with real or complex numbers as coefficients have solutions over the complex numbers. The distinction between rational and irrational numbers is explored in the Connected Mathematics Investigation 5, Irrational Numbers, in the 8th grade unit Looking for Pythagoras, and using the Wheel of Theodorus in Figure 11.2.



This is a nice construction to do in real time using Geometer’s Sketchpad or another dynamic geometry program. Starting from unit length sides, the constructed sides are either integers or irrational. We can construct infinitely many irrational numbers, and make the point that no rational approximation gives the exact value of the square root of a non-square integer. This figure is one of the slides in the mathematics presentation for Unit 6, on real numbers.

Figure 11.2. Wheel of Theodorus for constructing irrational numbers.

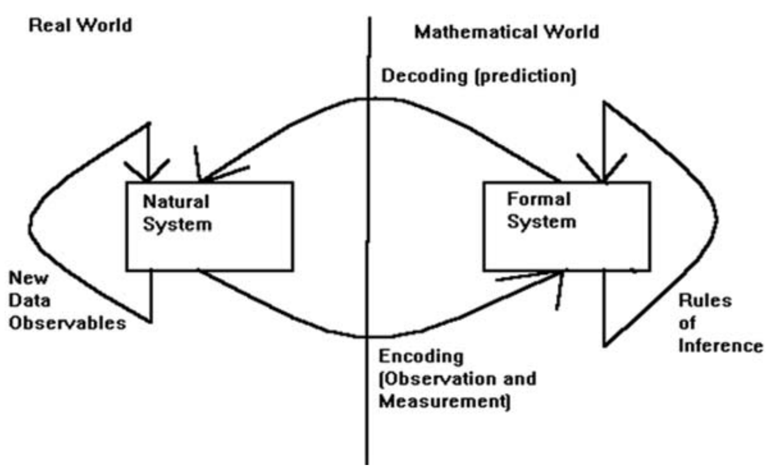
The focus of the first semester is on number and in the second semester is on algebra. Many middle school teachers are encouraged to develop a deeper understanding of algebra to teach Algebra 1, so topics are chosen to be relevant to classroom practice in Algebra 1. The course was taught several times using the support text [7], and recently the syllabus was rewritten to use [5], both of which are algebra textbooks at the college level that emphasize functions and mathematical modeling. The topics are put in context of the mathematical structure underlying the applications. We adopted Functions and Change: A Modeling Approach to College Algebra [5] for its many real-world application problems that emphasize data analysis and technology (i.e., through tables of data and the use of calculators or spreadsheets). Throughout the book, the students build a firm understanding of functions, including combinations of linear, quadratic, cubic, exponential, and logarithmic functions. The book stresses that students should answer questions using complete sentences and proper mathematical notation.

The first unit in the second semester is on mappings and functions. A custom Java-based graphing utility is used that incorporates a representation of a real-valued function of a real variable in symbolic, graphical, and tabular forms. The research reading for this unit from the NCTM anthology [10] is "Statis and Change: Integrating Patterns, Functions, and Algebra Throughout the K–12 Curriculum."

The "elementary mathematics from an advanced standpoint," approach in evidence in the next units. The development of polynomial rings and division is a chance to review the abstract algebra terminology from the first semester. The new concept of polynomial rings leads to the arithmetic of polynomials, including a restatement of the division algorithm in which the size of the remainder is measured by the degree of the polynomial. A more immediate application to school mathematics is solving polynomial equations by factoring. Though a typical polynomial with integer coefficients is unlikely to factor over the rational numbers, factoring polynomials is a useful skill. A model lesson from Connected Mathematics is from Frogs, Fleas, and Painted Cubes (8th grade), Investigation 5 on painted cubes. The article about middle school children's problem-solving behavior by Pape [8] is a good reading for class discussion in the chat room.

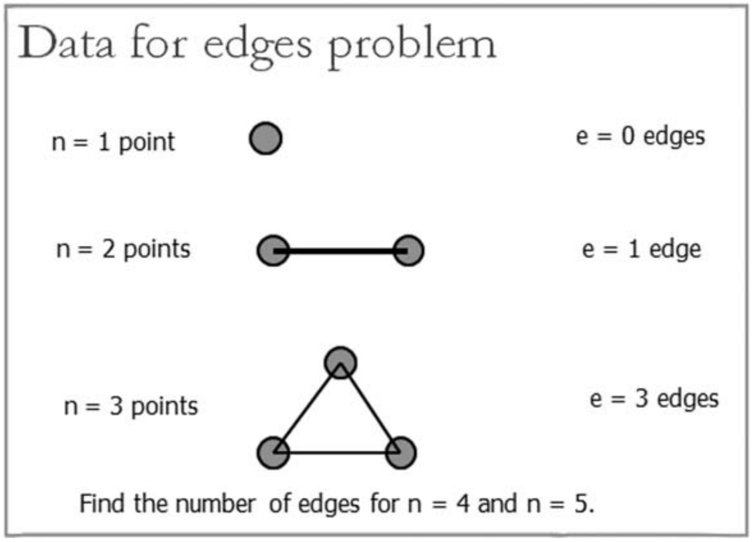
Mathematical modeling is a general theme of the second semester. Models can be built by exploiting a previously known or posited relationship. Otherwise techniques of data analysis are necessary since the relationship underlying the model is unknown. The modeling schema that is used is illustrated in Figure 11.3. The heuristic applied for modeling is called DECAL, for the steps

- *D*escribe the problem, identifying known relationships or laws. Here it is good to recall formulas from geometry or physics.
- *E*xplore. Critical thinking is incorporated as part of the heuristic, employing either inductive or deductive reasoning, working backwards from subgoals, and visualization.



This figure comes from the stage-setting discussion of modeling in the second semester of the course, in which the DECAL heuristic can be traced. It is used in the first mathematics session and recalled to put finite differences and least squares in context.

Figure 11.3. Modeling schema



A follow up slide gives data for $n = 4$ and $n = 5$, which is ample to discover the quadratic function of n governing the number of edges. Once a formula has been generated, the analysis proceeds to find an inductive proof that the relationship is correct. Finally, we link the problem to an earlier analysis of triangular numbers to see why the same rule of formation appears in both cases. Combinatorial problems tend to yield clean data for modeling with finite differences.

Figure 11.4. Edge counting in complete graphs, to generate data for finite differences.

- *C*reate a model that is relevant to the information and setting.
- *A*pply the model to answer the question
- *L*ink the model to new situations to provide a conceptual check of its validity and to make sure it answers the original question appropriately.

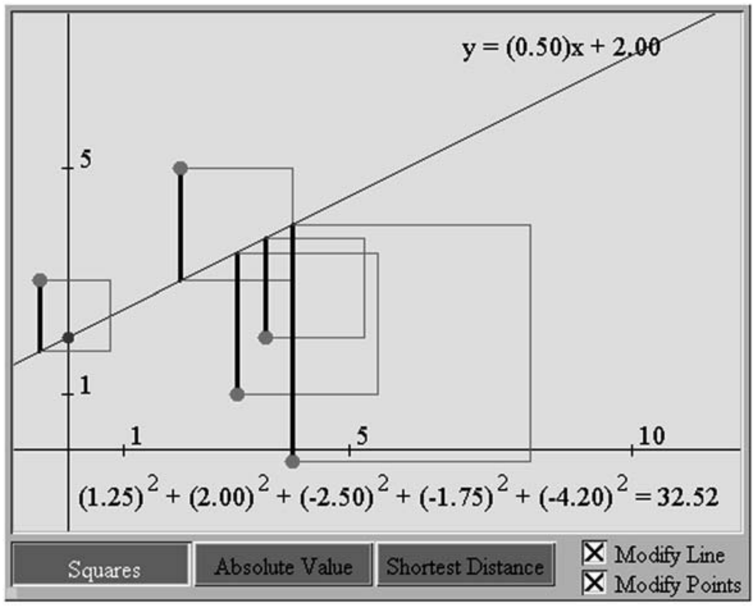
Finite differences is a useful modeling tool as long as the data collected follow a simple underlying mathematical law. It is more useful for combinatorial problems involving counting, or problems from physics in which there is not much error in experimental results, and less so for data from the social sciences. The first example we develop is for edge counting in the complete graph on n vertices (see Figure 11.4). For a problem involving a non-polynomial function, we generate data from the famous Tower of Hanoi problem. There are many online implementations of this disk-moving problem, e.g., `www.cut-the-knot.org/recurrence/hanoi.shtml`.

The approach to minimizing error in a constrained model by the method of least squares is the most sophisticated modeling technique that is developed. We find the NCTM Illuminations applet on least squares to be especially useful for a discussion of how to minimize error in fitting a line through data (see Figure 11.5). The applet, available at `standards.nctm.org/document/eexamples/chap7/7.4/standalone1.htm`, shows the squares of errors as geometric objects, and allows points and the line of fit to be dragged to look at various strategies for minimizing the sum of errors. It also provides for alternate measures of the error using the sum of absolute errors or the sum of shortest distances.

11.4 Sample Vignettes

We have found vignettes to be useful for bringing points into focus from the research readings and to spark discussion in the online forums and in class. Vignettes in the literary sense are short scenes that focus on one moment to give a trenchant impression about an idea. We find that by looking at a common misunderstanding in a classroom setting we can elicit discussion, especially from in-service teachers, about similar experiences they have had. Many of the vignettes were developed for the Number and Algebra course by Dr. Melanie Butler, currently a faculty member at Mount Saint Mary's College. Here is a vignette illustrating a common mistake in integer nomenclature:

Misconception 1: Students believe that a number can be both even and odd.



There are other least squares applets available online, but the NCTM version with squares having sides the vertical distance of the points from the suggested line of fit gives a strong geometric intuition why outliers affect the error so strongly. This applet is incorporated into the mathematics class for least squares modeling, and revisited in the C&I class with suggested classroom activities.

Figure 11.5. NCTM least squares applet.

Vignette 1: Jill argues that the numbers $2n$ and $2n + 1$ can both be either even or odd, depending on the choice of n. What misconception might Jill have? How would you address this misconception?

Another example from number theory:

Misconception 2: Students have trouble understanding prime numbers because students cannot see a way to generate prime numbers and do not understand that there are infinitely many of them.

Vignette 2: Ahmed says that there are finitely many primes because there are fewer and fewer as you go along. Is Ahmed right? What misconceptions might Ahmed hold?

In the discussion of real numbers, a common student mistake gleaned from the research readings is that students have trouble distinguishing between rational and irrational numbers.

Misconception 3: Students who are used to approximating irrationals with fractions (such as approximating π by 22/7) may start to think of irrational numbers as rationals or start to think of fractions as irrational.

Vignette 3: Sal states that 1.41 and 3.14 are both irrational numbers. What misconception is Sal demonstrating? How would you address the misconception?

The advantage of teaching with vignettes is that it invites in-service and pre-service teachers to share classroom experiences and anecdotes, and to rehearse effective teaching strategies before the classroom situation arises. Vignettes also spark rich sidebar discussions on other topics or concepts that middle school students have difficulty understanding, and the chat window in the online course allows the discussions to proceed without being disruptive or intrusive (see Figure 11.6). This allows deeper class discussions into how best to combat the misunderstandings. For example, we might be discussing the following misconception through a vignette.

Misconception 4: Students believe that the more digits a number has the bigger the number is because they are overgeneralizing properties of whole numbers.

Vignette 4: Mike thinks that 4.52 is bigger than 4.6 What is Mike's misconception? How would you address his misconception?

Here is a transcript of the discussion of this vignette from the Spring 2009 semester. It is viewable online at `www.math.wvu.edu/~mays/524/Streaming.html`.

Instructor: Discuss how would you address this misconception.

Student 1: Teach him to add zeros as place holders.

Student 2: Place value . . . from left to right.

Student 3: Maybe use a number line from 4.5 to 4.6.

Student 4: I would refer the number to money.

Student 5: Add 0 as a place holder and talk money.

Student 2: Yes, Student 1.

Instructor talks about the money viewpoint.

Student 6: Add a zero to the 4.6.

Student 7: To make both numbers have the same amount of digits by adding a zero to 4.6.

Instructor inserts on the whiteboard $4.6 = 4 + 6/10$ and $4.52 = 4 + 5/10 + 2/100$ and talks briefly about the notation.

Student 6: My students would say \$4 and 6 cents.

Everybody agreed that their students would say this too!

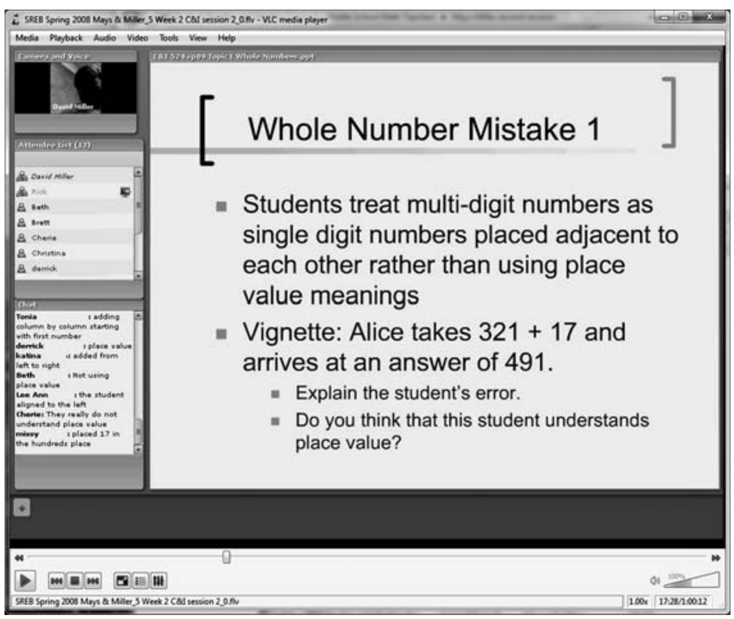
Instructor: Why are they saying \$4 and 6 cents? How are they reading this?

Student 1: I then have them show me what 6 cents and 60 cents would then look like—they're reading it as a single digit by place value.

Student 8: Anything after the decimal point is cents . . . they don't think about place value.

Student 6: They just see the 6 and know that it should be cents.

Student 7: There is only one number to the right of the decimal so they assume only single change.



The Chat box in the lower left indicates the first few student responses to the vignette. The discussion continued
Student 1: adding column by column starting with first number.
Student 2: place value.
Student 3: added from left to right.
Student 4: not using place value.
Student 5: the student aligned to the left.
Student 6: they really do not understand place value.
Student 7: line the numbers up on the left.
Student 8: she is lining the problem incorrectly with the hundreds instead of with the ones place.
Student 9: I found this to be a common problem this year.

Figure 11.6. Class session on whole numbers.

Student 5: Students have trouble understanding place value so a 6 by itself is 6 cents.

Instructor: How do you get students to 60 cents instead of 6 cents if you were connecting this to money?

Student 8: I try to teach them that the first place is the number of dimes.

Student 7: I ask them how many pennies it takes to make a dollar.

Student 6: I ask them is this the way we show money in the U.S.

Student 8: I have some students that would write .06 instead of .60.

Instructor talks about applying the general number line discussion from the Connected Mathematics Lesson to a number line focusing between 4.5 and 4.6.

Student 7: If this were measurement like 4.6 ft, my students would not realize that .6 would be half a foot not more than a half.

Another question that often comes up in our class discussion after this vignette is "How do I explain to my students that adding a trailing zero in the decimal representation doesn't change the value of the number?" This is especially interesting from the middle school perspective in a discussion of rounding and digits of accuracy.

11.5 Sample Resources

An important goal of the C&I part of the course is showing teachers technology resources. To do this we first list numerous resources as part of each unit's PowerPoint slides, available on the course website, and then cover a few, demonstrating them online using Adobe Connect. The demonstration can be accomplished by allowing the students to view what is on the instructor's desktop. This feature, available in many online presentation tools, is called appsharing. It is two way sharing: the instructor can relinquish control of his or her computer so that students can play with the resources themselves. The online resources are a great way for teachers to actively engage their students as they learn. Although we cannot mention every resource that we have used in the Number and Algebra courses, there are several websites and databases that are sources of many high quality resources. For example,

- **NCTM Illuminations**

  `illuminations.nctm.org/`: A NCTM website that has activities and lessons tied to the NCTM standards. They can be selected by grade bands (K–2, 3–5, 6–8, 9–12) , by subject (number and operations, algebra, geometry, measurement, and data analysis and probability), and by searching for words and phrases. The website has links to the NCTM standards and other NCTM web resources.

- **National Library of Virtual Manipulatives**

  `nlvm.usu.edu/en/nav/vlibrary.html`: A web site that houses virtual manipulatives. A list of manipulatives available is generated by selecting a grade band and subject (the same grade band and subject on NCTM illumination website). The National Library of Virtual Manipulatives is an NSF-supported project that began in 1999 to develop a library of interactive, web-based virtual manipulatives or concept tutorials, mostly in the form of Java applets, for mathematics instruction (K–12 emphasis).

- **Arcytech: Educational Java Site**

  `arcytech.org/java/`: The site was started in 1997 by Jacobo Bulaevsky with the purpose of providing Java applets that can be used as resources to help educate children. The site has ten mathematics java applets and one biology java applet. Interested teachers can reference `arcytech.org/java/letter.shtml`.

- **SHODOR: A National Resource for Computational Science Education website**

  `www.shodor.org/`: There are links for students, parents, and educators, and links to learn more about the website, and activities and lessons. In the activities and lessons are resources for middle and high school (Interactive, SUCCEED curriculum, and Computational Chemistry), high school and undergraduate (CSERD and Masters Tools), and Special Education Tools (DEAF STEM and Braille through Remote Learning). The resource Interactivate has been mentioned in the Number and Algebra courses and can be found at `www.shodor.org/interactivate/`, where the learner can choose links of activities, tools, or dictionary, and the instructor can choose links of lessons, discussions, and standards. The resources on the SHODOR website are part of the National Science Digital Library (NSDL).

- **The National Science Digital Library (NSDL)**

  nsdl.org/: NSDL is the nation's online library for education and research in science, technology, engineering, and mathematics (STEM). The middle school portal is at msteacher.org/.

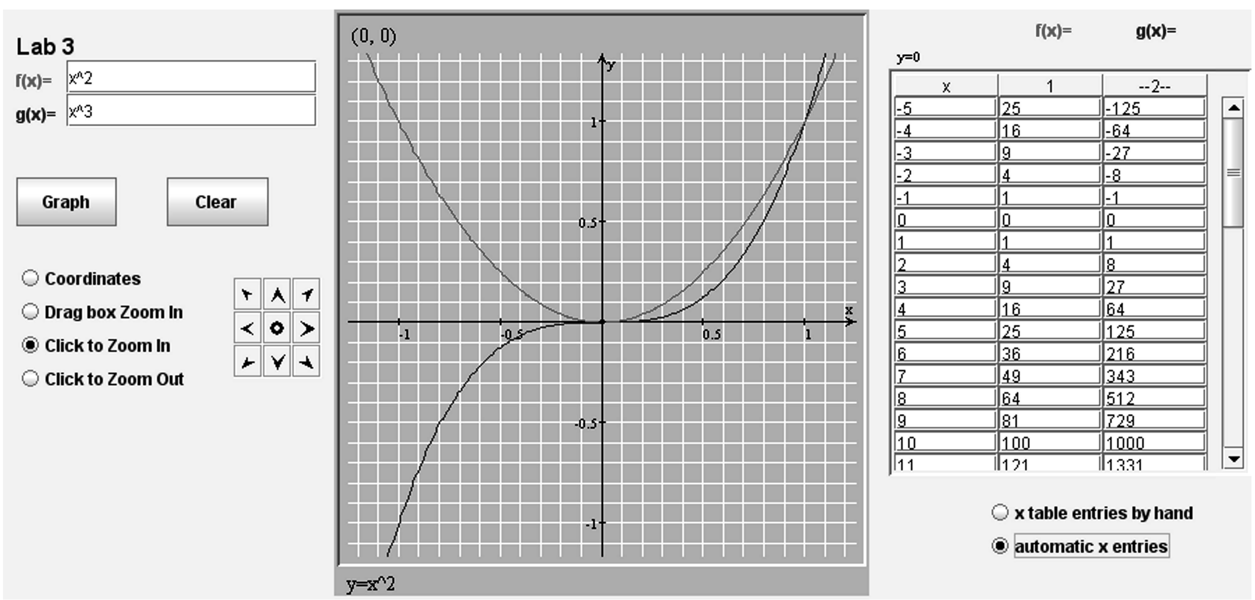
- **Math Forum**

  www.mathforum.org/: The Math Forum website is a resource for improving math learning, teaching, and communication. There is a middle school link at www.mathforum.org/teachers/middle/ that has classroom and career links.

Among these websites and databases, there are resources that go hand in hand with the Connected Mathematics lessons. For example, the resource www.nlvm.usu.edu/en/nav/frames_asid_162_g_3_t_1.html can be used when teaching Connected Mathematics unit Accentuate the Negative (7th grade) Investigation 3: Subtracting Integers. Students can use the websites at home in place of the manipulative that they use in class.

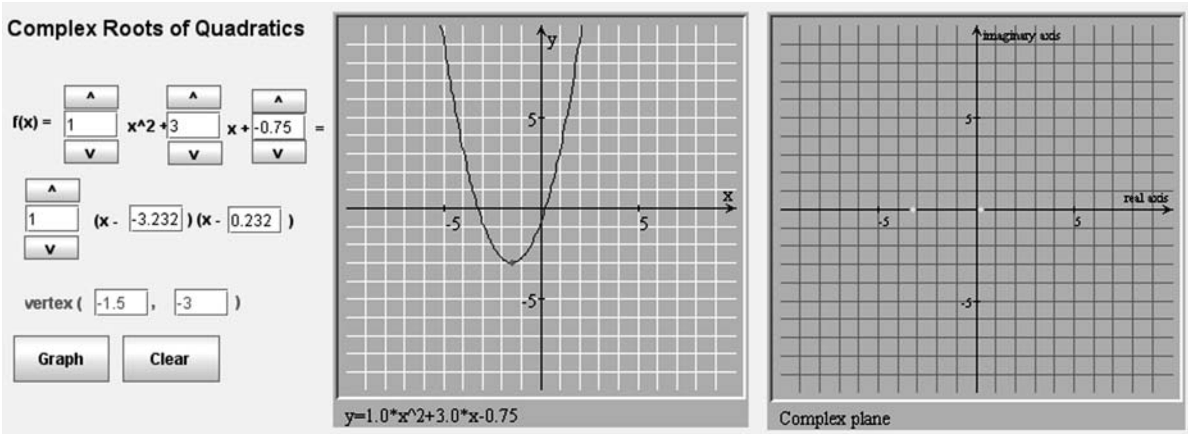
Another feature of the course is a set of custom java applets written to illustrate points in the course and to provide a coherent, stable user interface for students to explore problems. Many of the applets written for the Number and Algebra course were developed using the Mathematical Java Toolkit developed by Joe Yanik at Emporia State University (NSF award DMI-9950714). A full set of applets for this course, applets and activities for other courses, and class activity sheets and source code, is available for download or linking at math.wvu.edu/~mays/AVdemo/AVdemo.htm. Links to Professor Yanik's site and a mirror of the Math Toolkit development tools is maintained on the West Virginia University Institute for Mathematics Learning website at www.iml.sitespace.wvu.edu/javadevelopment. The most heavily used applet is a general purpose graphing utility (see Figure 11.7).

As an illustration of how the user interface works in more specialized contexts, we show a screen shot for a graphing applet on properties of quadratic functions (see Figure 11.8).



Here there are two math fields at the upper left where functions can be entered in calculator notation and graphed. The parser understands algebraic exponents, logarithms and exponential functions, trigonometric functions, and inverse trigonometric functions. The red $f(x)$ and blue $g(x)$ appear in the color of the graphs displayed and in the columns of the function table. Several zoom modes are implemented: the Coordinates radio button triggers a mode that displays the coordinates of a point clicked in the graph area. The Drag box builds a rectangle with sides parallel to the axes from the point where the mouse button is clicked to where the button is released, and uses the rectangle as the bounding box for a redrawn graph. The zoom in and zoom out buttons rescale the graph axes by a factor of two and replot the graphs, centering the new graph at the point clicked. Once a graph is plotted, it can be moved with the direction arrows on the direction rose. Clicking the center of the rose restores the graph to its original size and position. The x table entries-by-hand option under the table of values allows individual x values to be entered, and then $f(x)$ and $g(x)$ are displayed.

Figure 11.7. A general purpose graphing applet.



In this applet the coefficients of the quadratic term, the linear term, and the constant term can be adjusted by the up and down arrows or by entering a numerical value in the number field. The middle line, showing the factored form of the quadratic, appears only when the quadratic has x intercepts. When it is present, the outer coefficient and values of the roots can be adjusted with the arrow keys or entering numerical values in the number fields. The vertex (the red dot) can be dragged directly, which updates the numerical fields of the standard form and the factored form.

Figure 11.8. An applet for exploring complex roots of quadratics.

11.6 Remarks on On-line Instruction and Resources Available

Mathematics is difficult to teach on-line. Clear communication of mathematics depends on symbols, diagrams, and graphical or tabular representations. The abstract and algorithmic nature of mathematics is difficult to express in a stream of words, which is all that is available in an audio-only distance education class. Even the text-based environment of chat rooms does not easily allow for Greek letters and operator symbols, superscripts, subscripts, or stacked constructions such as summations or matrices.

Presenting content linked to pedagogy has similar challenges. On-line courses should model good face-to-face teaching in presenting multiple ways of understanding a concept and fostering discussion. It is important to choose appropriate software and lesson materials online. The point is not that distance education is superior to, or even necessarily as good as, face-to-face instruction, but that distance education makes it possible to reach students who otherwise would not be served at all. Such students include in-service teachers aiming to acquire new credentials and pre-service teachers enrolled at branch campuses or who are otherwise constrained. Fortunately, emerging technologies such as conferencing software that provides audio-visual streaming, application sharing, and whiteboards are providing solutions to the problems in distance delivery.

We are at the fifth of the five generations of distance education technologies as cited in [1]: print-based correspondence; broadcast TV, radio, and tapes; teleconferencing and hypertext; computer-mediated internet; and on-line interactive media, including internet-based access to the World Wide Web. As reported in [2], internet-based courses have many pedagogical advantages over other platforms: web based courses invite hands on learning experiences, so students can be more engaged; the structure of courses fosters the development of small learning communities, especially when the instructor encourages use of discussion boards; and students have greater access to faculty members by email or other asynchronous channels.

These features have been adapted in the higher education community in its search for greater efficiencies and new markets. In a 2006 study [9], it was noted that credit-granting internet courses were offered by more than 65 percent of 2-year and 4-year universities. They are included in many programs designed to be completed entirely through distance education. SREB maintains a clearinghouse of distance education courses appropriate for in-service or pre-service teachers at `theteachercenter.org/`, tied to state-by-state certification requirements.

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II

Courses for Middle School Mathematics Teachers

D. Precalculus and Calculus

12

Vertically Connecting Precalculus and Calculus with Middle School Mathematics

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Middle grades mathematics teachers have a unique role in the education of schoolchildren, helping students make the connections between the foundational concepts of elementary mathematics and the more sophisticated skills and understanding required in secondary mathematics. The University of Texas at Arlington has designed and implemented two capstone mathematical content courses for the pre-service middle school teachers to establish the connections. The courses are titled Precalculus for Mid-level Mathematics Teachers and Calculus for Mid-level Mathematics Teachers. Each course is designed to meet the content, pedagogy, and instructional goals of The Mathematical Education of Teachers [2] recommendations for middle grades teachers, and the certification requirements laid out by the state of Texas. Pre-service teachers enroll in the sequence of courses after completing all other mathematics requirements for their degree, which consist of elementary statistics and college algebra, and a three course sequence for elementary and middle grades teachers focusing on, respectively, number and operation, geometry, and data and modeling. Pre-service teachers complete the precalculus and calculus sequence in the two semesters before taking their middle level state content certification exam (the Texas Examination of Educator Standards in Mathematics 4–8) and pedagogy certification exam (Pedagogy and Professional Responsibility 4–8).

The two courses interweave key mathematical concepts with elements designed to aid in understanding student thinking, in particular, middle school students. The courses include collaborative activities, daily discussion and questions connecting the college level mathematics topics with school mathematics (often using excerpts from middle school mathematics curricula), reading and viewing of middle school case studies, and a final project. The final project, called the two-problem comparison paper, requires the pre-service teachers to analyze the unifying mathematical ideas between specific problems in college level and middle grades level mathematics.

The University of Texas at Arlington has an enrollment of 33,000 students with approximately 35 students graduating each year in the middle school mathematics education program. The capstone precalculus course typically runs one section in the fall semester, and the capstone calculus course runs one section in the spring semester. Enrollment in the courses is restricted to students who are seeking certification as middle school teachers. The courses are taught by a faculty member of the Department of Mathematics.

In this article, ideas for building similar courses for middle school math teachers are provided. The essential elements of this article should be adaptable to content courses for teachers of other disciplines, in which there is a desire to focus on vertical connections in school curriculum.

12.1 Format of the Courses

The format for both courses is activity and discussion based, with the goal of having the students learn mathematics concepts through guided discovery, and modeling teaching practices for their mathematics classrooms. Lectures are interspersed with collaborative activities (sources for which are in section 3.1), large and small group discussions, and informal student presentations of individual and group progress on problems. No part of the student's grade is based on participation. However, the willingness of the students to share their strategies and reflections on problems is a crucial element of the class culture that must be facilitated and encouraged by the instructor from day one. The students must be aware that it is their own sense-making of the mathematics that is driving the learning in the class. As an example, when a future teacher poses the question in class (to the instructor),"Am I right?" the instructor always defers to the class. It is the obligation of the class, not the instructor, to decide the correctness of an explanation, though the instructor is there to provide guidance.

Class discussions and assignments are often on excerpts from middle school mathematics curricula (described in section 3.4) that have connections to the mathematics studied in precalculus and calculus. The goal of the discussions is for the pre-service teachers to discover and appreciate the connections, so they can see how the concepts, strategies, and structures studied in middle school mathematics carry through to the college level. Awareness of these mathematical connections shows the relevance of a precalculus and calculus course for a future teacher of middle grades math.

Both courses use textbooks, primarily used as a source of homework problems and as a reference, rather than as a guide for the content or structure of the courses. In recent semesters, the textbooks used for Precalculus were standard college precalculus texts: *Functions Modeling Change: A Preparation for Calculus* [3] by Connally, Hughes-Hallett, and Gleason, or *Precalculus Essentials* [8] by Sullivan and Sullivan. The calculus text has been *Calculus Connections, Mathematics for Middle School Teachers* [5] by Harcharras and Mitrea.

The courses meet twice weekly. A class period of an hour and twenty minutes might include an opening large-group discussion (5–15 minutes) of student solutions to problem of the day (described in section 3.2). The majority of the class period is then small-group collaborative work on the new mathematical content. The collaborative work can be paused for instructor-led large-group discussion of an interesting development by one of the groups or to clarify a misunderstanding. During the large-group discussions students are asked to share their thoughts or progress on a problem. There is no extended lecture by the instructor, but the instructor may spend time at the board introducing an activity or acting as a (knowing) scribe for a student group that is trying to organize their ideas to explain them to the class. Every other class period ends with a video case study of a middle school mathematics classroom learning a topic with connections to the day's content. Homework is assigned at each class meeting and a sample of solutions is collected weekly for a grade to encourage the students to stay up-to-date. No more than ten homework problems are assigned each class period, they are chosen to provide the students with practice on the concepts that have been developed in their small groups. The collected homework problems are graded very broadly: each problem receives either a check, a check minus, or a zero if it is, respectively, correct, partially correct, or not done. The graded homework is used to let the instructor see where students are in their understanding. The students turn in journals with their explanations to selected questions of the day and their reflections on questions posed about the video case studies and middle school curricula excerpts. The students are encouraged to write their journals thinking of their peers in the class as their audience.

The students' grades are based on their submitted written work (homework and journals), three take-home exams (the problems on these exams are strictly mathematical, as on a test in a traditional precalculus or calculus course, though some may be phrased in a teaching context), and their final project (described in section 3.5). A sample syllabus for each course can be viewed at `www.uta.edu/faculty/tjorgens/middleschoolcourses.html`.

12.2 Course Content

12.2.1 Precalculus

Precalculus for Mid-level Mathematics Teachers serves to bridge the gap between algebra and calculus for future middle level teachers. The course has functions as a unifying theme. The goal is for students to develop a firm understanding of the concept of function, be able to analyze functional behavior, and understand covariation. This includes studying transformations and inverses, and solving equations. The ideas are applied to specific functions, including exponential, logarithmic, power, polynomial, rational, trigonometric functions, and functions that cannot be described by a formula. There is an emphasis on multiple representations of mathematical ideas: verbal, concrete, pictorial, tabular, symbolic, and graphical. For most students, the course introduces them to trigonometric functions and their applications. Throughout, the mathematical connections between precalculus and school mathematics are highlighted using the course elements detailed in the following.

12.2.2 Calculus

Calculus for Mid-level Mathematics Teachers serves to introduce the basic concepts of calculus to middle school teachers. Its primary goal is to develop understanding of the mathematical ideas in calculus and gain insight into the topics in the middle school curriculum that are related, such as proportionality, dimensional analysis, rates of change, and area under a curve. The key mathematical ideas are differentiation and integration, an understanding of their applications, and an understanding of their relationship.

Building on the strong understanding of covariation developed in the precalculus course, the ideas of calculus are explored in multiple contexts, and classes of functions are used that best suit each of them. To develop computational proficiency for differentiation and integration, power, polynomial, and exponential functions are studied. When looking at the geometry of the derivative and the integral, less friendly functions are used, often functions described only by a graph or a table of values.

Students in this course are expected to be able to explain the definition of the derivative as a rate of change and its geometric context. They must understand how the derivative can be used to approximate nearby values of a function using a tangent line. There is an emphasis (continued from precalculus) on using the new tool of differentiation to further analyze the behavior of functions, i.e., intervals of increase and decrease, concavity, and extreme values. Students should be able to sketch the graph of the derivative of a function given the graph of the function and to sketch a possible graph of a function given a graph of its derivative. For integration, students compute the area below the graph of a function by using a limit of a Riemann sum or by using a definite integral. Students are expected to be able to apply the Fundamental Theorem of Calculus to compute definite integrals and area, especially in problems that have their foundation in middle school mathematics.

12.3 Essential Course Elements

What sets the courses apart from standard precalculus and calculus courses and from mathematics courses for elementary (only) and high school (only) teachers, are the course elements that tie teaching mathematics to middle schoolers. They are the carefully chosen collaborative activities, the daily discussion (question of the day), inclusion of video case studies, highlighting mathematical connections with middle school curricula, and the two-problem comparison paper.

12.3.1 Collaborative activities

The mathematical content of the courses is developed through activities and guided discovery. The activities are chosen to unpack the ideas in the mathematics, so the students understand the concepts and their foundations in middle school mathematics, and appreciate themselves as mathematicians. The pre-service teachers are given the opportunity to understand the vertical mathematical connections of precalculus and calculus, so they can assimilate and understand the mathematics. Moreover, the students see how a rigorous mathematics course can be run without a teacher as lecturer format.

The primary source for the collaborative activities in the precalculus course is the Teacher Quality Modules [11], designed by mathematicians at universities in Texas to facilitate teacher training in mathematics content for teaching courses in middle school and secondary school mathematics. One set of activities that works particularly well with pre-service middle school teachers is from the Algebra II Teacher Quality Module [4], Topic 1.1,“Linking Foundations.” This sequence of activities has the students explore the connections between thinking of transformations on objects geometrically, which is how middle schoolers first encounter transformations, to thinking of them functionally which is how they are viewed in precalculus.

The pre-service teachers start with a figure composed of triangles and line segments representing a sailboat, on gridpaper. The vertices of the figure are labeled with letters. After writing the coordinates of each point in the figure, the future teachers then transform the figure using patty paper as an aid (the square, lightly waxed paper that is used to separate hamburger patties and can be purchased at most restaurant suppliers). Patty paper is used as a manipulative for doing constructions and other explorations in geometry classes. For instance, the pre-service teachers are asked to translate the sailboat left 4 units and down 1 unit, and to record the coordinates of the images of the vertices. They describe the translation in words and in symbols. In words, the x coordinate decreases by 4 and in symbols, $x - 4$. Similarly, the y coordinate decreases by 1, or $y - 1$. They then generalize the observations they have made to coordinate notation. When the point $(4, 2)$ is translated to the left and down to $(0, 1)$, the corresponding coordinate notation would be $T : (x, y) \to (x - 4, y - 1)$. The notation matches with their intuition, that when you move a figure right (left), you add to (subtract from) the x coordinate, and when you move a figure up (down), you add to (subtract from) the y coordinate.

The future teachers then explore what happens when they horizontally or vertically translate the graph of the function $f(x) = x^2$. They describe the transformation and write it in coordinate notation. Then they try to write the translations in function notation, checking their work on a graphing calculator, and discover that horizontal translations represented in coordinate notation may cause confusion when using function notation.

Connecting the geometric notation with the functional notation for a horizontal translation requires pre-service middle school teachers to think deeply about what is different between the two perspectives, and how middle school students might be confused as they move from thinking geometrically (in early middle school) to thinking algebraically (in late middle school and high school). This is not a trivial exercise, and the pre-service middle school teachers struggle with understanding what makes the functional point of view "turn things around." However, once the connection is made, the pre-service teachers have a deeper understanding of functional notation and a capacity to prevent misunderstandings of the concept in their students.

Another collaborative activity that fits well in the precalculus course is Topic 6.1 from the Algebra II Teacher Quality Module [4], "Characteristics of Exponentials and Logarithms." It develops a conceptual basis for the properties of exponential and logarithmic functions by concretely focusing on their inverse relationship. The students discover how logarithms work when they use a qualitative approach (rather using the "log" button on their calculator) to solve exponential equations, such as $10^x = 60$, numerically . Students are asked to show how they obtain their solution graphically, by using guess and check, or by using a bisection method. Pre- and post-tests of the pre-service teachers, who had previously been introduced to logarithms in earlier classes, showed they had an improved understanding of the idea that a logarithm is an exponent, and of how the properties of logarithms followed from the properties of exponents.

For the calculus course, the collaborative activities come from worksheets developed as part of the course materials for the Math in the Middle Project at the University of Nebraska-Lincoln [10]. The worksheets consist of 18 problem sets that lead the future teachers through exercises on the basic concepts of calculus. Using roughly one worksheet each week, the problem sets move from limits to applications of integration in one semester. The worksheets are designed to help students discover calculus concepts. The future teachers work in groups during class time to complete the problem sets. The instructor will spend one or two minutes introducing each problem set, connecting it to previous learning and providing a framework for the new concept. After each problem set is completed, it is discussed as a class, with the students sharing solutions and the instructor guiding the discussion to ensure that the big ideas of each problem set are clearly identified and understood.

As an example, to develop the notion of average rate of change, the worksheet begins with the graph of a piecewise linear function that describes the motion of a remote-controlled car moving in a line, showing the distance (in feet) of the car from the person controlling it t seconds after turning it on. The pre-service teachers fill out a table that explains what the car was doing in the time intervals corresponding to segments of the graph, including beginning time, ending time, beginning distance, ending distance, and velocity. The next problem gives a continuous (but not piecewise linear) graph that shows a girl's distance from home (in miles) as a function of the time in minutes since she left her home to run errands. The future teachers must answer questions about the girl's average velocity between her stops on the journey. For instance, they find her average velocity from the time she left the hardware store (marked at point H on the graph) to the time she arrive at the auto parts store (marked at point A). Then they draw the line segment $\overline{HA}$, and compute its slope, comparing it their previous answer. Once they make the connection between average velocity (rate of change) and the slope of the line joining the corresponding points, the pre-service teachers look at a quadratic function and examine the intervals over which the average rate of change is positive, negative, and zero. They write the computations of average rate of change in function notation, leading into the next topic, the difference quotient. Working with three rich problems over one and a half class periods, the future teachers gain the insight into the concept of average rate of change that they will need to build the notion of derivative.

12.3.2 Daily discussion: Question of the day

The question of the day is a question or topic for discussion that connects the topics the pre-service middle school teachers are studying with middle school mathematics. Each day, usually at the end of the period, the pre-service teachers are presented with a question of the day. Most days, the instructor selects a topic or question based on experience. That is, knowing the content topic of the day leads to some natural questions or issues. Sometimes the question arises naturally during the course of a class period (in which case, the selected question may be discarded). The question will be discussed briefly as a group, and then assigned to the students to think about and to come back with an explanation. The questions often give rise to wonderful discussions, arguments, student presentations, and "aha" moments. The most fruitful questions of the day (in the eyes of the instructor) are chosen to be included in the pre-service teachers' journal write-ups. After the question has been discussed, the pre-service teachers are told to write up their own explanation and submit it as a journal entry. The journal is given: 7 points if effort was made but the response is incomplete or incoherent, 9 points if the ideas are coherent and communicated well, and 10 points if the writer goes above and beyond, showing deep reflection and insight. This counts toward the written work portion of the course grade, which makes up 25% of the grade.

Here are some question of the day examples and the context in which they were posed.

- Come up with a concrete example which will illustrate to a seventh grader why dividing by zero is an undefined operation, and another example which will show that dividing zero by a nonzero number does make sense. (The question arose during the study of domains of functions.)
- Does a radical sign always have a $\pm$ attached to it? Is $\sqrt{9} = \pm 3$ or 3? (This arose during discussions about solving quadratic equations.)
- There seems to be a contradiction between function notation for horizontal shifts and coordinate notation for horizontal shifts. Explain why this is not a contradiction. (This arose during the study of transformations of functions.)
- Why does it make sense to reverse the inequality when you multiply both sides of the inequality by a negative number? (This arose when solving for the set of x values for which a derivative was always negative.)
- What does the "-1" have in common, if anything, in the following expressions: $5-1$, $\frac{2}{3}^{-1}$, $\tan^{-1}(x)$? (This arose when developing the formula for the derivative of a inverse function.)

Thinking about the questions often forces the students to go back to first principles, and the foundations which they will be building upon in middle school. They are surprised how often the answers to these seemingly simple questions require deep understanding. For further questions of the day, see [9].

12.3.3 Video case studies of middle school classrooms

Perhaps the aspect of the courses that the pre-service teachers most appreciate is viewing of video case studies of middle school classrooms. It gives the students a chance to view mathematics teaching in action, within the context of their own mathematical learning. As stated by Shifter and Bastable in [6, p. 35], viewing and discussing video case studies of middle grades teachers in their classrooms models "how close attention to student thinking and a deep grasp of the mathematics at issue can lead teachers to a more coherent, hence more confident, approach to their instruction."

The video cases used in both courses are taken from the book and accompanying DVD *Connecting Mathematical Ideas* by Boaler and Humphreys [1]. They are excerpts, usually about 15 minutes long, from Humphreys' seventh grade mathematics classroom, which was videotaped for an entire year. Each is accompanied by written commentary from Humphreys, in which she discusses what her goals for the lesson were and gives her reflections on how the lesson worked for her students. There is also written commentary by Boaler, a mathematics educator, who puts the case study in the context of mathematics education research. While it is worthwhile just to watch the video case studies and discuss them (this was done the first time the precalculus/calculus sequence was offered), reading the commentaries enhances the classroom discussions and reinforces the mathematical connections that are the focus of the courses. So we suggest that the pre-service middle school teachers buy the book, and that the commentaries are assigned as readings prior to viewing the videos.

Watching the videos in the midst of a mathematics content course brings the focus back onto middle school students' mathematical thinking. Prospective teachers see how knowing the mathematics, centering the lesson on student thinking, and trying to find logic in students' mistakes can make a powerful learning environment for middle schoolers.

The mathematical content of the video cases matches up well with the course content of precalculus and calculus. The first case focuses on student understanding of algebraic representations, leading students from concrete examples to generalizable formulas using variables. The cases work well during the first month of precalculus, when much of the content is based on what functions are and how to represent them. The case study asks the question of how a middle school student understands the concept of a variable, and forces the pre-service teachers to consider their own concept of variable. Another video case, ideal for viewing while studying rational functions, is "Defending Reasonableness: Division of Fractions." For the calculus course, there are middle school case studies on proof, student understanding of volume and surface area, and the generation of geometric formulas.

After viewing each video case, small and large group discussions are held in class, and the students are required to turn in journals responding to selected prompts that are in the discussion guides from the book. The journals are graded by: 7 points if effort was made but the response is incomplete, 9 points if all the prompts were addressed fully, and 10 points if the writer goes above and beyond, showing deep reflection and insight.

Other sources of video cases of middle school classrooms that could be used in a precalculus or calculus course are the professional development materials at `learner.org` and the NSF-funded Developing Mathematical Ideas materials, seven seminars built around casebooks that track a mathematical theme from kindergarten through grade eight. For instance, see [7].

12.3.4 Connecting to middle school mathematics curricula

For each content topic, samples of middle school mathematics curricula are selected by the instructor that have mathematical connections to it. Sometimes the connection is obvious (such as comparing a middle school lesson on finding the slope of a line, given a table containing coordinates of points on it, with a calculus lesson on approximating the derivative using a table of values for a function), and other times it is less so (such as looking at the mathematical connections between a middle school lesson on finding patterns in sequences of numbers and a precalculus lesson on the periodicity of trigonometric functions.) The excerpts from middle school curricula are used primarily to show that the foundations laid in middle school mathematics are essential to understanding the mathematics that is to follow. The curricula excerpts are presented at the completion of study of a topic, and a large-group discussion takes place about the connections the pre-service middle school teachers can identify between the mathematics of the middle school lesson and the mathematics they have just studied. The students are not graded on the discussions, but the process of identifying vertical connections in the middle school curricula serves as necessary preparation for the students' final project (see the following section.)

In Texas, middle school certification covers grades 4–8. Middle school curricula that have been used as resources for the courses in the past include Investigations in Number, Data, and Space (which is used by local school districts where the pre-service teachers are likely to teach), Connected Math, and Math in Context. The latter two curricula have sample lessons available online (see for instance, `showmecenter.missouri.edu/showme/default.html`). Also, the Harcharras and Mitrea text, *Calculus Connections* [5], contains middle school curricula excerpts. For curricula that the students can't access, the instructor provides copies. The students are also encouraged to bring in the middle school texts that are being used in the classrooms from local districts they are visiting for their education courses, and they are used as a resource.

12.3.5 Two-problem comparison paper

As a final project in each capstone course, the students complete a major written assignment, the two-problem comparison paper. This is a tool for better understanding the ideas unifying college level and middle grades level mathematics, and for putting the mathematics in context at both levels. It serves as a summative evaluation of the pre-service teachers' mathematical understanding of the course. For the two-problem paper, each student selects a college-level problem from the course and selects a related problem from grade 4-8 mathematics that the student believes has concepts similar to those in the college-level problem. Both problems must be approved by the instructor. The goal of the paper is to show the mathematical connections between the problems. For some students, the common mathematics is superficial. Other students find rich, deep connections between their problems that vary from the algorithmic, to the structural, to the strategic.

In the two-problem paper analysis, the pre-service teacher must solve both her college-level problem and her related middle grades problem. A thorough explanation includes any strategies used to approach the problems, gives their context, and identifies the mathematics standards (Texas Essential Knowledge and Skills) that apply. She must explain the mathematical connections between the college-level and middle grade problems, clarifying what common ideas they share. A sample assignment sheet for the two-problem comparison paper and a checklist for students to follow when writing their paper can be found at `www.uta.edu/faculty/tjorgens/middleschoolcourses.html`.

The students often require guidance in selecting problems that are rich enough in content to merit the analysis required for the two-problem comparison paper. Because of this, we recommend that the students be required to submit their proposed problems early in the semester for approval by the instructor.

The problems compared in the two-problem comparison paper can come in many forms. Some examples from recent semesters follow.

- *Precalculus problem:* Graph the function $g(x) = 3(5x-10)^2+4$ by transforming the graph of the parent function $f(x) = x^2$.
  Middle grades problem: Evaluate the expression $3^2 + 6(5 + 4) \div 3 - 7$.
  Primary mathematical connection: The student viewed the problems as structurally connected in that the key to solving both was to understand the proper order of operations, though the order in which to do things in each problem is different.

- *Precalculus problem:* What are the zeros of the function $f(x) = x^3 - 5x^2 + 2x + 8$?
  Middle grades problem: The vending machine charges 75 cents for a can of pop. You have 53 quarters. How many pops can you purchase?
  Primary mathematical connection: The student focused on the algorithmic connection. The solution to the precalculus problem required polynomial division, whereas the middle grades problem required long division of whole numbers.

- *Precalculus problem:* Construct a possible formula for a rational function given its graph.
  Middle grades problem: Find the equation of the line that goes through two points graphed below (graph omitted.)
  Primary mathematical connection: The problems were strategically related in the eyes of the student. That is, a solution requires the ability to read graphical information and relate it to parameters determined by the type of function.

- *Calculus problem:* Of all lines through the point (5,2), find the line that cuts off the triangle of smallest area in the first quadrant.
  Middle grades problem: Sammy knows he is 5 feet tall. At 4 pm, he cast a thirteen foot shadow. He then measured the shadow of a tree in his yard to be 78 feet. What is the height of the tree?
  Primary mathematical connection: The student who compared the problems found multiple strategies for solving each problem. The most straightforward method in each case depended on recognizing and using a similarity relationship. The student emphasized the value of being well-versed in multiple solution techniques and savvy enough to find the best one.

12.4 Concluding Remarks

While there is no template for the ideal middle grades mathematics teacher preparation course, the elements we described provide means with which to use the content of precalculus and calculus to build on the mathematical and pedagogical knowledge that are needed for teaching. After the implementation of the capstone courses at the University of Texas at Arlington, our pre-service teachers have had greater success passing the middle level mathematics Texas content certification exam—first time passing rates have improved from as low as 40% in 2006 to over 95% passing on their first attempt in 2010. More importantly, our future teachers have strengthened their knowledge of mathematics and broadened their understanding of the mathematical connections between educational levels.

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[10] `scimath.unl.edu/MIM/coursematerials.php`, Concepts of Calculus course materials designed for University of Nebraska Math in the Middle Institute.

[11] `www.utdanacenter.org/highered/alg2/`, Teacher Quality Modules, funded by the Texas Higher Education Coordinating Board and the Texas Education Agency.

13

Visual College Algebra for Teachers

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At Western Oregon University, we offer a *College Algebra for Teachers* course using the *Visual Algebra for College Students*[1] [3] materials that we wrote and class tested. *Visual Algebra* is designed to help middle school teachers gain a deep understanding of basic algebraic skills. Students *visually* show algebra using a concrete model (algebra pieces), *verbally* describe the meaning of each algebra piece move, *symbolically* connect the ideas to standard algebraic algorithms and procedures, and *graphically* connect the ideas of the visual models and symbolic work. Overall, students think deeply about topics, do not rely on rote memorization or rote rules, and understand ideas so well that they can easily describe, model, and teach core algebraic ideas to their middle grade students in a variety of ways. This allows our future teachers to meet the needs of the different learning styles in their classrooms.

Visual Algebra takes students through modeling integer operations with black and red tiles to modeling linear and quadratic patterns with tiles and variable algebra pieces, looking at the general forms of the patterns and then connecting all the ideas to symbolic manipulation, creating data sets, graphing, and finding intercepts and points of intersection. Ideas are then extended to higher order polynomial functions and then to modeling complex number operations with black, red, yellow, and green tiles. Throughout the course, students are able to relate what they learned using visual methods to the standard methods and algorithms they will see every day in their classroom. For example, after factoring quadratic equations using visual methods, students learn factoring by grouping and factoring using the "ac"method.

Western Oregon University (WOU) is a bachelor's and master's degree granting public institution serving approximately 6,000 students. Originally a normal school, the College of Education still draws many majors. Each year, the WOU Mathematics Department faculty offers two sections of our *College Algebra for Teachers* course serving 25–40 pre-service K–8 education majors, including future middle school mathematics teachers. All pre-service middle school teachers (both mathematics and non-mathematics) are required to take a college algebra course. Although some students have had a traditional college algebra course, we advise the middle school education majors to take the *College Algebra for Teachers* course. In-service middle school mathematics teachers can choose to take a similar graduate level course that we offer every third summer. A year-long foundational *Mathematics for Elementary Teachers* sequence (number and operation, statistics and probability, geometry) is the prerequisite course work for both the undergraduate (pre-service) and graduate (in-service) course. The pre-service *College Algebra* course serves as prerequisite material

[1] *Visual Algebra* is a component of the *Algebra Through Visual Patterns* grades 6–10 curriculum [2] developed via the non-profit, Oregon based, Math Learning Center (`www.mathlearningcenter.org`). Starting in 2001, Burton revised and extended the materials. During her sabbatical in 2007, she adapted them into the text and supporting materials of *Visual Algebra for College Students* [3].

for our *Calculus for Middle School Teachers* course and as one of ten required mathematics courses for future middle school mathematics teachers at Western Oregon University [1].

13.1 Active Learning Approach

All of our K–8 Mathematics Education courses [1] are taught by mathematics faculty, are hands-on, are based on guided discovery and exploration, and are activity based. Our *College Algebra for Teachers* class meets twice a week with 110-minute sessions. Small groups of three or four students work in an interactive classroom environment. In a typical class, we begin by answering student questions and perhaps give a short quiz (Section 13.4.2, Quizzes). Students then break into small groups and read the beginning of an activity set (Section 13.2.1, Student Materials) as the instructor guides the beginning of the student work by presenting a simple activity for the class to discuss (Section 13.2.2, Instructor Notes). Students then work together through the activity set while the instructor circulates, answering questions (typically with another question), helping students use models and generally prompting student discussion. As the various stages of key ideas are completed, students are asked to share their work and observations with the rest of the class. Throughout the course, students are encouraged to use good group work and sharing strategies. A discussion of the strategies is found at the beginning of the student *Visual Algebra* text [3].

13.2 The Visual Algebra for College Students Materials

All of the *Visual Algebra for College Students* materials are freely available for use. The Math Learning Center has a product webpage (see [3] for the URL) where students and instructors can download a pdf version of the *Visual Algebra for College Students* text, order inexpensive algebra piece manipulatives and read an information note for instructors (which includes reproduction permissions). Instructors can contact Laurie Burton (`www.wou.edu/~burtonl`) directly to obtain complete instructor resource materials.

We have used the *Visual Algebra for College Students* materials multiple times for pre-service and in-service teacher courses. Burton initially designed the materials and Kruczek has tested them and provided suggestions as the materials developed. The collection of materials is classroom ready and can be used in a variety of formats. We use the first three (of four) chapters for a ten-week quarter course; a semester course can be taught by incorporating the fourth chapter. A graduate course can cover all four chapters and expand on the main ideas (see Section 13.3.3 for an example).

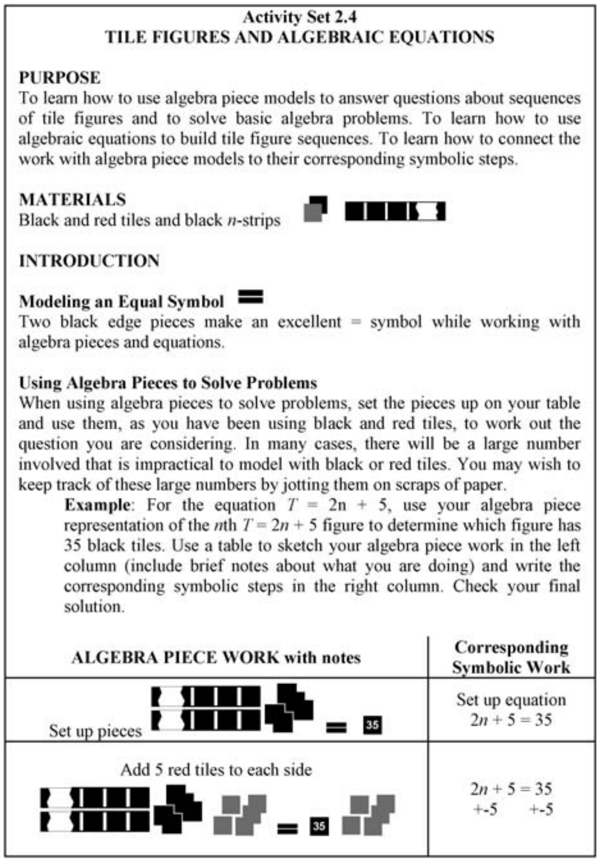
13.2.1 Student Materials

Visual Algebra for College Students, Chapter One, *Integers and Integer Operations*, is introductory. However, learning to work with tile models yields an understanding of black and red tile collections and their properties (Section 13.3.1). Chapter One sets the groundwork for the concepts covered in the subsequent chapters. Chapter Two, *Linear Expressions, Equations and Graphs* introduces students to effectively analyzing and describing linear patterns (Section 13.3.2). Chapter Three, *Real Numbers and Quadratic Functions*, extends the ideas from Chapter Two to quadratic patterns and focuses on symbolic manipulation, graphical display, and their connection with algebra piece models (Section 13.3.3). Chapter Four, *Polynomials and Complex Numbers*, completes the *Visual Algebra* materials (Section 13.3.4).

The text is designed for student discovery. Each chapter consists of a series of *Activity Sets*. Each *Activity Set* contains a paragraph describing its **Purpose** of the *Activity Set*, a list of needed (Student) **Materials**, and an **Introduction** that provides background information, definitions, examples, drawing guides, graphing calculator instructions (see Figure 13.1) and a set of exploration based activities with room for the students to write directly in the text (see Figure 13.2). A few short selected answers to the activity sets are at the back of the text.

Following each *Activity Set* is a homework set. Each chapter ends with a chapter *Review and Vocabulary Guide* and a *Student Practice Chapter Exam*. Answers to the practice exams are in the back of the text.

Student electronic files are also available for download at the Math Learning Center *Visual Algebra* product page [3]. They include graph and grid papers and a text file with embedded algebra piece images.



Activity Set 2.4
TILE FIGURES AND ALGEBRAIC EQUATIONS

PURPOSE
To learn how to use algebra piece models to answer questions about sequences of tile figures and to solve basic algebra problems. To learn how to use algebraic equations to build tile figure sequences. To learn how to connect the work with algebra piece models to their corresponding symbolic steps.

MATERIALS
Black and red tiles and black n-strips

INTRODUCTION

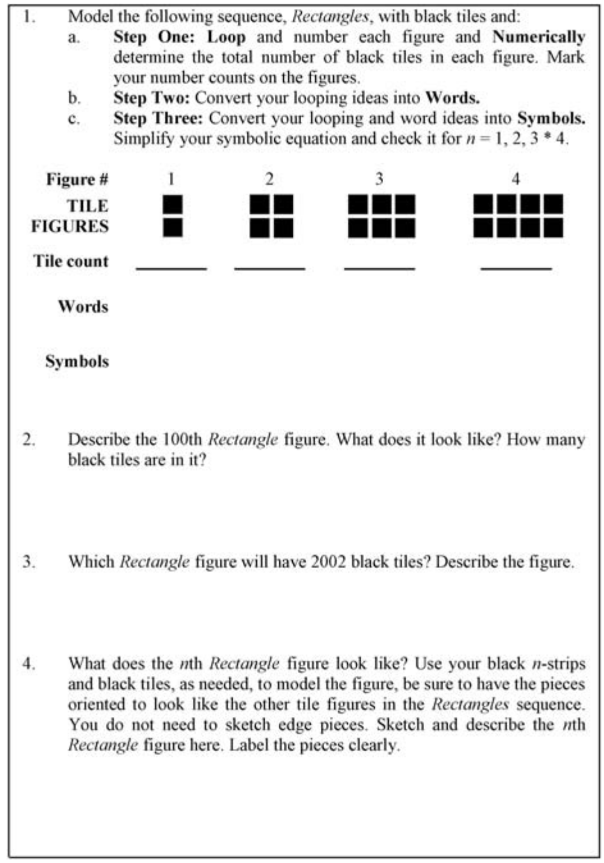
Modeling an Equal Symbol
Two black edge pieces make an excellent = symbol while working with algebra pieces and equations.

Using Algebra Pieces to Solve Problems
When using algebra pieces to solve problems, set the pieces up on your table and use them, as you have been using black and red tiles, to work out the question you are considering. In many cases, there will be a large number involved that is impractical to model with black or red tiles. You may wish to keep track of these large numbers by jotting them on scraps of paper.

Example: For the equation $T = 2n + 5$, use your algebra piece representation of the nth $T = 2n + 5$ figure to determine which figure has 35 black tiles. Use a table to sketch your algebra piece work in the left column (include brief notes about what you are doing) and write the corresponding symbolic steps in the right column. Check your final solution.

ALGEBRA PIECE WORK with notes	Corresponding Symbolic Work
Set up pieces	Set up equation $2n + 5 = 35$
Add 5 red tiles to each side	$2n + 5 = 35$ $+{-5} \quad +{-5}$

Figure 13.1. Beginning of an Activity Set (Red tiles are displayed as gray tiles.)



1. Model the following sequence, *Rectangles*, with black tiles and:
   a. **Step One: Loop** and number each figure and **Numerically** determine the total number of black tiles in each figure. Mark your number counts on the figures.
   b. **Step Two:** Convert your looping ideas into **Words.**
   c. **Step Three:** Convert your looping and word ideas into **Symbols.** Simplify your symbolic equation and check it for $n = 1, 2, 3 * 4$.

Figure #	1	2	3	4
TILE FIGURES				
Tile count	____	____	____	____
Words				
Symbols				

2. Describe the 100th *Rectangle* figure. What does it look like? How many black tiles are in it?

3. Which *Rectangle* figure will have 2002 black tiles? Describe the figure.

4. What does the nth *Rectangle* figure look like? Use your black n-strips and black tiles, as needed, to model the figure, be sure to have the pieces oriented to look like the other tile figures in the *Rectangles* sequence. You do not need to sketch edge pieces. Sketch and describe the nth *Rectangle* figure here. Label the pieces clearly.

Figure 13.2. Sample of student write-in section of Activity Set

13.2.2 Instructor Materials

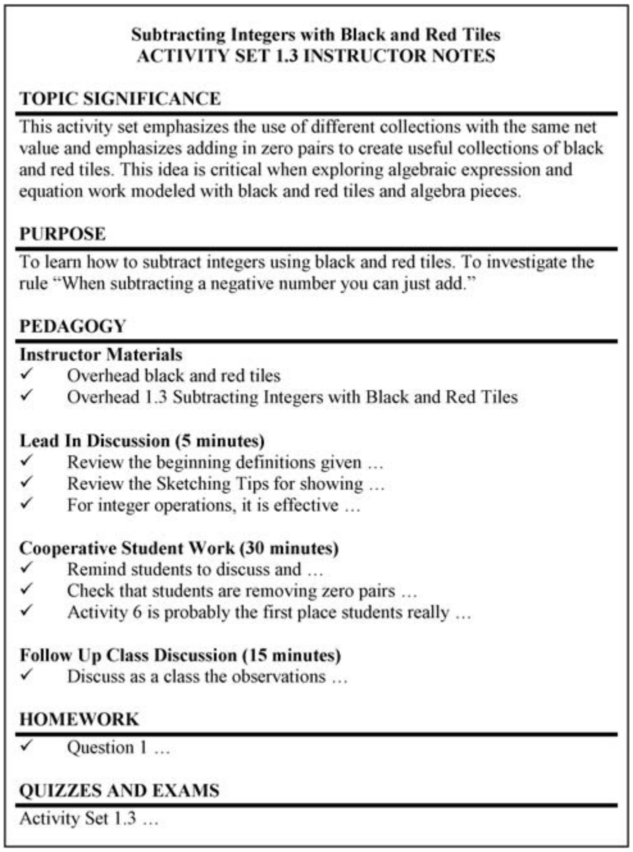
The Instructor Resource Manual (IRM) has activity and homework solutions. Each Activity Set in the student text has a set of Instructor Notes in the IRM. The Instructor Notes provide a paragraph on Topic Significance (relative to the rest of the materials), the Purpose from the student Activity Set, a list of Instructor Materials, a Lead In Activity with ideas and comments to set the stage for the Activity Set, pedagogical notes on Cooperative Student Work that gives information about working with the students as they work on the Activity Set, ideas for a Follow Up Class Discussion to wrap up the Activity Set work and comments on the Homework questions with assignment suggestions (see Figure 13.3). Most sets of Instructor Notes have overhead masters to use during the various stages of the class (see Figure 13.4).

Each chapter of the IRM contains an Instructor Chapter Exam (available in .doc form) and Instructor Chapter Exam solutions.

13.3 Sample Activities

13.3.1 Integer Operations

Visual Algebra for College Students begins with integer operations using a black and red tile model. The black tile has value $^{+}1$ and the red tile has value $^{-}1$. Students quickly see that different collections of tiles can have the same net value, and thus, collections can be made to accomplish a task. For example, students may not understand the integer subtraction rule, "when you subtract a negative, you just add." However, they quickly see how a well chosen tile collection can illuminate this idea for their students. As shown in the following example, students see by adding five matching sets of black and red tiles (zero pairs), after taking away the five red tiles, they have essentially added five black tiles to their original collection of three tiles. See Figure 13.5.



Subtracting Integers with Black and Red Tiles
ACTIVITY SET 1.3 INSTRUCTOR NOTES

TOPIC SIGNIFICANCE

This activity set emphasizes the use of different collections with the same net value and emphasizes adding in zero pairs to create useful collections of black and red tiles. This idea is critical when exploring algebraic expression and equation work modeled with black and red tiles and algebra pieces.

PURPOSE

To learn how to subtract integers using black and red tiles. To investigate the rule "When subtracting a negative number you can just add."

PEDAGOGY

Instructor Materials
- ✓ Overhead black and red tiles
- ✓ Overhead 1.3 Subtracting Integers with Black and Red Tiles

Lead In Discussion (5 minutes)
- ✓ Review the beginning definitions given …
- ✓ Review the Sketching Tips for showing …
- ✓ For integer operations, it is effective …

Cooperative Student Work (30 minutes)
- ✓ Remind students to discuss and …
- ✓ Check that students are removing zero pairs …
- ✓ Activity 6 is probably the first place students really …

Follow Up Class Discussion (15 minutes)
- ✓ Discuss as a class the observations …

HOMEWORK
- ✓ Question 1 …

QUIZZES AND EXAMS

Activity Set 1.3 …

Figure 13.3. Instructor Notes Outline

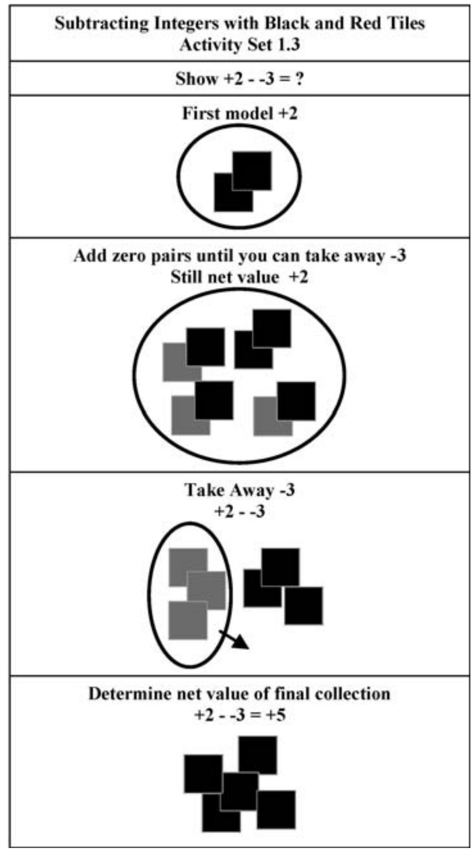


Figure 13.4. Instructor Overhead

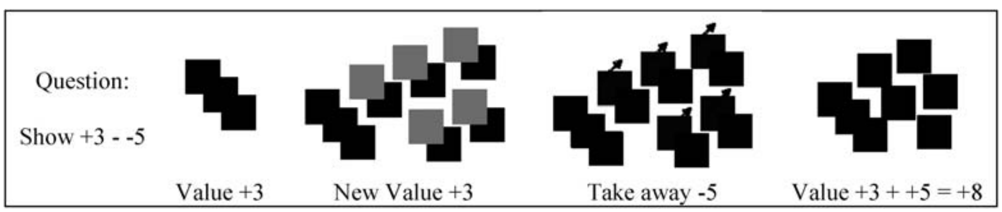


Figure 13.5. Subtraction example

13.3.2 Linear Expressions and Equations

Students start to use variable algebra pieces in Chapter Two, *Linear Expressions, Equations and Graphs*. To begin, students learn about the dimensions of black algebra n-strips, which represent the variable n, as in Figure 13.6. In this chapter, students work only with whole numbers. Thus a black strip represents an always positive variable number (n) and a red strip represents an always negative variable number ($-n$).

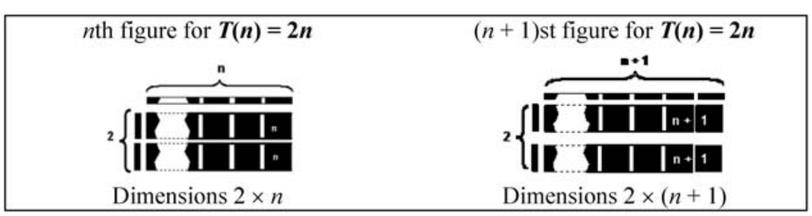


Figure 13.6. n-strips with dimensions

In Chapter Two, students focus on analyzing patterns. The example in Figure 13.7 shows a typical student analysis of a tile sequence that includes numerical, verbal, and symbolic work.

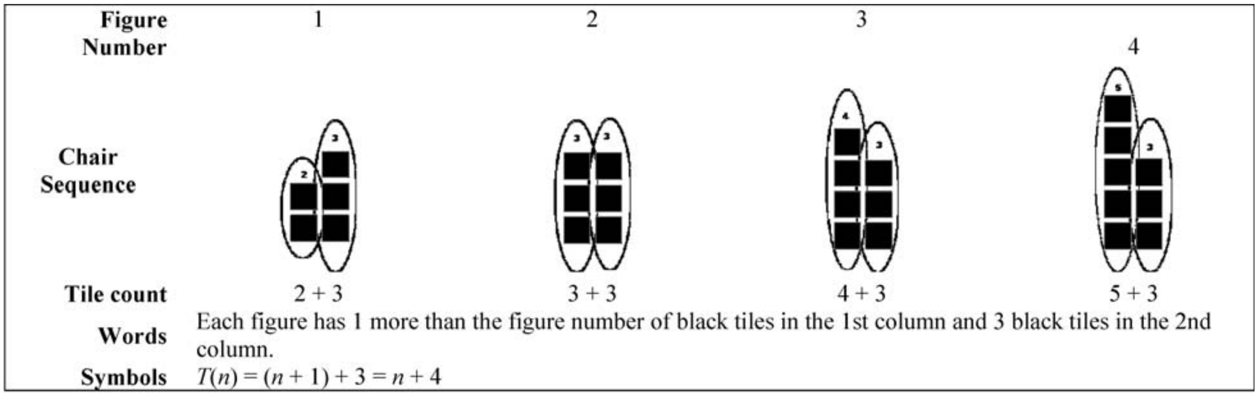


Figure 13.7. Analysis of a tile sequence

Once students have analyzed a tile sequence, they can easily answer a series of activity questions such as

Activity: Describe the 100th *Chair* figure. What does it look like? How many black tiles are in it?

Student: A column of 101 black tiles followed by a column of 3 black tiles for a total 104 black tiles.

Activity: Which *Chair* figure will have 2002 black tiles? Describe the figure.

Student: 2002 − 3 is 1999, 1999 is 1 more than 1998. It is the 1998th figure with 1999 black tiles in the first column and 3 black tiles in the second column.

Activity: What do the nth and the $(n + 1)$st *Chair* figures look like?

Student: nth chair—column of 1 n-strip and 1 black tile, then a column with 3 black tiles.
$(n + 1)$st chair—column of 1 n-strip and 2 black tiles, then a column with 3 black tiles. Only the n-strip grows for the $(n + 1)$st figure (See Figure 13.8).

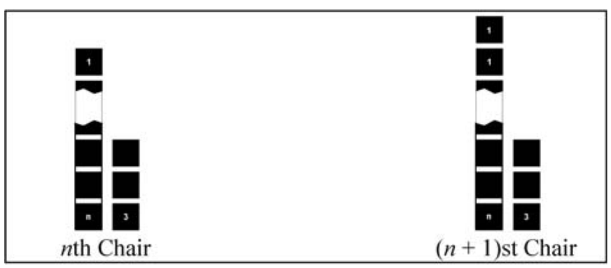


Figure 13.8. Algebra piece models of nth and $(n + 1)$st figures

Activity: If 17 black tiles are added to a certain *Chair* figure, there will be a total of 40 black tiles. Use your algebra piece representation of the nth *Chair* to help determine which *Chair* figure this is. Use the table and sketch your algebra piece work in the left column (include brief notes about what you are doing) and write the corresponding symbolic steps in the right column. Check your final solution.

Student: See Figure 13.9 for a sample student answer. Note students use scraps of paper to represent large numbers of tiles.

Due to the limitations of what can be modeled using algebra pieces, students initially focus on using algebra pieces and symbolic work to solve one-variable linear equations with integer coefficients and constants. After students become comfortable with this category of linear equations, they determine the steps needed to solve all one-variable linear equations, including those with rational coefficients and constants. Most students are familiar with the rule "put variables on one side and constants on the other," but when the equation contains fractions, getting the variables to one side can be difficult for some students. We discuss methods to handle this situation.

13.3.3 Quadratic Expressions and Equations

In Chapter Three, *Real Numbers and Quadratic Functions*, students get to the heart of the *Visual Algebra* materials. To use real numbers, students use white variable strips: x, which can be either positive or negative, depending on the value

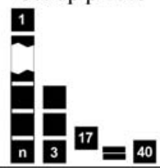
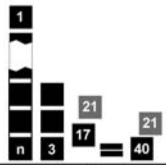
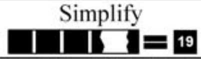
ALGEBRA PIECE WORK with notes	**SYMBOLIC WORK**
Set up pieces 	Set up equation $n + 1 + 3 + 17 = 40$
Add 21 red tiles to each side 	$n + 21 = 40$ $+\text{-}21 \quad +\text{-}21$
Simplify 	$n = 19$
Check: $n + 1 + 3 + 17 = 19 + 1 + 3 + 17 = 40\surd$	

Figure 13.9. Connecting algebra piece and symbolic work

of x, and opposite white variable strips: ^{-}x, which can be either positive or negative, depending on the value of x. Black variable squares (x^2) and red variable squares ($^{-}x^2$) are used to represent quadratic expressions and equations. The pieces are shown in Figure 13.10.

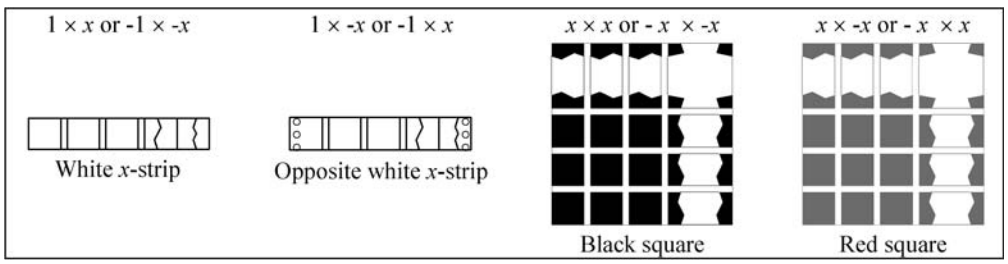


Figure 13.10. White and opposite white strips and black and red variable squares

The following activities and analysis of sample student responses show an analysis of a sequence of tile figures corresponding to a quadratic function (Figure 13.11) and then connects all the concepts numerically, verbally, symbolically, and graphically.

Activity: Analyze the extended sequence in Figure 13.11, give the symbolic formula for the sequence simplified into a $y = ax^2 + bx + c$ form and sketch the xth figure.

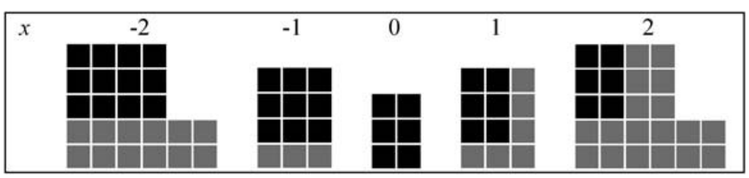


Figure 13.11. Sample extended tile sequence

Student: To analyze the sequence (see Figure 13.12), students notice there are a constant six black tiles in each figure representing a symbolic constant term of $c = 6$. This also means six black tiles for the xth figure. Next, the student notices the values of the tiles directly to the right of the six black tiles are 2×3, 1×3, 0×3, $^{-}1 \times 3$ and $^{-}2 \times 3$ for $x = -2, -1, 0, 1,$ and 2. The linear growth in the pattern is represented by the three opposite white x-strips in the xth figure and $b = -3$ in the symbolic form $y = ax^2 + bx + c$.

Students consider the remaining red tiles in each of the figures and see that they grow in two directions, which indicates quadratic growth. They can see the red tiles can be grouped into three sets of red square arrays, each of whose dimensions match the squares of the figure numbers $^{-}2 \times ^{-}2$, $^{-}1 \times ^{-}1$, 0×0, 1×1, and 2×2. Since the tiles are all red, with negative net value, the three sets of red tiles in each figure are modeled with red variable

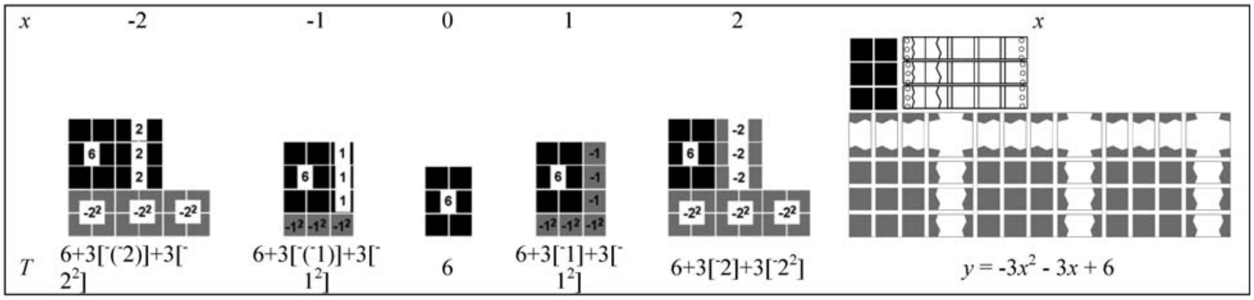


Figure 13.12. Numerical analysis of a tile sequence

squares. The xth figure therefore contains three red x-squares for the quadratic component and $a = -3$, in the symbolic form. Students finish with the symbolic form: $y = -3x^2 - 3x + 6$ and the general algebra piece figure shown in the last column of Figure 13.12.

Activity: For the quadratic function modeled in the tile sequence in Figure 13.11, find each of the following (if they exist): y-intercept, x-intercepts, turning point, range, factored form (algebraically and with algebra pieces). Sketch your models and the graph of the quadratic function and show your work.

Student: The y-intercept is the constant term already discovered in the tile analysis and the remaining items are found by first creating a factor rectangle, determining the edge sets and using them to determine the factored form of the quadratic.

To find the factor rectangle, students start with the six black tiles. They can be arranged in a 2×3 or 1×6 . Laying one opposite white x-strip above each of the red variable squares uses the three available opposite white x-strips. To create a rectangle while keeping the net value of the polynomial the same, an equal number of white x-strips and opposite white x-strips need to be added. Students easily sees that a 1×6 array of black tiles will not work because to match the red square edges and the three opposite white x-strips, they would need to add six white x-strips, not a set of zero pairs. The failed attempt is shown in Figure 13.13. A similar analysis shows a 6×1 array does not work either.

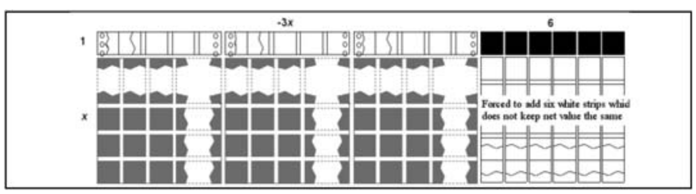


Figure 13.13. The 1×6 array arrangement of black tiles does not work

Turning their attention to the 2×3 array arrangement of the black tiles, students add zero pairs of white and opposite white x-strips until they create the following factor rectangle. They notice each region has the correct edge pieces:

1. The six opposite white x-strips have edges 2 and ^-3x ($2 \times ^- 3x = ^- 6x$).
2. The three red x-squares have edges of x and ^-3x ($x \times ^- 3x = ^- 3x^2$).
3. The six black tiles have edges of 2 and 3 ($2 \times 3 = 6$).
4. The three white x-strips have edges of x and 3 ($x \times 3 = ^- 3x$).

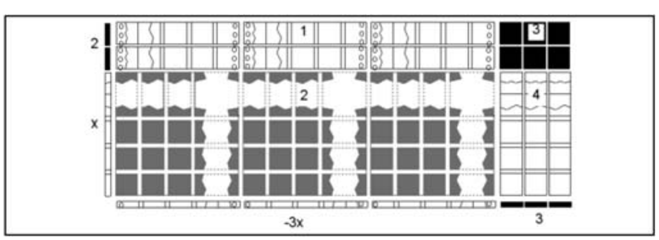


Figure 13.14. The 2×3 array arrangement of black tiles does work

For this correct rectangle, students lay in edge pieces to measure the dimensions of the rectangle and find the factored form of their quadratic: $y = ({}^{-}3x + 3)(x + 2) = {}^{-}3(x - 1)(x + 2)$.

Students then complete their work by filling out the table in Figure 13.15.

y-intercept	Factored Form	x-intercepts	Turning Point	Range
$(0, 6)$ Black tiles in xth figure.	$y = (-3x + 3)(x + 2) = -3(x - 1)(x + 2)$ Edge set dimensions in factor rectangle	$(1, 0)$ and $(-2, 0)$ Set edge sets $= 0$	$(-.5, 6.75)$ Halfway between x-intercepts	$y \le 6.75$ Below TP

Figure 13.15. Final student solutions

For an example of expanding on a topic for a graduate course; in-service teachers are typically able to gain deeper understanding of the big picture of factoring than pre-service teachers. While factoring a quadratic polynomial $y = ax^2 + bx + c$ with algebra pieces, they realize that if a and c have the same sign, additional algebra pieces do not need to be added to factor the polynomial visually (by creating a factor rectangle). If a and c have opposite sign, then they need to add an equal number of white variable strips (x) and opposite white variable strips (${}^{-}x$) in order to factor the polynomial with the algebra pieces.

The next section, *Multiplying and Factoring Polynomials*, discusses how to multiply polynomials and factor quadratic polynomials in one variable without using algebra pieces. Although many students are familiar with factoring, most have learned only how to factor using a guess-and-check method. We discuss the foolproof ac method, a variant of factoring $y = ax^2 + bx + c$ (with integer coefficients) by grouping. In the ac method, one tries to find two integers p and q whose product is ac and whose sum is b. Using the ac method should reinforce what was learned when factoring using algebra pieces. The next example illustrates this.

We showed how a student used algebra pieces to factor $y = -3x^2 - 3x + 6$. As seen in Figure 13.14, the student had modeled $y = -3x^2 - 6x + 3x + 6$. In the ac method, this would mean $p = -6$ and $q = +3$ since we are looking for two integers whose product is $ac = (-3)(6)$ and whose sum is $b = -3$. Using factoring by grouping, we get $y = -3x^2 - 6x + 3x + 6 = -3x(x + 2) + 3(x + 2) = (-3x + 3)(x + 2)$, which matches what was shown before.

13.3.4 Word Problems

At the conclusion of Chapter Three, in the section *Solving Word Problems*, students are asked to apply what they have learned to solve word problems. The word problems cover linear equations in one and two variables, and problems involving quadratic equations. This section was not initially in the course materials, but we felt a section on solving word problems was important so that students can develop skills connecting problem solving and algebraic thinking in later courses. We have included a few sample problems from the section.

Before attempting the next problem, we review with students how to set up and solve "distance $=$ rate $\times$ time" problems.

> Portland, OR and Guilford, CT are approximately 2975 miles apart. On November 30, I start driving from Portland at 7 A.M. PDT to meet my father, who began driving from Guilford at 7 A.M. EDT. Including stops for gas, food, and other breaks, on average, I drive 70 mph and my father drives 60 mph. At what time (PDT) will we meet each other if we drive straight through?

The next problem requires students to solve a system of linear equations.

> Before a 25% markdown, two shirts and one pair of shorts cost $60 at SuperMart. After a 25% markdown, four shirts and three pairs of shorts cost $108 at SuperMart. What is the regular price of a shirt at SuperMart? What is the regular price of a pair of shorts at SuperMart?

In this next example, students must use the fact that the extremum of a quadratic function occurs at the vertex, which they learned earlier.

> A sixth-grade class decides to enclose a rectangular garden using the side of the school as one side of the rectangle. The class has 28 feet of fencing available.
>
> a. If L is the length of the fencing parallel to the school, what is the width of the garden (in terms of L)?
> b. What is the area of the garden, in terms of L?

c. What should the dimensions of the garden be to maximize the area of the garden?
d. What is the maximum garden area that the class can enclose?

13.3.5 Complex Number Operations

Visual Algebra for College Students concludes with two activity sets working with complex numbers; operations and connecting these ideas to the complex roots of polynomials. The core set of black $(+1)$ and red $({}^{-}1)$ tiles is extended to include a green tile (i) and a yellow tile $({}^{-}i)$. Students quickly adapt to representations of complex numbers as collections of black, red, green, and yellow tiles and find addition and subtraction straightforward as they extend the idea of a black and red zero pair to a green and yellow zero pair. The next example shows a student summary of the multiplicative relationship between the four colored tiles (array edges to be precise) and the corresponding symbolic meaning.

Activity: Summarize the multiplicative and color relationships between 1, -1, i and $-i$ by filling out the provided table.

Student: See Figure 13.16 for a sample student response.

×	**B**	**R**	**G**	**Y**
B	B	R	G	Y
R	R	B	Y	G
G	G	Y	R	B
Y	Y	G	B	R

×	**1**	**−1**	***i***	**−*i***
1	1	-1	i	$-i$
−1	-1	1	$-i$	i
i	i	$-i$	-1	1
−*i*	$-i$	i	1	-1

Figure 13.16. Black, Red, Green, Yellow tiles and complex numbers

This example shows an application of these ideas by using a tile array model for complex number multiplication.

Activity: Use black, red, green, and yellow tiles and edge pieces to model $(2+3i)\times(1-2i)$. Simplify your solution to an $(a+bi)$ form.

Student: See Figure 13.17 for a sample student response.

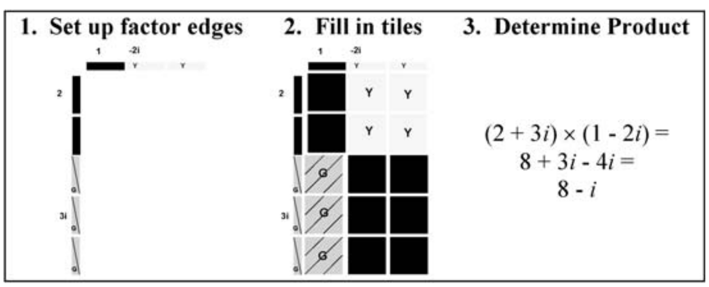


Figure 13.17. Array model for complex number multiplication

13.4 Assessment of Students

The typical assessment tools (homework, quizzes, and exams) are used in this course, and there is an oral final exam. Previously, students were asked to create a lesson plan incorporating some of the concepts used in this course, but we found student work was typically too similar to the text.

13.4.1 Homework

We assign and grade almost all the homework. The homework sets are thorough but do not repeat concepts. Each set consists of questions that have students apply concepts and generalize concepts from the section. Students are assessed on the accuracy and explanation of their work. For *College Algebra for Teachers*, effective student solutions include appropriate sketches, are focused on conceptual understanding instead of rote rule use, and are often written so that an upper elementary or middle school student can understand the explanation.

13.4.2 Quizzes and Written Exams

Regular, short written quizzes are effective while using these materials. Students typically work in groups in class, and many continue this practice by forming study teams to work on homework. Depending on the course structure, a one question quiz to start each day, or a two or three question quiz each week may be best.

We prefer to use written chapter exams in our courses. To encourage cooperative learning, it is possible to allow groups to discuss the exam (usually without writing) for five minutes before students individually write out their exam solutions.

13.4.3 Oral Final Exam

We have given oral final exams and find a 20 to 25 minute session with each student (with a five minute break between students) is ideal. We have a list of questions ready to ask and a sorted set of algebra pieces ready to use. For pattern and sequence analysis, we have pages for each student to mark on. We have students sketch graphs on a board or on a piece of graph paper. Student understanding typically builds while studying *Visual Algebra* and a cumulative oral final exam is an ideal way to discuss course concepts with students and to truly see their level of understanding. It is sometimes surprising. Students who have struggled with written homework will suddenly have an *Aha!* moment towards the end of the course as they finally put together all of the course concepts. Such students often excel in an oral exam environment where they can explain what they know. For every student, it is a chance to practice explaining mathematics and is generally a new and interesting experience. We keep a set of notes for each student during the exam, with scores for each question. Although scheduling the exams is time consuming, the final grading is usually quick as we summarize and record the interview notes.

13.5 Assessment of Course

The materials for *College Algebra for Teachers* have been modified over the years reflecting student and faculty comments. The main change has been the addition of sections on solving linear equations and factoring quadratic polynomials without the use of algebra pieces. In the first version of the materials, students almost exclusively used visual methods. Because we realized that students were not making the connection to the standard algorithms and techniques used in the middle school classroom, we added these connections. Recently, we added the section on solving word problems at the students request, as they realized they were still unable to set up the appropriate equations to solve word problems even though they were able to work with linear and quadratic functions and equations.

13.6 Concluding Remarks

Upper elementary and middle school students are frequently told a variety of rules in mathematics with no explanation for why these rules are true. These rules include “when you subtract a negative number, it is like adding a positive,” “the product of two negative numbers is a positive number,” and $(a + b)^2 \neq a^2 + b^2$. Frequently, the teachers themselves cannot give a clear explanation for why they are true. Modeling them with algebra pieces allows our pre-service and in-service teachers to understand core curriculum ideas and constructions and gives them the resources to explain and model with ease these basic operational and algebraic concepts.

Topics covered in our *College Algebra for Teachers* course are not as extensive as in a traditional college algebra syllabus, but the student depth of understanding of the materials covered, ideas our students will actually be teaching, is impressive. A frequent comment heard in our classes is, “I finally understand how this really works.” We have always found that the *Visual Algebra* students who take our next course, *Calculus Concepts for Middle School Teachers*, show an equal if not better mastery of core college algebra skills than students from a traditional college algebra course. They uniformly have better memories about their college algebra experience. It is clear the concrete experiences from *College Algebra for Teachers* have settled into a solid ability to explore, explain, and abstract. These pedagogical skills, the ability to explain concepts clearly and simply, and the algebraic concept mastery gained in *College Algebra for Teachers* makes these current and future teachers more prepared for effectively teaching and working with a variety of learning styles in their future classrooms.

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Tile Patterns & Graphing
Positive & Negative Numbers
Integer Addition & Subtraction
Integer Multiplication & Division
Counting Piece Patterns & Graphs
Modeling Algebraic Expressions
Seeing & Solving Equations
Extending Counting Piece Patterns
Squares & Square Roots
Linear & Quadratic Equations
Complete Sequences
Sketching Solutions
Analyzing Graphs
Complex Numbers

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II

Courses for Middle School Mathematics Teachers

E. Probability and Statistics

14

Probability and Statistics for Prospective Middle Grades Teachers

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University of Wisconsin Oshkosh — *University of North Carolina at Greensboro*

14.1 Introduction

> No reform of mathematics education is possible unless it begins with revitalization of undergraduate mathematics in both curriculum and teaching style. —National Research Council, [10]

The University of Wisconsin Oshkosh (UWO) is a comprehensive state university with approximately 11,000 undergraduate and 1000 graduate students. Elementary education is one of the largest majors at UWO, serving almost 1000 students. All elementary education majors at UWO (with the exception of special education majors) are required to complete three 3-credit mathematics content courses: *Number Systems*, *Geometry and Measurement*, and *Data Exploration and Analysis*. We prepare middle school teachers from this population by providing a middle school mathematics minor. Currently (fall 2010), 86 students are enrolled in the minor, more than one third of the Mathematics Department's total major and minor enrollment. Students in it take an additional three of four mathematics content electives and a capstone *Senior Seminar*, for a total of 24 credit hours of mathematics. Each course is offered approximately once a year, with an enrollment of 20 to 30 students. Detailed information about the middle school mathematics minor at UWO is presented in our other article in this volume.

The courses in the minor are designed for future middle grades teachers and elementary mathematics specialists. The four content courses are activity-based and encourage students to develop deep understandings of middle school mathematics, the mathematical thinking of children, and on practicing and developing mathematically powerful ways of thinking. Another of our objectives is for our students to become confident in tackling large problems and in assessing the quality of their arguments independently of their instructor. During the past decade we have written our own materials for each of these content courses based on our constructivist philosophy of learning, the research literature, and our work with future middle grades teachers. Our texts are titled *Big Ideas in Algebra* [3], *Big Ideas in Geometry* [4], *Big Ideas in Infinite Processes* [5], and *Big Ideas in Probability and Statistics* [6].

To prepare ourselves to write the texts, we studied three curriculum projects for middle school students, *Mathematics in Context* [7], *Connected Mathematics* [1], and *MathScape* [8]. All of these are activity-based and *National Council of Teachers of Mathematics (NCTM) Standards*-based curricula [9], so the middle school materials were written based on problem solving and understanding of concrete experiences. We believe such materials are better than the traditional materials to encourage and support the types of behaviors and thinking that mathematicians value. Furthermore, research suggests that schools that adopt *Standards*-based materials for more than two years show significantly higher test scores on even traditional measures of mathematical understanding than matched schools adopting traditional curricula ([11], [12], and [2]). We made certain that the ideas our prospective middle grades teachers encounter in the texts are connected to the mathematics curriculum of their future students, and we intended that the courses make the connections apparent.

In our courses, major topics are introduced with a *Class Activity* from the text—a problem whose solutions illuminate the important mathematics. Students are expected to work collaboratively and to develop the mathematics through class discussions moderated by the instructor. Each class activity is followed by a *Read and Study* section, meant to be read with pencil in hand, introducing, formalizing, and extending the mathematics generated by the class activity. The mathematical topic is then tied to the middle school curriculum in a *Connections* section. In many cases, pages from middle school curriculum materials are included. The final section is *Homework*, in which students tackle new problems, and not simply exercises.

In this paper we will describe Probability and Statistics, one of the four content courses we have designed for the middle school minor, and we will provide samples from materials we created and use in this course. Probability and Statistics focuses on an area that has received attention in the middle grades during the past decade as statistics has become important for understanding politics, science, and economics. In fact, NCTM (2000) [9, p. 48] asserts, “Students need to know about data analysis and related aspects of probability in order to reason statistically—skills necessary to becoming informed citizens and intelligent consumers.” Students learn the vocabulary and foundations of probability, data analysis, counting, and statistical inference. History is included throughout the course and real data is used whenever possible. A list of course topics for *Big Ideas in Probability and Statistics* (see Appendix A) shows the variety of content students encounter in the course. A syllabus for the course is in Appendix B.

We have organized this paper in sections that correspond to the four components of the text, illustrating each with examples and explaining how we would incorporate the component into the classroom experience. For our illustrations, we will focus on the development of the measures of center.

14.2 Class Activities

The *Class Activities* are intended to motivate new topics. They must have a rich mathematical content, but be accessible before a formal introduction to the ideas. The problems should be open to multiple problem-solving strategies. We begin each week with a new class activity. Students work in small groups toward a solution. We expect the work and the discussions to require a one-hour class period. Consider the following *Class Activity*, titled “The Best Answer” from [6]:

> Materials needed: a large bag (about 2 lbs.) of peanut M&Ms;
> a glass jar or clear plastic container.
>
> Empty the M&M’s into the jar. Then everyone should, individually, estimate the number of M&M’s in the jar. After the estimates have been made, write them on the chalkboard. Then, together with your group members, use the list of estimates to make a better estimate. Afterwards, each group should present and explain its solution to the rest of the class. What strengths and drawbacks can you find with the various methods? Ultimately, the M&M’s should be counted. Maybe the group whose answer was closest will get to eat the M&M’s?

There are many reasonable approaches to this problem. Both the mean and the median of the students’ individual estimates will probably be close to the true value, although the individual estimates may have a wide range. At this point in the course (about the fourth week), the terms “mean,” “median,” and “mode” have not yet been introduced; they will be defined in the *Read and Study* section that follows. However, most students are acquainted with the terms and during their group work, some will probably decide to compute them. They will also suggest other ideas, such as working only with the data generated within their group, or removing the highest and lowest values from the class data.

After allowing the students to work within their groups for twenty to thirty minutes to generate ideas, we ask them to rearrange their seats into a semicircle for a whole class discussion. Out of that discussion arises the need for notation and definitions (statistic, parameter, outlier), and it is our job as instructors to provide normative language. We also would raise natural follow-up questions if they had not already. How close does an estimate have to be to be considered “good”? How much would the groups’ estimates be affected if one or two students had made bad initial guesses? Would the methods work for plain M&Ms? What about for guessing the number of molecules of air in the jar? The value and depth of the class discussions will depend heavily on the skill and preparation of the instructor. We have attempted to help the instructor by providing an instructor’s guide that indicates our objectives and provides a line of questioning for each class activity.

14.3 Read and Study

In the text, each *Class Activity* is followed by a *Read and Study* section that is usually short—typically two pages—and is meant to be read slowly and carefully, with pencil in hand. In *Read and Study*, we discuss the mathematics elicited by the class activity, and make and explain the relevant definitions. The sections do not include worked examples. Instead, we ask that the students work and verify any examples they encounter. Questions for the students to answer and problems to work on within the section are written in italics, and this convention is explained during the first *Read and Study* section of each text:

> Hey, did you notice the italics? That is your signal that the question, no matter how stupid, is not rhetorical and that you should try to answer it. We mathematicians read our books with pencil in hand. We answer questions and verify anything authors claim to be true. (After all, who knows if they really know what they are talking about?) Start doing this too. The italics will help you remember to slow down and think while you read.

A purpose of the readings beyond developing specific mathematics content is to convey that the creation of mathematics is a human endeavor. Mathematical objects and processes were developed with specific purposes in mind. When possible, we try to describe their history. Here is an excerpt from the *Read and Study* section:

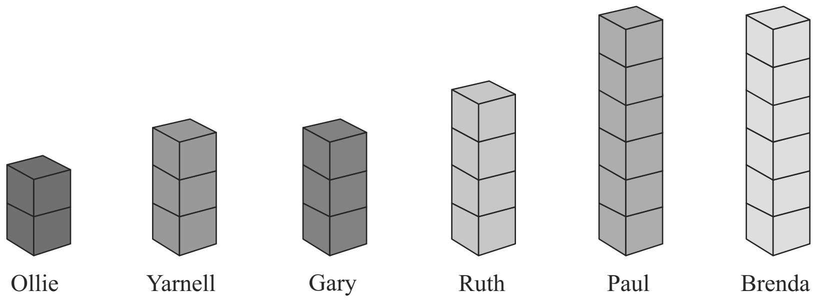
> Any measurement is subject to uncontrollable sources of error. Recognizing that even his own meticulous measurements fluctuated from one reading to the next, the astronomer Tycho Brahe was, in the late 1500’s, perhaps the first scientist to combine *repeated* measurements of the same quantity in order to produce a single reliable value. (You may recall from *Section 1* that Tycho’s student Johannes Kepler used his data to develop a mathematical description of planetary orbits.)
>
> Since that time, scientists have used a variety of methods to boil down a collection of data to a single value. As it was with Tycho, sometimes the goal is to identify a “best” value from a set of repeated measurements. Other times the goal is to use a single value to represent some aspect of a population of different things. (For example, the *World Book Encyclopedia* states, “The average weight of a baby born at term is 7 1/2 pounds.” *Are they referring to a particular baby?*)
>
> Two representative values are used in modern statistics: the *mean*, defined as the *average* of all the values; and the *median*, defined as the *middle* value upon listing the data in order from smallest to largest. (The median can be chosen as any value that is at least as big as at least half of the data, and at least as small as at least half of the data. *Can you think of an example of a data set for which there is more than one median?*) Another representative value used occasionally is the *mode*, defined as the most frequently appearing value(s).
>
> Which of these is better to use depends on what questions you are considering. The mode, for instance, can be used for categorical data, but the mean and the median do not make sense in that context. *Why? If you wanted to know how much money you would make working at Microsoft Corporation, would it be more accurate to look at the mean income of its employees, or the median income? Explain.*

Students are expected to complete the *Read and Study* sections at home immediately following the *Class Activities*. In the next class period, we discuss the reading. Active participation is required and is built into our grading policy, along with class attendance.

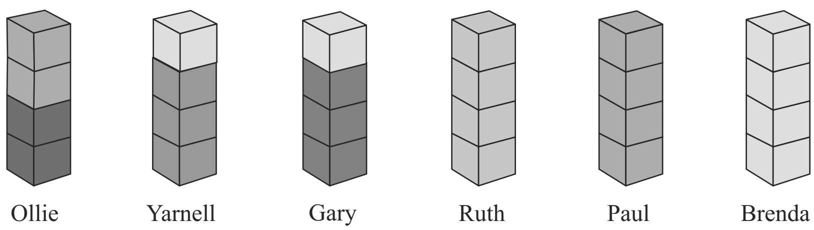
14.4 Connections to the Middle Grades

The *Connections* sections of the materials are designed to show students how the ideas they are studying are treated in the middle grades curriculum. The sections often include pages from children’s texts, and are followed by questions or homework that asks students to relate the content to ideas in the *Read and Study* section. For example, in the *Connections* section that follows the *Class Activity* “The Best Answer” (the M&M estimation problem), students are asked to conceptualize the mean as the number each group would have guessed if the guesses were shared.

> The *idea* of mean can be taught as “evening out” the data. Here is an explanation from *Connected Mathematics* in their text for Grade 6 titled *Data About Us*. The children have built towers to represent the sizes of their households and are now asked to find the average household size. *Study their explanation*.
>
> First, the students put the towers in order.



> The students then moved cubes from one tower to another, making some households bigger than they actually were and making other households smaller than they actually were. When they were finished moving cubes, their towers looked like this.



> Each tower now had four cubes. Notice that the total number of cubes did not change.

Before

Ollie	2 people
Yarnell	3 people
Gary	3 people
Ruth	4 people
Paul	6 people
Brenda	6 people
Total	24 people

After

Ollie	4 people
Yarnell	4 people
Gary	4 people
Ruth	4 people
Paul	4 people
Brenda	4 people
Total	24 people

> The students determined that the average number of people in a household was 4. The teacher explained that the average the students had found is called the **mean**. The mean number of people in the six households is 4.

Our classroom treatment of *Connections* is similar to that of *Read and Study*. Students read the section at home, work through the problems, and come to class prepared for a discussion of how the mathematics of our course is incorporated into the middle school curriculum. It is important for students to have these conversations in class because it helps them to appreciate the course content as relevant to their future work, and it strengthens their understanding of the underlying mathematics.

14.5 Homework

Homework problems (usually five to ten of them) are assigned on completion of the readings for each section. Our classes meet four days per week, one hour each day. This gives us enough time in class to allow students to engage in discussions of the problems with their classmates, after having worked on them at home. We assess homework in a variety of ways, including collecting written solutions or putting related questions on exams. Many problems are substantial enough to serve as topics for student presentations to the class.

The homework problems are never routine exercises. Our students quickly learn that they will not be able to solve the problems by mimicking a solution from their readings. In some cases the problems are designed to expand upon the treatment of a topic from the *Read and Study* section or from the *Connections* section. The first example below follows up on a discussion initiated in *Connections*; notice that the heights of the towers of Xs do not have the same meaning as the heights of the towers of cubes from the reading. This problem forces students to see the difference and to confront a possible misconception.

> Adapt the idea of “evening out” from the Connections section to find the mean of the data showing the number of pets in 16 households. Then explain how you did it.

	X	X						
X	X	X						
X	X	X		X				
X	X	X		X	X	X		X
0	1	2	3	4	5	6	7	8

Number of Pets

Other homework problems are more open-ended and less obviously connected to the reading. Here is an example from the same section of the text:

> *The Passion of the Christ*, a film that violently portrayed Jesus’ final twelve hours, was released in February of 2004. Within a month, two people had died while watching the movie. Do you find this unusual? You can look up some facts about the movie online at `www.imdb.com/`. News reports of the deaths can be found at `www.cnn.com/2004/US/Central/02/26/passion.death.reut/` and `news.bbc.co.uk/2/hi/entertainment/3559753.stm`.

The *median*, the *mode*, and particularly the *mean* have just been introduced in *Read and Study*. There is no direct indication to the students as to how they might be relevant to solving the problem. Students are expected to explore the connections. They are also expected to decide, after reading the reference materials, what information about the movie would be relevant. At this point in the course, students have been exposed to probability from philosophical and experimental points of view. The problem requires students to understand how the expected value relates to frequencies and it is directly preceded by the two problems:

a) Suppose you were to toss a coin a few times. Considering each toss to be a separate experiment, find a way of assigning numerical values to replace *head* and *tail*, so that the mean of the values would equal the proportion of heads among all the tosses.

b) Using the idea from (a), explain how the law of large numbers reconciles our theoretical view of probability from *Section 2* with our experimental view from *Section 3*.

The examples illustrate that we consider the homework problems to be foundational, rather than supplemental to the development of the course. Students are asked to look for and build on connections among the problems, and are cautioned not to skip any. For the *Probability and Statistics* text, we have prepared a guide for instructors. It is not a solutions manual, nor does it provide information that would be shared with students. Its use is to provide technical guidance so that student discussions may be appropriately directed by the instructor and to provide insight as to why we wrote the problem. The instructor’s guide includes the following comments about the problems we have posed:

a) Assign the values 1 for *head* and 0 for *tail*.

b) Conceive of an experiment in which an event A either occurs, in which case you assign the value 1 to the outcome, or does not occur, in which case you assign the value 0 to the outcome. As in (a), the proportion of trials for which A occurs is the sample mean of the numerical values. By the law of large numbers, if the number of trials is large, the sample mean should be close to the theoretical mean of a single trial, that being $1 \cdot \text{prob}(A) + 0 \cdot \text{prob}(\text{not } A)$, or $\text{prob}(A)$. That is, if a large number of trials is performed, then the proportion of trials for which A occurs should be approximately equal to $\text{prob}(A)$.

The problem regarding deaths in the movie theater is more open-ended, and students should develop various appropriate ways of approaching it. We encourage them to debate the merits and weaknesses of their arguments. The references provided contain enough information about the movie for conclusions to be drawn, but students need to decide for themselves what information they are looking for. The comments in the guide to the instructor are as follows:

> The first death occurred in February of 2004, and the second in March. By this time, the box office receipts were approximately $300 million, so around 50 million or so people had seen it. The movie is approximately 2 hours long. Over a 2-hour period, a certain proportion of people will die; for the population of Americans, its expected value should be $1/$(the number of 2-hour intervals in 75 years), just slightly less than $1/300{,}000$. So among 50 million people, we might expect more than 150 deaths. However, the movie-going population is not the same as the general population; for instance, the elderly and people in critical health do not typically attend movie theaters, so150 is bound to be high. Certainly, though, we would expect some deaths to occur in movie theaters, purely as a matter of chance.

Many problems in life do not lend themselves to statistical analysis, and we do not mislead students to believe that. In fact, we include frequent cautionary remarks and examples demonstrating the pitfalls of blindly applying statistical methods. In the case of *The Passion of the Christ*, however, at the time of the deaths many people were portraying this as a highly unusual coincidence; statistical reasoning would have indicated otherwise.

14.6 Class Projects

Group projects form a component of our course. They vary by instructor and can deal with probability, statistics, their history, or educational issues in probability and statistics. Students are asked to work outside of class on these projects in groups of 2–4, and sometimes to present their results in class. We will describe some group projects that have been used in the past.

In one project, students completed a statistical study of data they had gathered. The students decided on a population they wished to study and a research question, designed a survey or data collection method, and gathered data under several numerical and categorical variables. They then analyzed the data, employing the statistical measures and inferential techniques discussed in the course such as regression, confidence intervals, and hypothesis testing, and wrote a paper summarizing their findings.

In another project, students analyzed a game of chance (e.g., Craps), using experimental and theoretical probability. In this project students had to answer some probability questions using both large-scale simulations and theoretical probability techniques.

In a third group project, students read educational research and designed interview protocols to assess children's probabilistic misconceptions. Each group interviewed twelve children and created a poster summarizing their results. Their findings were shared in a class-wide poster session.

14.7 Assessment

Our students are assessed in a variety of ways such as with written exams, homework, group projects, and class participation. A syllabus, in Appendix B, describes a set of evaluation components. Some instructors may choose to collect and grade homework assignments, or homework may be graded indirectly through the exams. The following exam problem illustrates this:

> This is a follow-up to the Enigma [German encryption machine from World War II] problem from homework. Part of Enigma's encryption process was to use cables to swap pairs of letters. In our homework, we found that there are $25 \times 23 \times 21 \times \cdots \times 3 \times 1$ possible ways of using 13 cables to swap 13 pairs of letters. (One way, for example, would be A-K, B-L, C-M, D-N, E-O, F-P, G-Q, H-R, I-S, J-T, U-Z, V-Y, W-X.) Here's your problem:
>
> a) Explain why $25 \times 23 \times 21 \times \cdots \times 3 \times 1$ is the answer to the problem above.
>
> b) How many ways are there of using 12 cables to swap 12 pairs of letters? You would still see all 26 letters on the plug board, but you would be using 12 instead of all 13 cables, so you could swap only 12 pairs each time. Leave your answer unsimplified. Explain your reasoning.

Other exam problems might follow up on Class Activity discussions or on concepts from the reading. The following exam question was intended to ascertain whether students understood the concept of a confidence interval:

> Suppose you have obtained a statistic. Which is larger, the 95% or the 100% confidence interval around that statistic?

Some instructors weigh the class participation component heavily in determining course grades. Merely attending class is not enough. We expect students to engage in activities and discussions within their groups and with the overall class. It is vital to the class for all students to share their ideas and to interpret others' ideas.

14.8 Discussion

> Mathematics was seen as not just rules, computing, and memorization, but a magnificent world of patterns that have meaning behind every page. This class showed me that math did not need to be complicated and filled with rules and formulas, but a subject that is beautiful and full of logic and reasoning.
>
> —Student Amy Herman, 2008

The middle school mathematics curriculum is one of transition from elementary to secondary mathematics. It ought to be rich, at once exploring deeper and more rigorously the concepts of the elementary curriculum, and introducing and laying the foundations for the concepts of the secondary curriculum. It requires much of its teachers. The mathematics preparation of prospective elementary teachers is not sufficient for the mathematical demands of the middle grades curriculum, and the mathematics major required of future secondary mathematics teachers is too abstract and formal for prospective middle grades teachers. The preparation of middle school mathematics teachers is best served by a program that focuses on their needs.

In our middle school mathematics minor courses we attempt to help develop prospective middle grades teachers in three ways. First, we want our students have a deep understanding of middle school mathematics, its definitions, representations, and connections to the broader mathematical landscape. Second, we want them to view mathematics as the human activity of making sense of patterns, so they can see mathematics as something that they and their future students can do together. Third, we aim to help them to become sufficiently mathematically sophisticated so they can make judgments about their future students' conjectures, make mathematical arguments and construct counterexamples, and create examples (and non-examples) of mathematical objects.

To do this, we have adapted and developed the problems in the *Class Activity* and *Homework* sections. We use the problems to support and model pedagogy of inquiry. The materials are student-tested each semester and revised and updated as needed. In fact, the *Read and Study* materials are written versions of the classroom discussions between students and instructors as they did mathematics together. The courses have produced middle school teachers ready for the mathematical challenges of the classroom. We are now beginning the process of custom publishing these texts with McGraw-Hill, to share the content without violating copyrights on the *Connections* materials. Whether you are interested in designing your own activities or using ours, please feel free to contact any of the authors.

14.9 Conclusion

> I am in Kaukana at the middle school. We did an activity yesterday ... [The students] had a lot of fun with it and showed some amazing thinking and experimenting, so I promised I would make more lessons like that for them. It is really neat to see how all of my minor classes are contributing to my teaching.
>
> —Former student Tori Schneider, 2010

Between 2005 and 2010 one hundred and forty-four students completed the middle school program. Of those, sixty-five provided career information, and the data suggests that our graduates fare well: fifty-eight of the sixty-five (89%) are currently employed as teachers.

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Appendix A
List of Topics for Big Ideas in Probability and Statistics

Chapter 1: Conceptual Foundation

Section 1: Introduction to Data and Correlation
Sample vs. Population
Types of Data
Scatterplots
Positive and Negative Association
Correlation
Introduction to Regression
Cause and Effect Misconceptions

Section 2: The Nature of Probability
Philosophical Foundations of Probability
Standard Probability Models
The Language of Chance
Odds

Section 3: Frequency
The Experimentalists' View of Probability
Data Displays (Pie Charts, Line Plots, Bar Graphs, Histograms)
Probability Distributions (Simple Discrete, Uniform Continuous)

Section 4: Measures of Center
Mean, Median, Mode
Sample Mean versus Population Mean

Expected Value
The Law of Large Numbers

Section 5: Measures of Spread
Boxplots
Range, Interquartile Range
Variance and Standard Deviation

Section 6: Conditional Probability
The Conditional Nature of the Sample Space
Independent and Dependent Events
Compound Events
Probability Tree Diagrams

Chapter 2: Counting

Section 7: Set Permutations
Set Permutations and the General Multiplication Rule
Counting Tree Diagrams

Section 8: Multiset Permutations
Combinations and More General Multiset Permutations
The Binomial Distribution

Chapter 3: Inference

Section 9: An Introduction to Inference
Foundations and Misconceptions Regarding Inference

Section 10: Bayes' Theorem
Frequency Approach to Bayes' Theorem
Common Fallacies and Abuses Involving Conditional Probabilities

Section 11: Hypothesis Testing
Basic Principles of Hypothesis Testing
Introduction to the Normal Distribution and the Central Limit Theorem

Section 12: Confidence Intervals
Basic Principles and Misconceptions Regarding Confidence Intervals
Sample Bias

Appendix B
Fall 2009 Course Syllabus for Big Ideas in Probability and Statistics

Textbook *Big Ideas in Mathematics for Future Middle Grades Teachers: Big Ideas in Probability and Statistics* by Beam & Szydlik.

Course Description In this course we will study three important areas of mathematical thinking:

1. Probability
2. Combinatorics (counting techniques)
3. Statistics, both descriptive and inferential.

We will study the historical development of probability and statistics, will learn to evaluate statistical claims using real data, and will examine problems and ideas from upper elementary and middle school curriculum materials, as guided by the National Council of Teachers of Mathematics, in order to understand probabilistic and statistical thinking in children.

Attendance, Participation and Homework Most of our class time will be focused on problem-solving activities and discussions of the reading and homework problems. Your attendance and participation is important for yourself and your classmates, and will account for 20% of your course grade. By participation, I mean that I expect you to have completed the assigned reading every night, and to have attempted every homework problem that was assigned. I will judge your participation each day based on how involved you are in class discussions, and on whether I see evidence that you have worked hard on your homework.

Projects There will be two group projects, each worth 15% of your course grade. One project will focus on the teaching of probability and statistics in the upper elementary or middle grades, and the other will require you to collect and interpret data.

Exams Two exams, each worth 25% of your course grade, are tentatively scheduled as follows:

Exam 1:	Friday, October 30
Exam 2:	Friday, December 18

Grading Scale The grading scale will be as follows (after rounding to the nearest percent):

A	93–100% of the weighted course points
A−	90–92%
B+	87–89%
B	83–86%
B−	80–82%
C+	77–79%
C	70–76%
D+	67–69%
D	60–66%
F	below 60%

II

Courses for Middle School Mathematics Teachers

F. Combination Courses

15

Vermont Middle Level Mathematics Initiative: Courses and Materials for Mathematics Educators

George Ashline and Marny Frantz
Saint Michael's College and Vermont Mathematics Partnership

15.1 Introduction

In this paper, we highlight successful courses and classroom materials developed in recent years for in-service fifth through ninth grade mathematics educators by the Vermont Mathematics Partnership (VMP). The VMP's Middle Level Mathematics Initiative (MLMI) is a professional development program designed to enhance mathematics instruction to help all middle level students succeed in mathematics.

The VMP is a targeted math and science partnership funded by the National Science Foundation and the US Department of Education [20]. Building on the successful Vermont Mathematics Initiative [19], a content-focused graduate program for elementary mathematics teacher leaders, the VMP has worked directly with partner school systems and developed courses, programs, and materials for educators across the state. Launched in 2004, the MLMI implemented a new series of graduate courses in mathematics content, assessment, and pedagogy for middle level mathematics educators, teacher leaders, special educators, and administrators. Offered primarily as week-long summer institutes with fall follow-up sessions, courses include *Proportional Reasoning in the Middle Grades*, *Modeling Change: Using Algebra to Analyze Change*, *Making Informed Decisions: Data, Statistics, and Probability*, and *Geometry in the Middle Grades*.

This new series of graduate courses has been designed to satisfy several licensure needs for middle level educators in Vermont. To fulfill the middle level state endorsement, educators must demonstrate that they meet knowledge and performance standards for the mathematics core content and the professional knowledge of middle grades education. Middle grades educators applying for this license must show evidence of knowledge in several areas, such as the NCTM middle level content strands, mathematics history, and mathematics pedagogy for middle level classes [18]. Our courses can help to provide this evidence. In addition, to maintain their license, Vermont teachers must earn nine credits every seven years; depending on the school district, they may come from workshop or conference participation, independent study, or (most commonly) graduate courses. Many of the in-service teachers taking our middle level courses have used them to help fulfill the requirements.

After consulting the Vermont grade level expectations [17], we aligned course components with state mathematics content requirements for the fifth through ninth grades. Some of the grade level expectations addressed throughout the courses include concepts in arithmetic, number, and operations; geometry and measurement; functions and algebra; and data, statistics, and probability.

In this paper, we describe the methods and process that we used to develop the courses and some of the common course content and structure, which were grounded in our statewide grade level expectations and national standards. To make available some of our course materials, we have posted to the VMP website [20] a selection of activities and a resource document for each course. To illustrate our course features, we provide an analysis of our initial *Proportional Reasoning* course, which we have successfully offered several times. To describe the impact of the MLMI on educators, we highlight a survey of more than forty middle level educators who have taken at least two of our courses. In it, many respondents said how course insights, materials, and capstone investigations have enhanced their own teaching.

15.2 Course Structure and Course Content

In the past in Vermont, mathematics professional development has been for all grades, K–12, or specifically designed for elementary or secondary level educators. The VMP saw the need for professional development tailored for middle grades mathematics educators. The VMP conducted a needs assessment of middle level education in Vermont in 2003. We explored the challenges of helping all middle level students succeed in mathematics. When asked to offer suggestions for the content and structure of professional development, middle level educators asked for week-long summer courses at convenient locations, with ongoing discussions at fall follow-up sessions. They suggested courses on advanced mathematics for adult learners and integrating differentiated instructional strategies and other materials in the classroom. They hoped that courses would include an overview of the development of mathematical ideas across grades and research on how adolescents construct understanding.

The MLMI's course design teams based their work on information gathered from the needs assessment and a review of research and data, middle level mathematics programs, and national standards and reports. The primary goal in our courses and materials has been to help middle level educators enhance their instruction and assist their students in meeting Vermont standards and grade level expectations in mathematics. We illustrate a variety of pedagogical approaches and strategies through which educators could successfully share mathematical content with their students.

We conducted a review of ten documents, including six standards-based mathematics programs, the Best Practices Institute program information [15], the National Educational Technology Standards or NETS for teachers [8], the NCTM standards [12], and *The Mathematical Education of Teachers* [5]. Our assumption was that teachers will be able to teach mathematics more effectively if the depth of their mathematical understanding significantly exceeds the level at which they are teaching. For instance, a teacher who is teaching about function machines should have a deep understanding of linear functions, including dependent and independent variables, slope of a line, intercepts, and so on.

As we reviewed materials, our goal was to answer the question "What do middle level mathematics educators need to know to be able to teach mathematics effectively to students in the middle grades?" Within the review process, we determined major mathematical topics and themes present. Then we considered the context for the mathematics, potential misconceptions, and content worthy of inclusion in a professional development course for middle level teachers. Our document review helped to guide our course development. Two quotes summarize our findings well:

> Teaching middle grades mathematics requires preparation different from, not simply less than, preparation for teaching high school mathematics, and certainly reflecting more depth than that needed by teachers of earlier grades. —*The Mathematical Education of Teachers* [5]
>
> The important mathematical foci in the middle grades—rational numbers, proportionality, and linear relationships—are all intimately connected ... —*Principles and Standards for School Mathematics* [12]

Our MLMI courses are independent of each other and not sequential. We want to ensure that educators demonstrate increased content knowledge and can teach a full range of middle level students. The professional development courses are intended to help in-service teachers meet state licensing requirements. One issue that we faced was the granting and administration of credits for our courses. We explored several options and eventually chose to collaborate with a local state college to offer graduate level credits for our courses, with enrollment fees covered by our grant and local school district professional development funds.

In course development, we grappled with several fundamental questions. What should educators know and be able to do? How would we assess and monitor understanding? How could we adapt problems to extend and probe this understanding and meet the diverse needs of educators? What readings and resources would help educators achieve a deeper understanding? How do the courses build across middle grades, trace back to K–4 roots, and extend to high school? How are materials related to national standards and state grade level expectations? What should be the balance between content and pedagogy?

As we addressed these questions, we realized that mathematics content knowledge and pedagogy are of primary importance, and that one area should not dominate the other. We emphasized in our courses and materials an understanding of middle level mathematics content and an awareness of how it connects with elementary and secondary level concepts. In some cases, middle level teachers may have procedural facility, but lack a deep conceptual understanding. On the pedagogical side, we designed courses and materials to demonstrate equitable classroom practices, apply current research on middle level mathematics teaching and learning, provide proficiency with assessment of student learning and using assessment to inform instruction, and address student language struggles in mathematics.

After considering adult mathematics content knowledge, we analyzed how best to facilitate an understanding of the content and created a variety of activities, experiences, and problems to meet our goals. A development team planned the content and framework for each course and created a syllabus and schedule for concepts and pedagogy. Then, an instructional team (which overlapped with the development team) implemented each course. Although our courses have not had textbooks, we have assigned relevant articles and manuscripts to accompany course activities and discussions [20].

We will later discuss more specifics about the common pedagogical emphases of our courses, but first here is a brief description of their content:

- *Proportional Reasoning in the Middle Grades* (described with more detail in a later section)

  We deeply investigate middle level proportional reasoning content and identify proportionality in each National Council of Teachers of Mathematics (NCTM) Standards strand: Number and Operations, Geometry and Measurement, Algebra and Functions, and Probability and Statistics. We identify, compare, and contrast proportional and non-proportional relationships in many applications and problem-solving contexts. We examine ways in which students build understanding of ratios and proportions, and how to correct common misconceptions.

- *Modeling Change: Using Algebra to Analyze Change*

  We examine change as it occurs in middle level mathematics and in many real world applications. We look at iterative processes and limits in ways that connect to middle level mathematics. We also analyze task complexity with an eye toward providing problems that involve the mathematics of change for all levels of students. We provide opportunities to deepen understanding of classifying, modeling, and predicting change and to examine strengths and limitations of mathematical models for change.

  In contrast to *Proportional Reasoning*, *Modeling Change* contains a great deal of mathematics content, in particular precalculus content, which may not be appropriate for middle level students. The course also contains a fascinating development of the mathematics of change beginning in elementary level mathematics and continuing through middle level mathematics. In this course, educators make extensive use of graphing calculators and Internet applets.

  Modeling Change topics include growth and decay comparisons among different types of functions (such as linear, quadratic, exponential); constant versus changing rates of change, importance of scale factors, ratios, rates, and underlying units; limits and iterative processes; and representing change with manipulatives and graphs.

- *Making Informed Decisions: Data, Statistics, and Probability*

  We emphasize a well-informed understanding of the analysis, representation, and interpretation of data. We explore the use and possible misuse of measures and representations of data. We distinguish among association and causation and generalized trends in raw data. We also apply counting techniques and other methods to determine probabilities.

  In comparison to *Proportional Reasoning*, *Making Informed Decisions* focuses more on content, as middle level educators are often less familiar with the probability and statistics topics. Although there is substantial focus in *Proportional Reasoning* on considering and analyzing student work, in *Making Informed Decisions* there is a greater emphasis on investigating student misconceptions about data and statistics. Also, Internet datasets and applets are components of this course.

Making Informed Decisions topics includes statistical language and dilemmas that require critical thinking; probability, randomization, surprising results, streaky behavior; measures of central tendency and variability; graphical and tabular representations, correlation, lines of best fit; and data analysis and historical studies.

- *Geometry in the Middle Grades*

  We explore such topics as similarity, transformations, spatial reasoning and visualization, and making and testing conjectures. We use the dynamic geometry software *Geometer's Sketchpad* [7], and online applets. We distinguish between convincing arguments and proofs, and create and critique inductive and deductive arguments. We identify and classify two and three dimensional figures. We construct geometric figures using various methods and materials.

  To help address the concern of middle level teachers that students often show marked deficiencies in geometry on standardized assessments, *Geometry in the Middle Grades* considers area, perimeter, and volume and focuses on conjecture, proof, sketching, and construction.

  Geometry topics include properties of geometric structures (including two- and three-dimensional figures); derivation of formulas for the area and perimeter two-dimensional figures, and the surface area and volume of three-dimensional figures; conjectures, theorems , postulates, convincing arguments, and proofs; similarity and congruence; transformational geometry; spatial visualization; strategies for representing three-dimensional figures in two dimensions; and geometric constructions versus drawings of geometric figures.

15.3 Essential Course Elements

15.3.1 Common course content and pedagogy

Several features are common to all our courses. These include:

- pre and post-assessments (see Conclusion for more information),
- problem solving through multiple approaches,
- activities that involve more than one mathematics content area,
- educator sharing of problem solutions and of teaching strategies with the whole group and in smaller grade-level groups and study groups,
- formative assessment, including exit questions,
- differentiating instruction,
- addressing students' mathematics language issues, especially through the use of graphic organizers,
- final projects that involve action research in educators' classrooms,
- extensive use of technology, and
- analysis of student work.

These features can be described in two broad categories, the dynamic, constructivist classroom and the links between mathematics content and mathematics pedagogy.

15.3.2 The dynamic, constructivist classroom

We have developed activities that lend themselves to investigation, the sharing of strategies, and multiple approaches. Many of our investigations have involved more than one mathematics content area. For example, in our *Geometry and our Making Informed Decisions* courses, educators encounter connections to algebra. In our *Proportional Reasoning* course, we address each strand of the NCTM standards and its connection to proportional reasoning. To support our investigative approach to mathematics learning in all our courses, we make extensive use of graphing calculators, Internet data sources, mathematics applets, and computer software, especially *Geometer's Sketchpad*. We integrate hands-on experiments and mathematics manipulatives wherever possible.

Figure 15.1. Educators collaborate to solve challenging problems.

To facilitate collaboration among educators, we introduce two kinds of discussion groups, namely study groups and grade level groups. Mixed-grade study groups meet every morning for about an hour to discuss the previous night's assignment, exit questions, and other topics relevant to the day's work. Grade-level groups meet to perform a task or to hold a discussion specific to their grade levels. Educators have found this collaboration to be invaluable. As one educator responded in our survey, "There was a variety of learning environments throughout the day. Group discussions, presentations from a variety of instructors, group sharing ... groups were mixed regularly ... " To answer a survey question about MLMI impact, one educator wrote, "One of the biggest concepts I took away from both classes was instructional strategy: Posing a big problem ... in a way that students can access at different levels; using guiding questions to encourage collaborative inquiry; posing an exit question or reflection question."

15.3.3 The links between mathematics content and mathematics pedagogy

Throughout our courses, we model ways that educators might effectively share mathematical concepts with their own students. In some cases, investigations can be used directly with middle level students. Adult learning materials, intended to strengthen educators' mathematical understanding, would need modification for use with middle level students. We regularly remind educators of this. For example, in *Geometry*, we show Eratosthenes' calculation of the circumference of the earth ([20] and [4]). It involves geometry and trigonometry beyond the scope of middle level curricula. In our discussion, we consider how to modify materials for the middle level classroom.

Since many educators face heterogeneous middle level classes, we also provide differentiated activities and experiences to expand their understanding and to add relevant activities to their repertoire.

We emphasize formative assessment in our courses. At the beginning of each course, educators take a pre-assessment on course content and pedagogy, and take it again at the end. We also model formative assessment by using exit questions twice a day. (See the Conclusion and Appendix for more information about our pre- and post-assessments and exit questions.) We also ask educators to provide daily feedback about what has been successful and what should be changed in the day's work and to give final feedback at the end of the course. Because students' mathematical challenges often arise from difficulties with mathematical language, we give attention to this in our courses. We incorporate many graphic organizers, such as concept maps and cluster-word webs, to help teachers organize and communicate their ideas about course topics. We also discuss how the tools might be applied with middle level students. We have used another graphic organizer, the Frayer Model, to help analyze and describe important terminology under consideration [3]. For example, considering the term "proportional," educators create a Frayer model containing essential and inessential characteristics and also examples and non-examples for the term. Several educators have successfully used this organizer with their students.

In most of our courses, we devote time to review student work, analyze student errors and misconceptions, and discuss how this might inform instruction. Educators in all of our courses complete a final project in which they explore some aspect of their own students' work. For example, here are the specifications for the final project in *Proportional Reasoning in the Middle Grades*:

- Create and administer a pre-test to assess your students' level of proportional reasoning. Analyze results and summarize your findings. Discuss how results will affect your instruction of proportionality throughout the year.
- Select one item from your pre-test and select pieces of student work that illustrate the variety of class responses, including both correct and incorrect solutions. Discuss the numerical relationships in the problem, the context, and what questions you might ask to follow up.
- Choose three students, possibly at different levels of mathematics readiness, and create a display of their work on the pre-assessment as a whole. Discuss students' level of proportional reasoning and how you will address their needs in your instruction.

15.3.4 An in-depth look at one course

To illustrate our MLMI courses, we will discuss our *Proportional Reasoning in the Middle Grades* course in greater detail. Additional course information, including a syllabus and daily working agendas, is available online [20]. Proportional reasoning is the most significant theme throughout middle level mathematics. Students with a good grounding in proportional reasoning are better able to achieve success in mathematics. According to Susan Lamon "This form of reasoning opens the door to high school mathematics and science, and eventually, to careers in the mathematical sciences." [9, p. xiii] Based on anecdotal evidence from course educators, the grant survey, and reflections, our *Proportional Reasoning* course has had a significant impact on educator understanding of mathematics, pedagogy, syllabi, and student learning. We have presented this course several times as a summer institute, in-service sessions, an independent study course, and as an evening course during the academic year. The intensive summer institute approach seems to be the most successful; the other approaches have also been useful.

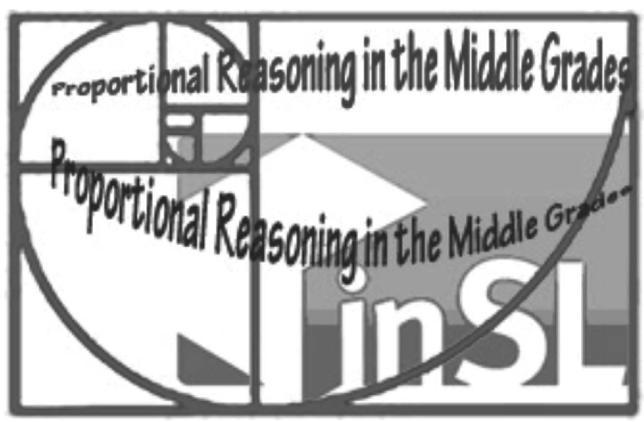

Figure 15.2. *Proportional Reasoning* course golden ratio spiral title.

We describe each day of the *Proportional Reasoning* course and mention a few activities.

Day 1 The theme for this day is the meaning of proportionality. This lays the foundation for the rest of the week and illustrates the prevalence of proportional reasoning throughout middle level mathematics and its connections to each NCTM strand [12, 13]. We emphasize proportional relationships and contrast them to non-proportional relationships. (See the Appendix for a discussion of an exit question for this day.) In an activity called "I Spy" [20], educators working in pairs are given cameras and instructed to take several photos around or outside of the building that show some kind of constant change. Each pair then chooses two photos to analyze, in terms of change and proportionality. Some draw their photos "by hand" and take measurements on printed copies of the photos or on the computer screen. Others import their photos into *Geometer's Sketchpad* to conduct their analysis. For example, Figure 15.3 illustrates one group's photo of a handrail on a nearby building and their *Sketchpad* analysis superimposed on the picture. They located the coordinates of five different points along the rail and determined the rail's linear rate of change in terms of the underlying units of distance.

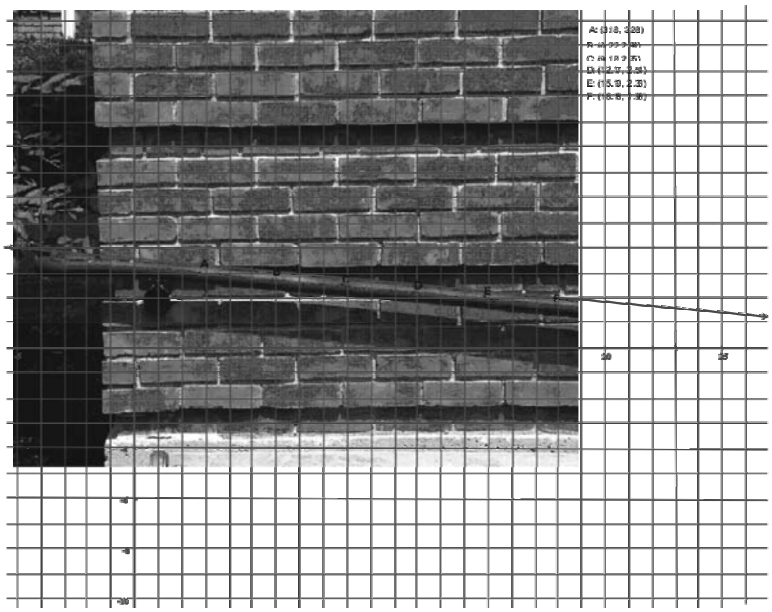

Figure 15.3. One group's "I Spy" photo with *Geometer's Sketchpad* analysis.

Day 2 This day's theme is proportional reasoning in number and operations. The day involves work with ratios, fractions, percentages, and decimals. The work focuses on several important number and operation and data analysis middle level expectations. The golden ratio is considered in several activities. In an activity called "What Can Spinners Tell Us?" [20], we use an Internet applet from the eNLVM interactive lessons [6] to investigate some topics in probability. The applet enables the user to adjust the number of sectors in a spinner, the size of the sectors, and the number of spins in a run. Educators determine theoretical probabilities and expected results for various spinner configurations and then run simulations. This is a good exercise in determining spinner probabilities and lets educators observe the Law of Large Numbers in action.

Day 3 The theme for this day is proportional reasoning in geometry. Grappling with a variety of middle level geometry, algebra, and measurement concepts, educators enhance their understanding of perimeter, area, and volume, and analyze student work on similarity problems. For example, in small groups, educators consider various similar right triangles and analyze the patterns of ratios of respective sides and corresponding angles. This leads to a discussion of trigonometric ratios, their properties and relevance. Groups then estimate the height of a nearby building. (During the initial offering of this course, which took place in our state capitol of Montpelier, groups gathered information to estimate the height of the capitol dome.) Educators estimate the angle of elevation to the top of the building with handmade clinometers and the distance to the building [16]. They use similar triangles and trigonometric ratios to complete summary worksheets and estimate the overall dome height [20].

In another activity, educators investigate what happens to the surface area and volume of three-dimensional figures when their dimensions are doubled, tripled, etc. They are encouraged to build the figures with cubes in order to have a

Figure 15.4. Educators create clinometers, gather data, and use proportional reasoning to estimate the height of the Capitol building.

Figure 15.5. Educators use mathematics manipulatives to analyze a proportional reasoning problem.

hands-on and visual way to see the results. We also work in groups to solve the classic soccer ball problem: how many soccer balls would it take to fill the room? Analyzing student work and discussing instructional strategies suggested by the student work is a powerful experience for educators. As one educator put it, “It’s great to look at what students write in depth and to try to figure out their thinking.”

Day 4 This day’s theme is proportional reasoning in functions and algebra. Although functions and algebra have been seen previously in the course, on this day we carry out an in-depth investigation of rates of change by looking at distance versus time and velocity versus time graphs. We look at students’ understandings of the equation $d = r \cdot t$, and we conduct an investigation of an inversely proportional relationship, “The Penny Experiment: More Workers, Less Time?” [20] to answer the question, “How does the number of workers doing a job relate to the time it takes to do the job?” This illustrates a well-known inversely proportional relationship.

Day 5 On this day, with a continuation of the proportional reasoning in functions and algebra theme, we conclude the work on rates of change and continue to work on middle level concepts relating to algebra. We also continue to discuss ways to solve problems involving proportions and the pros and cons of teaching students to solve proportions by cross multiplication. We build upon their understanding of rates of change through a “scooter” activity. There are two versions, one which involves comparing one scooter with a constant velocity to another with a constant acceleration and another more scaffolded version that involves analyzing the distance traveled by one scooter under various conditions [20]. The activities help educators to distinguish between proportional and non-proportional relationships and to deepen their understanding of time, distance, and velocity interrelationships.

Follow-up session The follow-up session, held in the fall, is devoted to educators’ presentations of research projects. Discussions are typically lively and engaging. We have found, however, that holding follow-up sessions during the academic year is difficult to arrange. We are exploring the idea of conducting project presentations online in a discussion board format or in a virtual world such as “Second Life.” [14]

Assignments Educators receive assignments each day of the course. They involve reading Lamon’s *Teaching Fractions and Ratios for Understanding* [9] and other articles from such sources as the NCTM middle school journal, *Mathematics Teaching in the Middle School*, and solving problems that involve proportional reasoning. In addition to action research projects, educators complete a final problem set.

Our other courses are comparable to *Proportional Reasoning in the Middle Grades*. We maintain a strong commitment in all our courses to presenting and discussing the links between mathematics content and mathematics pedagogy. All our courses offer some adult learning material beyond the scope of middle level curricula and also material that is suitable for the middle level classroom. And all our courses emphasize the dynamic, constructivist classroom and the links between mathematics content and mathematics pedagogy. Each course has a course resource document, listing books, articles, and websites [20].

15.4 Conclusion

In this paper, we have highlighted several successful courses and classroom materials developed for middle level mathematics educators through the VMP's MLMI. Altogether 138 educators have participated in MLMI courses. Many enrolled in more than one course, so there have been 208 enrollments in total. We saw this high rate of return participation as an indication of the success of our program. We conclude the article by describing some of the challenges that we faced in this initiative and ways we addressed them. We also describe the impact of the MLMI on teacher pedagogical content knowledge and share a few other results from our survey of 40 educators who have taken multiple MLMI courses over the past several years.

As course developers and instructors, we faced a number of issues in designing and implementing our materials and courses. They included maintaining intellectual rigor of adult learning while still offering direct classroom applications and activities, helping educators see the linkages between adult content-knowledge and its roots in the K–12 curriculum, and countering the belief that elementary and middle level educators need only a rudimentary understanding of mathematics.

We have addressed these challenges in several ways. For instance, we have incorporated features of the Equity Framework [20] in our course design and made them explicit to educators. The framework, developed in 2003 by VMP leadership in collaboration with Dr. Rachel Lotan of Stanford University, is based on decades of research on achieving equity in heterogeneous classrooms. We have focused on important mathematical ideas and on building skills and concepts in meaningful contexts. We have engaged groups in complex problem solving that requires collaboration, and we delegated authority for learning, requiring independent research, and application. We have focused on research on how students learn the relevant mathematics content, and where they have misconceptions. Furthermore, we intentionally have addressed language challenges, built a strong understanding of vocabulary, and guided instruction with ongoing, formative assessment, including exit questions and daily study groups to review course work. (See the Appendix for more formative assessment information.)

In MLMI course sections for which pre-post measures of teacher pedagogical content knowledge are available, educators showed statistically significant gains (at the .05 level), 80% showing gains of 9% or more. Pre-assessments were administered at the beginning of these courses and post-assessments were administered at least three months later. These pre-post measures contain items from: (1) the Learning Mathematics for Teaching Project at the University of Michigan [10], a National Science Foundation Research, Evaluation, and Technical Assistance (RETA) project in support of Math Science Partnerships such as the Vermont Mathematics Partnership, (2) Assessment Resource Tools for Improving Statistical Thinking [2], and 3 items developed by MLMI course development teams.

Educator reaction to our courses has been positive. In our survey, many educators shared how the MLMI has affected their mathematics content and teaching strategies. Some comments concerning the *Proportional Reasoning* course, mentioned by several educators include, "It helped me to see how important proportional reasoning is in middle school. I never realized how many different contexts there were to consider. Proportional reasoning is a high priority topic in the middle grades. I learned how it connected with many other strands of math and it is essential that students understand this to make connections. I was able to see the connection between proportional reasoning and so many topics in the curriculum that I teach. I also found strategies that I was comfortable to use and incorporate in my teaching. The *Proportional Reasoning* course deepened my conceptual understanding so my teaching strategies became more flexible."

We end with two comments that illustrate how the MLMI has been received by middle level educators. One educator noted, "The courses taught and reinforced content for me. They improved my understanding and helped me to analyze problems in more depth. These courses validated the need to teach these topics and showed me the connections." Another put it this way, "I think most of the open exchange of ideas and practices with instructors and fellow practitioners throughout the state. This is so enriching to the teaching experience and so rarely happens in our hectic day to day existence."

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Appendix
Formative Assessment Strategies

In MLMI courses like the *Proportional Reasoning Course*, we have used daily homework groups where educators shared their understanding of problems and reading assignments. We have regularly incorporated exit questions (usually at the end of morning and afternoon sessions) to measure educator facility with major concepts. We also have gathered feedback from educators after some activities and discussions. After gauging the level of comprehension and number of questions or concerns raised, we then planned subsequent activities and sessions to better meet the needs of the educators.

Exit question illustrations

In a recent "Assessment Issues" column of the *NCTM News Bulletin* [1], we described the purpose and application of exit questions in our course for middle level educators.

In the *Proportional Reasoning* course one exit question asks educators to define "proportional relationship." Many responses included statements that follow from a more succinct definition. We have had good discussions with teachers in both small and whole groups about the features of such a definition. For example, several educators have stated that the graph of a proportional relationship between two quantitative variables must include the origin $(0, 0)$; discussions of how this is a consequence of a good definition have been helpful.

Two examples from our *Making Informed Decisions* course highlighted in the column that measure educator understanding of risk comparisons and graphical displays are:

- In the United States, if about 40,000 people die each year in motor vehicle accidents and fewer than 1,000 die in airplane crashes, what would you report about the risk in each mode of transportation? Why?
- Suppose a teacher gave a challenging test, on which students did not perform well. Provide a graphical display that would represent student scores on the test. Explain why your graph indicates that the test was challenging.

If educators had difficulties with exit questions, we revisited and clarified the underlying concepts, either in whole group or homework group discussion. If the questions are well-received, then we have more confidently built upon the underlying concepts in the progression of our courses.

Table of contents for online materials (VMP, 2009)

- Resource documents for our four middle level professional development courses: *Proportional Reasoning, Geometry, Modeling Change,* and *Making Informed Decisions*
- Syllabus and daily working agendas for *Proportional Reasoning* course
- Activities for *Proportional Reasoning* course
  - Anna Scooter
  - Data Collection Table and Two Measurement Diagrams
  - Eratosthenes (used in *Geometry* course)
  - I Spy
  - Penny Experiment
  - Twin Scooters
  - What do Spinners Tell Us?
- Equity Framework, used in all of our courses

16

Challenging Yet Accessible Mathematics Courses for Middle Level Teachers

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This article describes three challenging yet accessible mathematics courses designed for middle school teachers and offered by the Department of Mathematics at the University of Nebraska-Lincoln (UNL). Their descriptions are based on the courses as we have taught them as part of the *Math in the Middle (M^2) Institute Partnership* , a National Science Foundation Math Science Partnership (MSP) program that works with practicing teachers. As a grant funded MSP, we take seriously the responsibility to share information about our program and the courses we have created. Readers are directed to our website [1] for information about our program and specifically to our course materials [2] for a link to additional information about the courses described in this article as well as other courses that we offer.

The aim of the M^2 *Institute*, and the university—public-school partnership that created it, is to develop intellectual leaders in middle level mathematics (grades 5–8). A core strategy that guides Math in the Middle is to offer teachers content rich mathematics courses that are accessible and useful. Practicing teachers who are admitted to the M^2 *Institute* earn 36 graduate credit hours over several years, resulting in a Master's Degree. More information on the M^2 *Institute* is in the companion article [3] in this volume.

We describe three *Math in the Middle* courses: Mathematics as a Second Language (MSL), Experimentation, Conjecture and Reasoning (ECR), and Number Theory and Cryptology for Middle Level Teachers (NT&C). They are the first, third and fifth mathematics courses in our graduate program. Previously, ECR and NT&C appeared slightly earlier in the program. With appropriate attention to the participants' mathematics background, ECR and NT&C can be offered independent of the previous courses in our program. In fall 2008, UNL offered ECR as a distance education course for teachers who are not part of *Math in the Middle*, and a version of the course has been offered five times to undergraduates whose goal was to become elementary or middle level mathematics teachers. Versions of NT&C have been offered to secondary mathematics teachers and to undergraduates who intend to become middle level mathematics teachers. In summer 2009, we successfully offered NT&C to middle level mathematics teachers who are not part of *Math in the Middle*, as part of the Nebraska Math and Science Summer Institutes [1]. The instructor had been involved in teaching several M^2 *Institute* courses, including NT&C. Four of the students who took NT&C that summer have chosen to pursue a Master's Degree through UNL's Department of Mathematics.

[1]The authors acknowledge the support of the National Science Foundation (EHR-0412502) in the development and teaching of the courses described in this article. All ideas expressed in this paper are our own and do not reflect the views of the funding agency.

Six cohorts of teachers have participated in the *Math in the Middle Institute*. When the final group of teachers received their Master's Degrees in August 2011, that brought the total to 156 teachers who completed the three courses as part of earning their Master's Degree with 92% success rate—only 13 teachers dropped out. Three of them failed to complete the first course, MSL, and one failed to complete ECR. In the beginning, class size was 33–35 teachers, but because of resource limitations, the last two groups were smaller. Our version of Mathematics as a Second Language was inspired by Herbert and Kenneth Gross, who created Mathematics as a Second Language for the Vermont Mathematics Initiative (VMI) and wrote a text with that name. Professor Kenneth Gross has come to Nebraska four times as the lead instructor for the M^2 *Institute* course that uses his materials. The course focuses on a deep understanding of the basic operations of arithmetic as well as the interconnected nature of arithmetic, algebra, and geometry. Experimentation, Conjecture, and Reasoning is, as the name implies, about the cycle of doing mathematics. It emphasizes problem solving, reasoning and proof, and mathematical communication. Number Theory and Cryptology for Middle Level Teachers gives participants key ideas of number theory that have applications to the middle level mathematics curriculum.

There are two types of courses taken by *Math in the Middle* participants: blended distance-education courses (taken during the school year) and on-site courses (completed during the summer months). Academic year courses begin with two days of face-to-face class meetings followed by 12–15 weeks of online instruction and interactions over Blackboard; the amount and frequency of Blackboard instruction and interaction varied by course and instructor. On-site courses, offered during the summer, meet eight hours a day for five days with homework assigned each evening.

The courses teach mathematical content that is relevant to middle level mathematics based on the recommendations of *The Mathematical Education of Teachers* [4] and have as a goal to help participants become productive mathematical thinkers with a toolbox of skills and knowledge to experiment, conjecture, reason, and solve problems. Developing "mathematical habits of mind" (e.g., [5, 6]) means helping learners acquire an understanding of and experience using these skills. Mathematical habits of mind are marked by great flexibility of thinking, including the use of indirect arguments and making connections between knowledge the mathematical thinker possesses and the problem being considered. Although a complete mathematical toolbox includes algorithms, a person with well-developed habits of mind knows why algorithms work and when they will be effective. Mathematical habits of mind also include ease of calculation and estimation and persistence in solving problems. A person with well-developed habits of mind has a disposition to analyze all situations and the belief that he or she can find a solution. Such a person also engages in metacognition by monitoring and reflecting on the processes of reasoning, conjecturing, proving, and problem solving. The pedagogy courses taught by faculty in the Department of Teaching, Learning, and Teacher Education have as a goal to help participants develop pedagogical practices that will support the development of a set of mathematical habits to guide the learning practices of their middle level students [7].

16.1 Mathematics as a Second Language

We decided to begin our institute with a course whose goal is to give participants a sound understanding of the basic operations of arithmetic. That is, we want our participants to gain a good understanding of the rational numbers and have an introduction to irrational numbers. Course goals include understanding the basic operations of arithmetic, developing number sense, and introducing algebra as a means of communicating mathematical ideas.

Participants who begin the course thinking, "I know all this," quickly learn that the course challenges their understanding of the rational numbers and of mathematics. It differs from standard "Arithmetic for Elementary School Teachers" courses. From the first problem that participants encounter (Figure 16.1) [8], the course stresses the interconnections of arithmetic, algebra, and geometry. We can stress that many problems can be approached in a variety of ways and that different approaches can lead to different insights. We can extract from teachers' work several approaches to solving the problem, including mental math, adaptive guessing, a table of values, an algebraic approach, and a graph. We are not surprised when eighth grade algebra teachers are first surprised and then impressed by elementary teachers' problem solving abilities using tables of values.

A second emphasis of the course is on understanding mathematics as a language using Herbert Gross's viewpoint that numbers are adjectives that modify nouns (or sometimes other adjectives) and that understanding the noun that an adjective modifies is important. For example the digit "5" is not an object, but rather an adjective modifying the noun "ones." When the noun that "5" is modifying is not "ones," then it must be specified, such as "5" hundreds or

The Film Problem

> There are two photography stores in town that do custom film developing, Perfect Picture (abbreviated PP) and Dynamic Developers (abbreviated DD). At PP the cost to develop one roll of specialty film is $12, but additional rolls of film cost only $10. At DD, the cost of developing one roll of film is $24, but each additional roll is developed at a cost of only $8. For what number of rolls of film is the cost of developing the same at PP and DD?

Figure 16.1. Opening Problem in Mathematics as a Second Language [8]

“5” fourths. Considering numbers as adjectives and units as nouns seems to help elementary and middle level teachers with measurement, such as adding numbers of inches and feet, or combining hours and minutes. Considering numbers as adjectives also applies to place value and adding and subtracting algorithms: numbers can be combined only with other numbers with like nouns. This adjective-noun framework is particularly useful for fractions. Teachers are less likely incorrectly to add fraction denominators when they use adjectives and nouns: “5” fourths plus “3” fourths is clearly “8” fourths. The teachers we have worked with enjoy this approach and believe it is useful in helping their students build their arithmetic skill.

We named the course “Mathematics as a Second Language” (MSL) after a course that is offered by the Vermont Mathematics Initiative and the textbook written by Kenneth and Herbert Gross [8], and our delivery is based on the Vermont model. Two innovations in our course are our greater use of problems designed to develop “the habits of mind as a mathematical thinker” and the emphasis we place on understanding rational numbers and the real numbers. We thought that the modifications are appropriate for middle level teachers. We used what we found in the Mathematics as a Second Language textbook, as problems like The Potato Race, The Fly and the Spider, and Wine and Tea [8] were perfect for what we wanted to accomplish.

16.1.1 Mathematics as a Second Language: Description

MSL typically is offered as a summer institute course of 40 hours over five days. In the first and fifth year, when the timing of grant funding prevented a cohort beginning in the summer, MSL was then offered during the academic year. The first day focuses on three themes: Algebra vs. Arithmetic, From Hieroglyphics to Place Value, and Understanding Addition through Arithmetic. The second day focuses on subtraction and signed numbers. Multiplication, area, and the Pythagorean Theorem are in Day 3, and Day 4 focuses on division and fractions. In Day 5 we move from arithmetic with fractions to decimal representations mostly of rational numbers. Throughout the course, there is attention paid to comparing arithmetic and algebra, solving equations, understanding place value and the history of counting, understanding of inverse processes, looking at the geometry of multiplication, comparing rational and irrational numbers, and understanding of the one-dimensional geometry of numbers. There is a course outline in Appendix A. More information on MSL is in [8].

The course is a careful study of arithmetic and its connections to algebra and geometry, and it lays a foundation for developing mathematical habits of mind. Our approach is to use what our participants have come to call, “Habits of Mind Problems.” The problems need not be connected to the mathematics being studied but they should be solvable. Ideal problems involve experimentation or consideration of special cases and require careful reasoning and creativity to solve. An example, accompanied by a diagram, is:

> Downtown Mathville is laid out as a 6×6 square grid of streets. Your apartment is located at the southwest corner of downtown Mathville. Your math classroom is located at the northeast corner. You know that it is a 12 block walk to math class and that there is no shorter path. Your curious roommate (we’ll call her Curious Georgia) asks how many different paths (of length 12 blocks—you don’t want to backtrack or go out of your way) you could take to get from your apartment to the math class. It should also be clear that no shorter path exists. Can you solve Curious Georgia’s math problem?

Our participants are not familiar with Pascal’s Triangle or permutations and combinations. After trying to draw all possible paths and getting frustrated, they decide that they should ask a simpler question (e.g., they should find the number of shortest paths to a 2×2 grid, a 2×3 grid, etc.). Most, see a pattern and some realize there is a need to explain why it will continue. Other examples of “habits of mind problems” are available at [1].

Most participants approach ideas concretely. For example, if asked what it means to say that addition of whole numbers is commutative, they give an example such as $3 + 4 = 4 + 3$. The course helps participants go from thinking about numbers to thinking about systems of numbers. By the end of the course, participants might be given the question: Can you argue that between any two real numbers (rational or irrational) you can find both a rational number and an irrational number?

16.1.2 Mathematics as a Second Language: Instructional Style

Our approach to teaching MSL emphasizes active learning. Participants are given a problem to solve, and working in groups they often discover that there are several ways to solve it. The class size is 32–35 and we try to have five instructors in the room at one time. The instructional team often consists of two university mathematicians, two graduate students in mathematics, and one master teacher. If we sense that someone is struggling with the material, we will have one member of the instructional team offer suggestions to help the participant get over an intellectual bump in the road and continue to make progress.

Once a problem has been solved by most participants in the class, we select some to present their solution to the class, often having two or three participants present a solution to show different approaches. Following the presentations, there is often a mini-lecture by the lead instructor explaining what the problem was designed to teach.

Each night participants have three to four hours of homework, with members of our instructional team joining study groups to assist them. We try to provide significant support for our participants and we have found that they are willing to work hard to understand the material. Indeed, if we provide too much help, they admonish us, “Don’t help me now, I need to figure this out on my own.”

On Tuesday of each week, we begin the morning with small group meetings with Staff Advisors, where participants explain the homework they have done the night before, and in which participants can learn that there are often many ways to solve a problem. At the end of the course, participants are given an End-of-Course Assignment and four to six weeks to complete it. The end-of-course assignments are comprehensive take-home finals, and include three components: new mathematics problems covering course content, a “super six” (or seven), and a reflection component in which participants reflect on their growth in mathematical understanding. The “super six” are polished solutions to problems that participants completed during the course of which they are particularly proud or which demonstrate their mathematical growth. Participants include a short description of what the “super” problems represent for their mathematical learning. This reinforces topics studied in MSL and benefits the long-term retention of information.

Assessment in MSL is more formative than summative because our goal is to provide useful feedback to teachers to help them develop their ability to solve problems and to meet our expectations. The active nature of the course provides feedback each time teachers present their work to the class. Each morning, they have a small group discussion of the previous night’s homework with their staff advisor. After submitting their homework, graduate students provide written feedback. Occasionally, we ask teachers to rework and resubmit a solution or meet with them privately to say that the work they submitted is not satisfactory. More typical advice is to encourage teachers to revise solutions for their own benefit so that their work will be useful if looked at later. The “super six” set of problems submitted at the end of the course reinforces the idea that draft solutions should be replaced by final solutions. The End-of-Course assignment is the primary determiner of course grades. We offer written comments on problems without providing a point total. We use a rubric that assesses the teachers’ achievements, their progress in the course, and their commitment to learning. Because this is the first course in our graduate program, once we decide on a grade we offer teachers who are disappointed by it the opportunity to rework solutions to meet our standards and merit a higher grade.

In the first course of the Institute, we try to begin to understand who these participants are as learners of mathematics, what their mathematical strengths and needs are, and how best to meet their needs. Participants begin the Institute with differing mathematical backgrounds and teaching experience. While some enter with a college mathematics major and teach grades 7–12 (including a few who teach calculus), the majority have degrees in elementary education and many may have only taken one or two college mathematics classes. Expectations are rigorous. We have been pleased to learn that high school teachers (in small Nebraska schools, the same person often teaches students in grades 7–12), find the depth of understanding gained in MSL to be beneficial to their teaching, especially when they work with students who are struggling to learn basic mathematics.

16.1.3 Mathematics as a Second Language: Connections to Middle Level Curriculum

Connections that are made between mathematical concepts developed in the course and the middle level curriculum, and the use of concrete examples and representations, serve several purposes. Because our participants use many curricula, the connections we draw to middle level curricula are not specific to any one. The university faculty who have taught MSL are all familiar with the middle level curriculum, so are able to draw connections to MSL topics and problems. The purpose of the course is to deepen participants' understanding of middle level mathematics related to the basic operations of arithmetic. The connections developed among arithmetic, algebraic, and geometric representations, along with developing number sense, are other purposes. Participants often mention they had never before considered the connections among arithmetic, algebraic, and geometric representations of problems. As teachers, they see the value of having several solutions to a problem presented in class; some participants seek to then incorporate this type of instruction in their classrooms. They are excited by "habits of mind problems" and most seek to bring them to their students. They often are surprised how little adaptation of the problems is needed to make the problem solving experiences rich for middle grade students. Participants find the metaphor of numbers as adjectives to be quite powerful and many bring it to their middle level classrooms.

16.2 Experimentation, Conjecture, and Reasoning for Middle Level Teachers[2]

The goal for Experimentation, Conjecture, and Reasoning (ECR) is to continue to develop mathematical understanding and mathematical habits of mind. The course emphasizes the process of doing mathematics. As its name suggests, we want participants to grapple with a problem or a mathematical question, getting their hands dirty as they try special cases, to collect data and to experiment. The first goal is to come up with a conjecture about what is true. Then the participant attempts to solve the problem and provide the careful reasoning that justifies the conjecture or shows it is not true.

The course emphasizes strategic competence, adaptive reasoning, and productive disposition [9]. Steven Dunbar created the course for the M^2 *Institute* and taught it five times, including once when it was offered independent of our grant. For a textbook, the course uses *The Heart of Mathematics* [10]. We are fond of the book and believe it has contributed to the success of the program. When the course has been taught to undergraduates, the instructors chose different material from the book but remained true to the goal of exploring approaches to doing mathematics.

16.2.1 Experimentation, Conjecture, and Reasoning: Description

Experimentation, Conjecture, and Reasoning (ECR) has been offered as a graduate level academic year course in the M^2 *Institute*. After two on-site eight-hour days of instruction, the online instruction is broken into four independent units: Numeric Patterns, Geometry, Counting and Probability, and Conditional Probability. A course outline, instructor notes, and a syllabus are in Appendices B–D. On-site time is spent on problem solving and orienting the participants to the tools and technology of the course. Three of the four online sessions are about three weeks in length and have two sub-sessions, each one to two weeks in length. Geometry is an extended online session, containing five sub-sessions, each about a week in length.

As the director of the Mathematical Association of American's American Mathematics Competitions (AMC) [11] which is housed at UNL, Dunbar used the AMC's extensive resources to enrich the curriculum offered in the course. The AMC has three mathematics competitions designed for grades 6–8, 9–10, and 11–12. Dunbar used questions from previous AMC8 and AMC10 exams to enrich the course by helping participants develop problem solving skills. While AMC items are multiple choice, after the first year, Dunbar adapted the items to become free-response by removing the choices. An AMC8 Exam may have about 10 questions that are accessible to a good student and other questions are interesting and challenging to the most gifted students. The problems are non-standard because they do not correspond to a topic that has just been studied. They combine facts or skills from two or more areas of mathematics

[2] While visiting Cal State Sacramento, Lewis observed a class for future elementary teachers titled Experimentation, Conjecture and Proof. He was impressed by the active learning and thought that the title captured the essence of doing mathematics. He and UNL colleagues designed a version of the course that was offered to undergraduates. Later, Dunbar developed the course for middle level mathematics teachers as part of the Math in the Middle graduate curriculum.

and often require creativity to solve. Therefore, they counter the notion that mathematics is a strict sequence of topics and algorithms to master. The problems are multi-step and multiple-choice, and are drawn from the general U.S. curriculum. Because the AMC8 has no problems that require algebra, our middle level (grade 5–8) teachers see them as relevant to the mathematics that they teach and the questions help them form a vision of what it means to offer challenging courses and curricula to their students. Faculty at other institutions who want to adapt ECR for use on their campus may use Dunbar's bank of questions.

16.2.2 Experimentation, Conjecture, and Reasoning: Instructional Style

ECR is best described as a blended distance-education course. There is a two-day, eight-hour-per-day, on-campus component and a distance education component. For the distance education portion of academic year courses, we use Blackboard, PC NoteTaker, email, and the Adobe Connect communication software. We use the Blackboard classroom content management system for course documents, announcements, information, and participant discussion. It also serves as a vehicle for questions and answers among participants, instructors, and teaching assistants. The course uses the MapleTA online course and homework assessment system (formerly known as EDU) for administering and scoring the bi-weekly quizzes that use the AMC8 questions. Adobe Connect (formerly known as Breeze) is used for group teleconferencing weekly. While there were technology difficulties (not all areas of Nebraska offer more than dial-up modem internet connections), we did not lose any participants because of them. Graduate students serving on the course instructional teams were helpful in resolving technical issues, such as getting webcams and speakers to work, accessing necessary materials on Blackboard, and installing PC NoteTaker correctly. Some participants with slow internet connections would turn off the video features in Adobe Connect, and use only audio in teleconferencing.

Sub-sessions of the course follow similar formats. In most, participants complete four activities. Participants take a Pre-Test on MapleTA that consists of five mathematics problems. The problems are drawn from prior years' AMC8 (or occasionally AMC10) contest problem sets. A sample problem that has been used in ECR is in Figure 16.2.

After completing the Pre-Test, participants read material on a topic, mostly from the textbook. The Geometry and Probability sections have additional reading and study material, available on our website [2]. Participants solve approximately eight problems from the text and submit them for grading by faxing them. Five of the eight problems exercise basic calculation and reasoning skills. Two of them require more calculation and reasoning skills, and persistence. The final problem requires flexibility of thinking and making connections between knowledge and the problem. The sub-session concludes with a Post-Test that uses MapleTA, consisting of another five mathematics problems drawn from prior years' AMC contest problem sets. At the end of the course, the participants complete an End-of-Course Problem Set (i.e., a comprehensive take-home final) on the content of the course. Examples of these are also available on our website [2].

> Ms. Hamilton's eighth-grade class wants to participate in the annual three-person-team basketball tournament. To make the whole team, the class chose n students. In how many ways can three starters be chosen?
>
> [note: n is substituted randomly to be 4, 5, or 6]

Figure 16.2. Sample AMC item (2004 AMC8, problem 4)

Technology plays a central role in the course, especially in the study of geometry. Participants are introduced to GeoGebra software [12] and use it to solve problems in classical Euclidean geometry that are taken from prior AMC contests. Two examples are:

> *Initial task:* In rectangle $ABCD$, $AB = 5$, and $BC = 3$, Points F and G are on $\overline{CD}$ so that $DF = 1$ and $GC = 2$. Lines AF and BG intersect at E. Find the area of $\triangle AEB$.
>
> *Later task:* Spot's doghouse has a regular hexagonal base that measures one yard on each side. He is tethered to a vertex with a two-yard rope. What is the area, in square yards, of the region outside the doghouse that Spot can reach?

For each problem, the participants first do a construction of the problem with GeoGebra. In the doghouse problem, the construction is not trivial, as participants need to consider the portions of the region in which the leash wraps around one side of the doghouse in addition to the portion of the region in which the leash can be straight. Then the participants measure the solution numerically to a couple of decimal places using the algebraic part of GeoGebra [12].

The participants then make a conjecture about the solution, which usually expresses a geometric construct, such as a length or area, using mathematical constants. The participants then solve the problems symbolically, using similarity or area formulas. The pattern of experimenting, conjecturing, and then reasoning is an excellent way to structure the course exercises. The reasoning takes the form of explicitly solving the problem, and expressing the answer analytically or symbolically, rather than a logic proof. By the end of the course, participants are prepared to reason and explain a solution, but do not yet have the tools, experience, or enough examples to construct a mathematical proof.

ECR assessment is based on the work submitted for a portfolio of items from Solidifying Ideas, New Ideas, and Habits of Mind solutions, the Post-Tests, and the Essays on Proof. Feedback offers comments on how solutions can be improved. Teachers can redo assignments. Homework from the sub-sessions total 50% of the final grade, the post-tests count for 25% and the Teaching and Learning Activity and the End-of-Course Problem set each count 10%, and responses and critiques posted on the Blackboard discussion lists count for 5%.

16.2.3 Experimentation, Conjecture, and Reasoning: Connections to Middle Level Curriculum

The course includes Learning and Teaching Project (LTP) that supports the goal of learning mathematics for teaching. It explores learning and teaching from two perspectives.

> How can you embed the mathematics that you learn in your classroom?
> What is the relationship between how well you understand the mathematics and how successful you can be as a teacher?

Prior to the LTP, an assignment asks participants to work two problems on the Fibonacci sequence. Part (a) or *Baby Bunnies* [10, p. 58] is about pairs of baby bunnies who mature and have an additional pair of baby bunnies each month. Part (b) or *Late Bloomers* [10, p. 59] is a variation in which it takes bunnies two months to mature.

For the LTP, teacher participants are asked to design a lesson (or lessons) for their students based on the problems and to write a report about their experience transferring challenging curriculum topics into their classroom. The main components of a LTP include a solution of the mathematics problem, an explanation of how the teacher adapted and taught the problem, copies of student work, a videotape of the lesson, and a written reflection based on the video and the students' work. The written reflection includes how the students did on the problem, how they were assessed, and whether anything was surprising about how students responded. More details about our use of a Learning and Teaching Project can be found on [2], which includes the rubric we use to assess the materials that teachers turn in for the LTP.

We have found that the problems are adaptable to a range of student ages and courses. Participants use various representations in solving the Fibonacci problems. Many use a family tree diagram, sketching out lines between generations of pairs of bunnies. Others use manipulatives and record their data in charts. Some use only charts to represent their solutions. While participants were asked to generalize the solutions, not all could represent a generalization algebraically. Some used words, and others were not able to generalize at all.

In our graduate program the course is taken in the fall following a participant's first summer in our program. Teacher participants naturally return to the style of teaching that they have been comfortable with for many years, although we find that many begin to experiment with different strategies. When teachers teach the lessons in their LTP, what we have asked was very close to some teachers' practice, while for others it was far outside their normal routine. For some, the LTP created a desire to incorporate more group problem solving in their teaching while, for others, it was one of the few times students worked in cooperative groups. For some participants, the LTP was one of the few (or only) times they provided their students with manipulatives to use. For others, cooperative group work and manipulatives were part of daily work.

When looking at the quality of K–12 students' solutions to the LTPs, we noticed a strong correlation between the quality and depth of a teacher's solution and the quality and depth of his or her students' solutions. Students whose teacher was not able to generalize did not themselves attempt any generalizations. Students whose teachers did not represent a solution algebraically did not use an algebraic representation. We believe it is important to strengthen teachers' mathematical knowledge if they are to be successful in offering challenging mathematics to their students.

16.3 Number Theory and Cryptology for Middle Level Teachers[3]

The course Number Theory and Cryptology for Middle Level Teachers (NT&C) focuses on the number theory that is necessary to understand the RSA cryptography algorithm, which is used to send secure information via the Internet. The goals of the course are to introduce participants to basic results of elementary number theory, mathematical definitions, reasoning, and proof, the application of number theory to cryptology, and the connections between number theory and the middle school curriculum. Materials needed and additional information are available online [2] (Appendices E and F have a course outline and instructor notes). The information that is available online is organized into folders as follows.[4]

A. *Course Overview and Information Folder:* contains a course description and explanation of course requirements, including a sample End-of-Course Problem Set.

B. *Course Notebook, Handouts and Solutions Folder:* contains electronic copies of documents needed in the course notebook along with a recommended organization. Examples of Problem Sessions for each section along with homework assignments are in this folder. Solutions to problems are available upon request.

C. *Section Folders:* intended for instructors and contain descriptions of the sections included in the course outline along with suggested strategies.

D. *Helpful Hints and Feedback from Past Instructors Folder:* contains a document with questions and answers provided by past instructors of the course. It includes a form requesting feedback from future instructors.

16.3.1 Number Theory and Cryptology for Middle Level Teachers: Description

Number Theory and Cryptology for Middle Level Teachers has always been offered as a one-week summer course in the M^2 *Institute*. The number theory portion of the course promotes a deep understanding of the integers and their properties in connection with multiplication and division. As theorems that form the foundation of number theory are developed, connections to middle level curricula are emphasized (examples are described below). The proofs which are included in the course are selected so that they are relevant and accessible to middle level teachers.

The cryptology portion of the course includes elementary ciphers that introduce cryptology and lay a foundation for understanding the RSA algorithm and appreciating its significance. The cryptology activities are adaptable as enrichment activities for middle level students. As part of an independent project, two of our participants used a prepublication version of [13] to teach a unit on cryptology to their fifth grade students. The connection of number theory to the RSA encryption algorithm allows the participants to see and understand a relevant, real-world application of mathematics. Understanding the algorithm is an accomplishment in which teacher participants can take pride, as few people understand the mathematics behind internet security. One of the secondary goals of the course is to convince participants they are mathematicians, capable of understanding complex mathematical ideas.

16.3.2 Number Theory and Cryptology for Middle Level Teachers: Instructional Style

The course is designed in an interactive-lecture style (similar to using the Socratic Method) with problem sessions, examples, and cooperative learning activities distributed throughout. It can be adapted to a variety of schedules with 40 hours of contact time. In *Math in the Middle*, it is offered as a week-long, five day course that meets eight hours each day. Homework is assigned each evening, from which selected problems are submitted for daily feedback from the instructional team. Appendix E has the Course Outline.

The primary text is a course notebook containing section outlines, activities, and other handouts, available on our website [2]. Other resources include *The Mathematical Universe* [14], in which Chapters A, F and P are assigned reading, and three NOVA movies, *N is a Number*, *The Proof*, and *Decoding Nazi Secrets*.

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[3]Course developers are Michelle Reeb Homp, Research Assistant Professor, Center for Science, Mathematics and Computer Education, University of Nebraska-Lincoln, and Kristin Pfabe, Associate Professor of Mathematics at Nebraska Wesleyan University. A complete description of NT&C is available on the Math in the Middle website [2]. The course was inspired by Ron Rosier (Georgetown) who created a senior seminar entitled *The Joy of Numbers* and by Judy Walker (a Haimo Award winner) who adapted *The Joy of Numbers* for a freshman honors course for non-majors and for a course for future middle and elementary teachers at the University of Nebraska.

[4]Our website [2] contains complete descriptions of all M^2 courses, and course materials for a majority of the courses.

Assessment consists of daily homework completion grades, participation in problem sessions and presentations conducted in class, and an End-of-Course problem set (or take-home final exam). Participants are encouraged to work in groups when solving the problems but they submit their own work. Assessed problems range from those that are computational, to those that require computation along with written explanation, to straightforward proofs, to proofs that require significant insight and effort. Feedback is designed to help teachers strengthen their mathematical reasoning and writing.

16.3.3 Number Theory and Cryptology for Middle Level Teachers: Connections to Middle Level Curriculum

Connections that are made between number theory and the middle level curriculum serve two purposes. The first is to deepen the participants' understanding of middle level mathematics and the second is to use participants' existing knowledge of middle level mathematics to support the learning of number theory. Because number theory is the study of integers, participants have prior knowledge on which the new number theory results can be built. This aids in the understanding of the new material as well as reinforces its relevancy to the middle level teacher.

An example of how number theory results are built upon participants' prior knowledge is in Linear Diophantine Equations. Diophantine equations are compared with continuous linear equations in y-intercept form ($y = mx + b$), so integer solution pairs to the Diophantine equations are graphically represented as the integral lattice points along the line. This allows for a simple explanation of the general form of all solutions to a Diophantine equation as the fractional form of the slope, m, which most middle school teachers recognize as rise over run (provided the rise and the run are both integers), translates into the formula for all integer solution pairs.

An example of a concrete representation (which helps to keep the content within the grasp of the middle level teacher) occurs in modular arithmetic. The section begins with the question:

Today is Wednesday. What day of the week will it be in 11 days? in 95 days? in 320772 days?

To see congruence classes and modular arithmetic in this problem, boxes are drawn for each day of the week along with numbers that fit into each box. A comparison is made to general cases, in which each of the boxes has a name (the least residue element modulo m determines the names of the boxes) and there are infinitely many numbers contained in each. For any modulus each integer has one box into which it fits. Thus, solving congruences and constructing proofs involving modular arithmetic can be compared with determining the name of the box containing the solutions, or analyzing the types of numbers contained in a given box. This has been a tremendous aid in developing middle level teachers' understanding of modular arithmetic.

Additional examples where number theory content is connected to the middle level curriculum are in the following problems, which are included in some form in the course materials [2]:

1. Divisibility rules for $d = 3, 4, 8, 9$.
   Write the divisibility rules, as you would explain them to your students.
   How do you write that an integer n is divisible by d using congruence statements?
   How does the statement $n \equiv 0 \bmod d$ translate into a divisibility rule?
   Determine a divisibility rule for 11 and explain why it works.

2. As a class, define what it means for an integer to be even or odd. Then prove that the product of two odd integers is odd, and that the product of an even integer times an odd integer is even.

This activity demonstrates (in a tangible way) the value of definitions and their role in mathematical proof.

While it may seem that asking teachers to prove lemmas about even and odd integers is not at the same level as RSA encryption, teachers are not fluent in formal proofs and need experiences proving easy statements. Many teachers have memorized facts about products of even and odd integers, but few of them can explain why the products work the way they do, much less give a proof. The RSA encryption algorithm requires several number theoretic results whose proofs are significantly more complex than those about products of even and odd integers. The Unique Factorization Theorem and Fermat's Little Theorem are used in the justification of the RSA algorithm but rather than lead the teachers through rigorous proofs (which they will not need to teach), we give them only the key ideas in the proof, or show them examples that provide evidence to make the theorems meaningful and believable.

16.4 Mathematical Growth Across Courses

At the beginning of the Institute, participants are not able to communicate mathematical reasoning in writing. Figure 16.3 has an example of a fifth grade teacher's work on the homework problem that was assigned about halfway through our first course, Mathematics as a Second Language: *Show that* $(7, 24, 25)$ *is a Pythagorean triple. Can you find other Pythagorean triples?* The participant simply asserted that $(7, 24, 25)$ is a Pythagorean triple, and she found five additional Pythagorean triples. She did not notice or comment on the pattern (a is always odd and c is always one more than b), nor did she search for a formula that would generate additional Pythagorean triples. Other than the labeling of a, b, and c, there were no words in her solution.

> There is a jug of wine and a kettle of tea. A spoonful of tea is taken from the kettle and poured into the jug. The mixture is thoroughly stirred and a spoonful of the mixture is taken from the jug and poured into the kettle. Is there more tea in the jug or more wine in the kettle?[5]

$(7, 24, 25)$

$7^2 + 24^2 = 25^2$

$a =$ leg

$b =$ longest leg

$c =$ hyp.

a	b	c
3	4	5
5	12	13
7	24	25
9	40	41
11	60	61
13	84	85

Figure 16.3. Participant's solution to the Pythagorean triple problem

On this same homework set, the same participant drew a picture of the situation modeled by the following problem and wrote "I really have no clue."

As an example of the growth seen in so many participants, Figure 16.4 has a solution given by the same student to a problem that appeared four math courses later on the NT&C End-of-Course Problem Set.

While a participant's solution should work with a general ISBN code rather than a particular one, this participant recognizes that she can analyze the problem using a specific case. She then extends her argument to the general case by considering the largest possible differences between a correct and an incorrect ISBN code (in which the prescribed adjacent digits are transposed) that can be obtained using the error detecting scheme. The generalization allows the participant to conclude that the difference between a correct and incorrect code in which digits a_4 and a_5 are transposed is not divisible by 11 and will therefore be detected by the scheme.

Unlike the teacher's earlier work, which did not use any words, her solution to this problem is predominantly text in which she explains her thinking and justifies her conclusions. Though she begins the problem by examining a specific case, she is able to observe and generalize a pattern (the possible differences between correct and incorrect ISBN codes), and then uses it to draw a conclusion and justify her solution.

Another example of the mathematical growth middle level teachers can make as they progress through our Masters' program can be seen in The Triangle Game, a problem that we have assigned in our first course, Mathematics as a Second Language. For The Triangle Game, which was adapted from a problem in *TriMathlon* [15], participants are asked to place the numbers 1, 2, 3, 4, 5, and 6 on the vertices and midpoints of an equilateral triangle so that the sum along each side (two vertices and a midpoint) is the same. Participants are asked whether there is more than one solution and, if so, to find all solutions, providing an argument for why a number is the smallest or largest Side Sum. Then, the question asks for generalizations of the game and solutions to a generalized version.

This problem is accessible to our participants and parts of it could be assigned to fourth grade students. Using trial and error, participants can find solutions such as $\{1, 6, 2, 4, 3, 5\}$ or $\{6, 1, 5, 3, 4, 2\}$ where the sequence of numbers starts at a vertex and the numbers are placed on the points clockwise. The problem becomes more challenging if participants provide the reasoning that 9 is the smallest and 12 is the largest Side Sum or they try to define what it means for two solutions to be equal if they are to find all solutions. The problem becomes very challenging if a participant invents The Square Game or The Pentagon Game and tries to find all solutions.

[5] The inclusion of problems like this is one reason that we are fond of the Mathematics as a Second Language text that we use in MSL.

Problem: International Standard Book Number (ISBN)

An ISBN is a 10-digit code $a_1a_2a_3a_4a_5a_6a_7a_8a_9a_{10}$that identifies a book. The check digit a_{10} is chosen so that the number

$$10a_1 + 9a_2 + 8a_3 + 7a_4 + 6a_5 + 5a_6 + 4a_7 + 3a_8 + 2a_9 + a_{10}$$

is a multiple of 11. Show that the ISBN encoding scheme detects all transposition errors in which the distinct digits a_4 and a_5 are swapped.

Written Solution: This property of the ISBN code (being divisible by 11) should work for any book. Thus, I will use as an example *Guess How Much I Love You*, written by Sam McBratney. The ISBN for this book is 1-56402-473-3. Therefore,

$$\begin{array}{ccccccccccccccccccccc} a_1 & & a_2 & & a_3 & & a_4 & & a_5 & & a_6 & & a_7 & & a_8 & & a_9 & & a_{10} & & \\ 10(1) & + & 9(5) & + & 8(6) & + & 7(4) & + & 6(0) & + & 5(2) & + & 4(4) & + & 3(7) & + & 2(3) & + & 3 & = & 187 \end{array}$$

and 187 should be divisible by 11, and indeed it is. $187 \div 11 = 17$. This is a correct ISBN code.

However, if the a_4 and a_5 digits were switched, the ISBN code would be 1-56042-473-3, and would not satisfy the equation to be divisible by 11.

$$10(1) + 9(5) + 8(6) + 7(0) + 6(4) + 5(2) + 4(4) + 3(7) + 2(3) + 3 = 183$$

and $183 \div 11 = 16.66666\ldots$. Therefore, 183 is not divisible by 11, and the ISBN error would be detected. This will be true for any ISBN code, since the switch of the numbers is changing the total value. Furthermore, any number change will not result in a total [error] equal to 11. There are a total of 10 digits from 0–9. If you look at the change in the value to the entire number

$$7(0) + 6(9) \to 7(9) + 6(0)$$
$$54 \to 63 \qquad \text{difference of } 9$$

or

$$7(0) + 6(9) \to 7(1) + 6(0)$$
$$6 \to 7 \qquad \text{difference of } 1.$$

The smallest change to the number would be 1, and the greatest change to the number would be 9. Any other number swaps would result in a change between 1 and 9. Thus, a change in 1 or a change in 9 would not make the number divisible by 11.

Figure 16.4. ISBN problem and participant solution

Some of the best examples of M^2 *Institute* participants' mathematical growth are in their action research papers and mathematics expository papers [1]. In one of the expository papers, an elementary-certified fifth grade teacher was asked to revisit The Triangle Game and see if she could generalize the problem by working with squares, pentagons, and other polygons [15].

> Take a regular, n-sided polygon (i.e., a regular n-gon) and the set of numbers, $\{1, 2, 3, \ldots, (2n-2), (2n-1), 2n\}$. Place a dot at each vertex of the polygon and at the midpoint of each side of the polygon. Take the numbers and place one number beside each dot. A *side sum* is the sum of the number assigned to any midpoint plus the numbers assigned to the vertices on either side of the midpoint. A *solution* to the game is any polygon with numbers assigned to each dot for which all side sums are equal, i.e., for which you have equal side sums. When we want to be specific about the number of sides, we will refer to The Triangle Game, The Square Game, etc. The most general problem we might state is, "Find all solutions to The Polygon Game." Your assignment is to learn about and report on what you learn about The Polygon Game.

While a complete solution to The Polygon Game was far beyond the teacher's reach (it is an open question), we wanted the teacher to think about the importance of careful definition, to use algebra or geometry to solve problems, to reason carefully and to communicate mathematics to her peers. We hoped that she would find all solutions to The Square Game and some for The Pentagon Game and The Hexagon Game.

To our delight, the teacher did this and more. The paper is a remarkable product for a middle level teacher and many readers of it may find it hard to believe that it was produced by a fifth-grade teacher. The teacher noted that for any n-gon, given one equal side sum solution there is a dual solution given by replacing a value i by $(2n+1) - i$ at a vertex or midpoint. By combining an algebraic approach to reduce the number of possible solutions for a polygon to a manageable number, with hard work and a productive disposition, she found all solutions to The Square Game, The

Pentagon Game, and The Hexagon Game. She showed that The Pentagon Game has solutions for equal side sum 14 (and its dual, 19) but not for 15 or 18. For equal side sum 16 (and 17), she found two distinct solutions that are not transformations of each other. She found a pattern for minimum and maximum equal side sum solutions for n-gons up to the octagon.

Two other achievements stand out. The teacher used modular arithmetic to show that for an n-gon with n odd, there is an equal side sum solution $S = 5(n + 3)/2$. The solution is found by beginning with 1 on the first vertex, then choosing each subsequent (clockwise) value for a vertex by adding $(n - 1)/2$ and reducing modulo n, moving around the polygon consecutively (using n rather than 0 for the appropriate vertex). This process assigns the numbers $1, 2, \ldots, n$ to the vertices. The midpoints are assigned by placing $2n$ between 1 and $(n+1)/2$ and then moving around the polygon counterclockwise, reducing the value by one at each point. As an example, the solution for a heptagon is $\{1, 14, 4, 8, 7, 9, 3, 10, 6, 11, 2, 12, 5, 13\}$. Finally, the teacher made a conjecture that for any n-gon, there is a solution $S = 3n + 1$ and provided examples for $n = 3, 4, 5, 6, 8$ and 10.

This example, when compared with examples of participants' work on The Triangle Game from our first course, illustrates the intellectual growth and increase of mathematical capacity that we regularly see in *Math in the Middle Institute* participants. Understanding how the mathematical knowledge acquired in the *Institute* translates into more thoughtful teaching can be seen by reading teachers' action research papers. We have posted both teachers' action research papers and their expository mathematics papers on [1].

16.5 Conclusion

We have given descriptions of three courses that are offered as part of the *Math in the Middle Institute*. While there is not enough space to include information to replicate them, more materials are available on [2] and can be had by contacting the course designers (`msquared@unl.edu`). We hope these descriptions and materials are helpful in creating similar courses to aid middle level mathematics teachers in deepening their knowledge of mathematics, developing the habits of mind of mathematical thinkers, and using them in their classrooms. We found that developing participants' habits of mind as mathematical thinkers is a unifying theme throughout the *Math in the Middle Institute* courses. Such content rich courses, intended to develop participants' mathematical and pedagogical knowledge, have served to build participants' capacities as teachers and prepared them to be leaders among their peers.

The Learning and Teaching Project is valuable as an assignment in encouraging participants to apply the mathematics they have learned to their teaching. Many participants think that the material is not adaptable to the classroom or will interfere with required teaching material. But most find it to be a valuable teaching experience and almost all learn something about themselves, their students, and their own teaching. Discussion of three participants' experiences with the LTP from Experimentation, Conjecture, and Reasoning is given in three case studies of participants from the second cohort [16]. Because of our experiences with the LTP in the *Math in the Middle Institute*, Learning and Teaching Projects are extended into our current work with K–3 and secondary mathematics teachers. The process of teacher change is difficult (e.g., [17, 18]), but we have found Learning and Teaching Projects to be valuable in causing participants to initiate and reflect on changes in their teaching.

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Appendix A
Mathematics as a Second Language Course Outline

Day 1: Arithmetic, Place Value, Addition
Algebra vs. Arithmetic
From Hieroglyphics to Place Value
Understanding Addition Through Arithmetic

Day 2: Subtraction and Signed Numbers

Day 3: Multiplication, Area, Pythagorean Theorem

Day 4: Division and Fractions

Day 5: Rational Numbers
Arithmetic with Fractions and Decimals
Decimal representations of Fractions
Systems of Numbers (Rational Numbers)

Emphasize throughout:
a comparison of arithmetic and algebra

solving equations
an understanding of place value and the history of counting
an understanding of inverse processes
the geometry of multiplication
different ways to think about division
a comparison of rational and irrational numbers
an understanding of the one-dimensional geometry of numbers

Appendix B
Experimentation, Conjecture, and Reasoning Course Outline

Beginning Two-Day Workshop

To experience ideas in as many ways as possible, to take ideas from one domain and explore them in another, and understand simple things deeply.

Session A: Numeric Patterns
To learn to look for patterns, find hidden, underlying structure, and understand simple things deeply.
Fibonacci Numbers I [10, pp. 49–63]
Fibonacci Numbers II (Additional material)

Session B: Geometry
To experience ideas in as many ways as possible, to take ideas from one domain and explore them in another and understand simple things deeply.
Pythagorean Theorem [10, pp. 208–217]
The Golden Rectangle [10, pp. 232–247]
The Platonic Solids [10, pp. 269–288]
GeoGebra Session I (Additional Material)
GeoGebra Session II (Additional Material)

Session C: Counting and Probability
To understand simple clear cases let us apply principles that we can apply widely.
General Counting [10, pp. 554–567]
Probability [10, pp. 523–540]

Session D: Conditional Probability
To make it quantitative and beware of unintended consequences.
Conditional Probability (Additional Material)
Bayes Theorem [10, pp. 645–662]

End of Course Review
A comprehensive look at the course.
End of course review

Appendix C
Experimentation, Conjecture, and Reasoning Instructor Notes

Math 804T: Experimentation, Conjecture and Reasoning
Instructor: Steve Dunbar
Text: *The Heart of Mathematics*, M. Starbird and E. Burger

The course syllabus can be viewed online at

`www.math.unl.edu/~sdunbar1/ExperimentationCR/experimentation.shtml`.

The site contains the syllabus in both HTML and PDF format and includes lesson plans that were used for organizing and partially presenting the information.

The course was primarily taught online, but included an initial on-site component. The textbook, *The Heart of Mathematics*, was the primary resource for reading, course topics, and exercises. The course concentrated primarily on problem-solving with some emphasis on mathematical exposition and theorems. I supplemented the problems in the text with online MapleTA-format homework-quizzes on the same general topic drawn from past year's copies of the American Mathematics Competitions AMC 8 and AMC 10 contests. The course did not deviate from the written syllabus.

I recommend that the final section of the course on Data and Distributions should be changed to a section with increased emphasis on conditional probability and Bayes' Theorem. The students found these topics to be challenging and they were not adequately explained in the course of developing the material on Data and Distributions. Another minor change in the course is to expand and slow down the development of the section on Counting. Again, the topics were not completely developed in the text.

Solutions to problems assigned from the text can be located in its accompanying supplementary instructor's manual *Instructor Resources and Adjunct Guide* for the second edition by E. Burger, M. Starbird, and D. Bergstrand. It is useful to instructors by providing ideas for additional activities, indications of difficult points, and lesson plans.

In each approximately week-long sub-section of the course, individual students worked eight assigned problems from the end of the corresponding chapter. The assigned problems are listed on the course syllabus. Five of the assigned problems were from the "Solidifying Ideas" section of problems in the text. They exercise calculation and estimation skills and encourage persistence. Two of the assigned problems were from the "New Ideas" section of problems in the text. They are designed to exercise students' disposition to analyze all situations as well as the self-efficacy to believe they can make progress toward a solution. I also assigned a "Habits of Mind" problem from the "Further Challenges" section of the text. These problems exercise students' flexibility of thinking and make connections between student knowledge and the problem.

Individual students submitted problem sets every week to two weeks by faxing the solutions. The students discussed online in the Blackboard discussion area many of them, especially the most challenging ones, but there were no regular formal presentations of homework problems. At the next courses' beginning workshop, students presented problems from the End-Of-Course problem set.

In each approximately week-long section of the course, students completed two instances of a quiz presented by the online MapleTA web-testing system. The Pre-Test consisted of five mathematics problems drawn from prior years' AMC contest problem sets related to the topic of the sub-session. The Pre-Test was not for grading, but it was intended to assess where students started as problem solvers in the topic. The problems exercise calculation and encourage persistence. At the end of the week-long session, students completed a Post-Test, also five mathematics problems drawn from prior years' AMC contest problem sets. The Post-Test is drawn from the same database of problems as the Pre-Test. Students could take the Post-Test until they reached mastery of the topic.

The problems assigned from the text vary in difficulty, and some are quite challenging. In some cases the mathematical facts necessary to solve or finish the problems are not presented in the text. As a result, I needed to supplement the text with additional facts about the similarity of triangles, medians of triangles, and so on. Examples are Problems 18 and 19 about side lengths of dual solids from the chapter on Platonic Solids. I suggest that future instructors work all problems first and supplement their instruction with appropriate mathematical background.

In the first section on Geometry, students needed a good deal of specific information about triangle geometry. They needed information about congruent and similar triangles and theorems about when triangles are similar and congruent. Students also needed facts about medians, angle bisectors, and altitudes of a triangle and their intersection points. They had a reasonable grasp of area and perimeter formulas for common geometric figures. Students were reasonably familiar with the Pythagorean Theorem before studying the section on the Pythagorean Theorem, but problems involving similarity seemed to be unfamiliar. Facts about the side-length ratios of 45-45-90 and 30-60-90 triangles are necessary many places in the course, especially in the section on Geometry, but the students have only a weak understanding of them.

In the section on Fibonacci numbers, some students had difficulty with subscript notation for sequences and the function concept. Students also had trouble with irrational numbers and the definition of irrationality, which arose in connection with the Golden Ratio.

In the sections on Counting, Probability, Data and Distributions, students had good background understanding but frequently needed additional algebraic skills. Students had some confusion in counting about when to add and when to multiply, but this was addressed with instruction and additional materials. There was variation in readiness to use Venn diagrams and tree diagrams to aid in counting.

Students need more facility in expanding algebraic expressions and in solving quadratic equations. Both appear regularly in all sections of the course.

For the reasoning part of the course, students lack a clear understanding of the converse and contrapositive of a theorem, and recognizing and using syllogisms to derive new statements. They have a shaky understanding of “for all…” and “there exists…” statements and reasoning from them. Finding and using a counterexample to disprove possible theorems is also a skill the students have not developed. It is thus impossible to bring the students to the level of writing even simple mathematical proofs, so I worked toward other levels of reasoning about mathematical issues. I emphasized a level of practical proof represented by addressing mathematical misunderstanding by peers.

The background knowledge issues may have been exaggerated because for Cohort 1, the course immediately followed the first course, Mathematics as a Second Language. For Cohort 2, the Experimentation course is the fourth course in the sequence following Functions, Algebra and Geometry Course (Math 802T). In any case, I attempted to build background knowledge by having a review session of geometry and algebra facts and posting the review materials on the course website.

The students in the course are hard-working and dedicated to learning the material in the course. All the students will work at each problem, even doggedly, and will often assist each other. I do not recall ever having a course before where virtually every student in the course worked hard on *every* problem and turned in *every* assignment on time.

The content of the course is ideal for middle-school teachers because it covers mathematical problem solving from geometry, Fibonacci sequences, sequences in general, counting and elementary combinatorics, and probability theory. The topics lend themselves well to experimentation and conjecture, but the students are often perplexed about trying a number of test cases to determine the truth or falsity of a conjecture. The students often expect a procedure or algorithm to be available for each problem. The students have a good-natured approach to experimentation, and are willing to do it, but the discovery of an approach to a problem not covered in the text is difficult for them. Structure in a problem helps. For example, the construction of tables of numbers of faces, edges, and vertices for polyhedra suggests Euler’s formula holds in all cases, not just for regular polyhedra. Construction of systematic tables leads to conjectures about identities for Fibonacci numbers. The book helps to create the right environment for these problems since many of the New Ideas and Habits of Mind (Further Challenges) problems develop ideas not explained fully in the text. Most of these problems have some and additional structure is a good idea.

The students especially liked experimentation that involved physical manipulatives and concepts that could be immediately used in their classroom.

Duties for assistant instructors included:

- answering daily questions arising on Blackboard discussion lists,
- leading small group problem sessions, sometimes in person and sometimes via computer or telephone,
- grading weekly homework (and provide feedback),
- assisting with creating problems for the End-Of-Course Problem set, weekly essays, MapleTA problem databases, and
- writing and distributing detailed solutions to weekly problem sets.

Provide assistant instructors with a copy of the course book Instructor's Manual and copies of solutions to all relevant problems (Evening Homework problems, Problem Session problems, problems corresponding to reading assignments, etc.) prior to the beginning of the course.

For the beginning workshop, we needed a VCR (for the video "Platonic Solids" and MathVantage videos) and cardstock, scissors, and rulers (for the triangle angles activity). For the other activities at the beginning workshop, we used the manipulatives and materials that are included with the course kit that is included with the Berger and Starbird book. It includes straws and pipe cleaners for constructing platonic solids, a card triangle kit for demonstrating the Pythagorean theorem, and additional manipulatives for other parts of the course. At the beginning workshop, we used the TI-84 calculator for collecting and analyzing data. For the demonstrations, we needed a calculator projection system. We also needed access to a computer with a projection system to demonstrate using the course web-pages and we needed access to a computer lab (with Internet access and with login information for all the students) to demonstrate and practice logging in to and using the MapleTA test administration system.

Provide students with a copy of the supplementary *Instructor Resources and Adjunct Guide* for the second edition by E. Burger, M. Starbird, and D. Bergstrand.

With the course being taught by distance education, it is challenging to maintain enthusiasm and contact with the students. It is difficult to know when students are behind or do not understand something, or when they are simply quietly working alone. The lack of personal interaction and body language clues about student involvement is challenging to maintaining course pace and level. Describing detailed, difficult, or deep mathematical information over a channel like email or Blackboard discussion threads is hard to do efficiently and effectively.

Students found writing about mathematical ideas in the Essays on Proof not something that they felt very comfortable with.

Communication of concepts in geometry is difficult over the Blackboard system, particularly in the three-dimensional platonic solids chapter where visualization is critical.

The difficulty of the problems is uneven, and some problems in the New Ideas sections are more difficult than the "Habits of Mind" problems. The difficulty of the problems from Chapter to Chapter is uneven, and the problems in some chapters are easier than the problems in other chapters. This caused some frustration for the students, who felt that they did not know what to expect.

A personalized grade letter to each student is a good way to end the course with individualized assessment.

Appendix D
Experimentation, Conjecture and Reasoning Syllabus

Prerequisites: Math 800T or Math 802T in the Math in the Middle Program or equivalent.
Required Text: *The Heart of Mathematics*, Second Edition, by E. Burger and M. Starbird, Key College Publishing, 2005 (ISBN 1-931914-41-9).

Strategic Objectives for the Course

The goal for this course is to bring you to a new level of your mathematical habits of mind. A productive mathematical thinker has skills and knowledge to experiment, conjecture, reason, and solve problems. "Mathematical habits of mind" means understanding and using the skills and knowledge. Mathematical habits of mind are marked by flexibility of thinking and the belief that precise exposition of solutions is important. Flexibility of thinking includes using indirect arguments and making connections between knowledge the mathematical thinker possesses and the problem being considered.

A person with well developed habits of mind, when presented with a problem to solve, will collect information, assess it, find multiple pathways to the answer, and explain the solution to others. Although a complete mathematical toolbox includes algorithms, a person with well developed habits of mind knows why algorithms work and when they will be effective. Mathematical habits of mind are also marked by ease of calculation and estimation and persistence in trying to solve problems. A person with well developed habits of mind analyzes all situations and has the self-efficacy to believe that he or she can make progress toward a solution. Such a person also engages in metacognition by monitoring and reflecting on the processes of reasoning, conjecturing, proving, and problem solving.

This is a condensed Math in the Middle statement consistent with the five strands from the National Academy of Sciences report Adding It Up [12]:

1. Conceptual Understanding
2. Adaptive Reasoning
3. Procedural Fluency
4. Mathematical Problem Solving Competence
5. Productive Disposition.

For further details about these strands, consult [12].

From another view, that of the Principles and Standards for School Mathematics [19], the course's strategic objectives focus on problem solving, reasoning, and proof and the associated processes:

1. Problem Solving
2. Reasoning and Proof
3. Communication
4. Connections
5. Representation

Three experiences teaching the course convince me that its content and approaches meet the objectives. To achieve the strategic goals of the course, I have divided the semester into four independent sessions. Each session will be 3–4 weeks in length, and will have two or three sub-sessions that will each take 1 to 2 weeks. In sub-sessions you will have a pattern of activities to complete that relate to the goals of the course:

1. Read the section of the textbook to prepare for reasoning, conjecturing, proving, and solving. The reading will give you precise definitions and algorithms.
2. Complete the Pre-Test on the Web. The Pre-Test consists of five mathematics problems drawn from AMC contest problem sets related to the topic of the sub-session. The Pre-Test is not for grading, it is to assess where you start as a problem solver. Trying the problems will exercise your calculation and reasoning skills and your persistence in pursuing solutions.
3. Complete the five assigned "Solidifying Ideas" problems in the text and submit them for grading. You will later include these in your divided portfolio provided by the Center office. The problems will exercise your calculation and reasoning skills and your persistence in pursuing solutions.
4. Complete the two "New Ideas" problems in the text and submit solutions for grading and ultimate inclusion in your portfolio. The problems will develop your disposition to analyze all situations and the self-efficacy to believe that you can make progress toward a solution.
5. Complete the assigned "Habits of Mind" problem from the Further Challenges section of the text and submit a solution for grading and ultimate inclusion in your portfolio. The problems will exercise your flexibility of thinking and making connections between knowledge you have and the problem being considered. A person with well developed habits of mind when presented with a problem to solve will collect information, assess it, find multiple pathways to the solution, and be able to explain the solution clearly to others.
6. A Post-Test, another five mathematics problems drawn from AMC contest problem sets. It is drawn from the same database as the Pre-Test. You can take the Post-Test until you have reached mastery of the topic. The Post-Test measures your progress in understanding and applying the mathematical habits of mind. Session Reflections: Choose a problem from each of Session A, B, C, and D and write a one-page reflection about it. Select a problem that you had some mathematical insight on while you were working through the solution or describe a problem you worked that you shared with a colleague, friend, student, small group of students, or your entire class. Describe what happened in the interaction. Reflect on what mathematical ideas you learned from working on each problem.

Teaching and Learning Project

This project explores learning and teaching from two perspectives. How can you use the mathematics that you learn in your classroom? Does there exist a relationship between how well you understand the mathematics and how successful you can be as a teacher of it? For your Learning and Teaching Project this semester we would like everyone to use problems 6 (Baby Bunnies) on page 58 and 9 (Late Bloomers) on page 59 as the basis for your lesson. We found these topics are adaptable to a wide range of student ages and courses. You should make the judgment where you want to start with your students, what you expect from them, and how far you can push them. More details about what to do in the Teaching and Learning Project on the page "Math 804T: Experimentation, Conjecture, and Reasoning, Math in the Middle, Learning and Teaching Project" in your portfolio. The materials you turn in for your Learning and Teaching Project will be assessed according to the rubric outlined on the pages labeled "Assessment Rubric for Written Reflections on Learning and Teaching Project" in your portfolio.

Detailed Course Schedule

Session A: Numeric Patterns To learn to look for patterns, find hidden, underlying structure and understand simple things deeply.

1. Fibonacci Numbers I (Chapter 2.2, pp. 49–63)
2. Fibonacci Numbers II (Additional material)

Session B: Geometry To experience ideas in as many ways as possible, to take ideas from one domain and explore them in another and understand simple things deeply.

1. Pythagorean Theorem (Chapter 4.1, pp. 208–217)
2. The Golden Rectangle (Chapter 4.3, pp. 232–247)
3. The Platonic Solids (Chapter 4.5, pp. 269–288)
4. GeoGebra Session I (Additional Material)
5. GeoGebra Session II (Additional Material)

Session C: Counting and Probability Simple clear cases let us discover principles that we can apply widely.

1. General Counting (Chapter 7.4, pp. 554–567)
2. Probability (Chapter 7.2, pp. 523–540)

Session D: Conditional Probability "Make it Quantitative" and "Beware of unintended consequences."

1. Conditional Probability (Additional Material)
2. Bayes Theorem (Chapter 8.2, pp. 645–662)

End of Course Review A comprehensive look at the course.

1. End of course review.

Grading

We ask that you have a well-defined sense of professionalism, that you always give assignments your best effort, and that you develop a sense of responsibility to your educational community (school, district, ESU). We ask that you give this course your best effort and that you exhibit a persistent desire to learn. It is our goal to provide you with support and we encourage you to ask for our assistance when it is needed. We are confident in your success.

Grading will be based on the work submitted for the Portfolio, the approximately weekly Solidifying Ideas, New Ideas, and Habits of Mind solutions, the Post-Tests, and the Essays on Proof from each section. Reading and writing will serve as the basis for some of your learning in this course. When you are asked to read or write something, the following is expected: Read so you can be an active participant in discussions of text. Write so you can respond to posed questions clearly and grammatically. The course instructional team will offer comments that will assist you in learning to solve each problem assessed. As you are all self-motivated, we recognize that you are putting forth your best effort. Our goal is to offer comments on how your solution can be improved. Your three-ring binder has a divider for your homework solutions for each section. The binder preserves your work and gives a sense of completeness. Feel free to include personal reflections on the mathematical skills and concepts that are clear to you and those that may still be confusing. You may keep all your work on a problem or only your final solution of a problem. Grading Scale: Each of the 10 or 11 sub-sessions has eight problems that will be graded on a scale of "1" for a good or satisfactory solution, "2" for a better or noteworthy solution, and "3" for an excellent or exemplary solution. Therefore each of the 10 or 11 sub-sessions will be worth up to 24 points total, and each of the homework sub-sessions will count for 5% of your grade for a total of 50%. The seven Post-Tests, one from most sub-sessions, will count about 3.5% each, making another 25% of the grade total. The Learning and Teaching Project and the End-of-Course Problem set will count about 10% and your responses and critiques posted on the Blackboard discussion lists will count about 5% of your grade.

Late Homework, and Re-Do Policy

Homework should be faxed to the Center Office on the due date. We understand that circumstances can arise in busy lives, but we need to complete our grading in a timely fashion, so contact us for special consideration if it is really necessary. Any problem on which you receive a grade of "1" can be re-done, and resubmitted up to one week after the homework is returned for a re-grade. We would rather that you learn to do the problems correctly and completely than to give punitive grades.

Grading Policy

Part of our responsibility is the assessment of each participant's achievement in each course. We recognize that teacher-participants are drawn from different grade levels, have different certifications to teach mathematics, and have different educational backgrounds. Thus, we believe it is appropriate to have an assessment system that values effort, teamwork, progress in learning mathematics, and the development of knowledge, skills, and dispositions for teaching and inquiry.

Expectations and typical characteristics of achievement at each grade level are

A+ The grade of A+ is honorific and will be fairly rare. It is evidence that the instructors have special admiration for the participant's achievements in the course.

A Achievement beyond the level needed to earn the grade of A−. Especially important will be evidence that the teacher has an excellent command of the mathematics studied in the course, the ability to transfer mathematics learned into the teacher's classroom, and far-reaching progress in developing the knowledge, skills and dispositions of educational inquiry.

A− Achievement beyond the level needed to earn a grade of B+. Especially important will be evidence that the teacher has a good command of the mathematics studied in the course, the ability to transfer mathematics learned into the teachers classroom, and progress in developing the knowledge, skills, and dispositions of educational inquiry.

B+ Regular class attendance, active participation, assignments submitted on-time, supportive and helpful to peers, admirable effort to complete assignments, evidence of good progress in learning mathematics and developing knowledge, skills, and dispositions of educational inquiry.

B Regular class attendance, reasonable participation, most assignments submitted on-time, cooperative with peers, reasonable effort to complete assignments, to learn mathematics and to strengthen knowledge, skills, and dispositions of educational inquiry.

B− A grade below B shows that the instructors do not believe that the teacher made a reasonable effort to use the opportunity provided by the Math in the Middle Institute to develop into a stronger teacher. Evidence may include one or more of the following traits: attendance problems, uncooperative behavior, failure to submit assignments, tendency to submit assignments late, or performance on assignments that indicate an inadequate effort to learn mathematics and to develop knowledge, skills, and dispositions of educational inquiry.

Appendix E
Number Theory and Cryptology for Middle Level Teachers
Course Outline

Section 1: Integers and Divisibility
Application: Divisibility rules, ISBN's
Exploration: Even and Odd Numbers

Section 2: Primes and Factorization
Introduction to Cryptology

Section 3: Linear Diophantine Equations
Other Number Bases

Section 4: Congruence
RSA Public Key Cryptography

Section 5: Linear Congruence Equations

Section 6: Fermat's and Wilson's Theorems

Section 7: Euler Phi-Function

Appendix F
Number Theory and Cryptology for Middle Level Teachers
Instructor Notes

The Course Outline should not initially be included in the notebook, but distributed to participants during the course as the appropriate sections are completed. They should initially be withheld from the notebook since participants are led to make conjectures to develop many of the ideas and theorems introduced in the course (the outline would give away too many). The outline should be distributed as the corresponding sections are completed during the course so that participants have a complete list of theorems and properties as reference. (The outline should be copied on one side only so that pages can be distributed as soon as possible at the completion of a section.)

Hard copies of solutions to problem session problems, evening homework problems, and problems associated with the reading assignments should be included in the instructor's notebook, but not the participants' notebooks.

Evening problems from the appropriate section/s on the "Evening Homework Problems" list (see notebook), along with a reading assignment and related problems were assigned for homework as indicated on the daily schedule. Participants were required to submit a written solution to two of the problems each day and they were to consider one additional problem to discuss in small groups. The remaining problems served as optional presentation problems, solutions to which participants had the opportunity to present to the class the next morning. The problems that were assigned and discussed can be found with the course notebook handouts.

Topics covered in the Functions, Algebra and Geometry Course (Math 802T) were beneficial in preparing participants to succeed in Number Theory and Cryptology. The multiplication tables for modular arithmetic (in mod 11 and in mod 12) were useful for solving linear congruences, and were referred to more than once. In addition to number theory topics, the review of linear equations, and slopes and graphs were beneficial to participants during the discussion of linear Diophantine equations. Discussions of prime numbers, prime factorizations and greatest common divisors in Math 802T were also beneficial.

For many of the topics, be concrete when explaining them. Suggestions on how to do this are included in the course outline. One term that is not included in the outline but which comes up repeatedly throughout the course is "counterexample." To make it concrete the following explanation was given:

> Suppose I made the statement that "All frogs are green," and to prove this to you I show you a green frog, then another green frog, and then hundreds more green frogs. Does this prove that all frogs are green? What would it take to disprove my assertion that "All frogs are green"?
>
> One possible answer: "It would take *only one blue frog*."

From then on, participants would identify the concept of a counterexample with the concrete notion of a blue frog.

Assistant instructors were asked to lead a brief discussion or present a lesson during the course. Some of the options from which they had to choose include:

- lead class discussion of solutions from one of the problem sessions
- present the Euclidean Algorithm and Extended Euclidean Algorithm
- present the lesson in other number bases
- present the divisibility rule for 9, 3 and Casting Out 9s

Additional duties for assistant instructors:

- lead small group problem sessions
- grade daily homework and provide feedback
- keep a record of participants who give presentations, in small groups and before the entire class
- write theorems and definitions given in class on poster paper to display in the classroom
- provided assistance on homework problems in the evenings (at the hotel where participants were staying).

Provide assistant instructors with a copy of the course notebook and copies of solutions to all problems (Evening Homework problems, Problem Session problems, problems corresponding to reading assignments, etc.) prior to the beginning of the course.

MAA
MATHEMATICAL ASSOCIATION OF AMERICA

MAA NOTES #80

The collection of articles in this volume is in response to the *Mathematics Education of Teachers* (MET) document which made it critical that special programs and courses for mathematics middle school teachers emerge. The articles are the result of gatherings of mathematics educators and mathematicians training middle school teachers. The articles appearing in this volume were chosen to disseminate various middle school programs' structures, to detail methods of teaching specific middle school teacher content courses, and to share materials and resources. The articles provide a rich set of readily available, classroom-tested resources.

CHERYL BEAVER, LAURIE BURTON, MARIA FUNG, KLAY KRUCZEK, Editors

RESOURCES FOR PREPARING MIDDLE SCHOOL MATHEMATICS TEACHERS

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