MAA AMC
American Mathematics Competitions

Official Solutions

MAA American Mathematics Competitions

38th Annual

AMC 8

Tuesday, January 17, 2023 through Monday, January 23, 2023

These official solutions give at least one solution for each problem on this year's competition and show that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

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The problems and solutions for this AMC 8 were prepared by the
MAA AMC 8 Editorial Board under the direction of:

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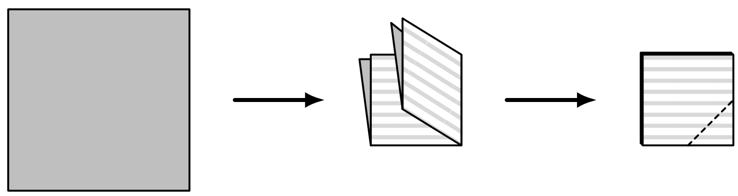
1. What is the value of $(8 \times 4 + 2) - (8 + 4 \times 2)$?

   (A) 0 **(B)** 6 **(C)** 10 **(D)** 18 **(E)** 24

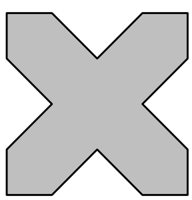
   Answer (D): By order of operations, multiplication has higher precedence than addition, so

   $$(8 \times 4 + 2) - (8 + 4 \times 2) = (32 + 2) - (8 + 8) = 34 - 16 = 18.$$

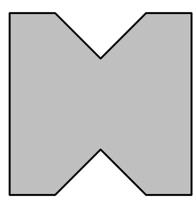
2. A square piece of paper is folded twice into four equal quarters, as shown below, then cut along the dashed line. When unfolded, the paper will match which of the following figures?



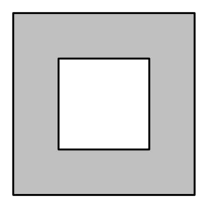
(A)



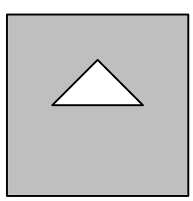
(B)



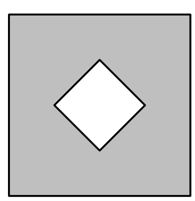
(C)



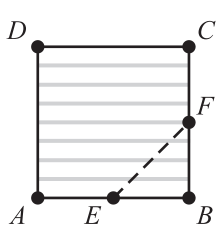
(D)

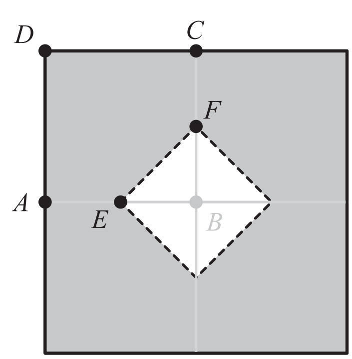


(E)



   Answer (E): Label the vertices of the folded square $ABCD$, as shown below on the left. Label the endpoints of the cut E and F. The cut along $\overline{EF}$ will remove vertex B and leave a diamond shape in the middle of the original square when the paper is unfolded, corresponding to choice **(E)**.





3. *Wind chill* is a measure of how cold people feel when exposed to wind outside. A good estimate for wind chill can be found using this calculation:

$$\text{(wind chill)} = \text{(air temperature)} - 0.7 \times \text{(wind speed)},$$

where temperature is measured in degrees Fahrenheit (°F) and wind speed is measured in miles per hour (mph). Suppose the air temperature is 36°F and the wind speed is 18 mph. Which of the following is closest to the approximate wind chill?

(A) 18 **(B)** 23 **(C)** 28 **(D)** 32 **(E)** 35

Answer (B): Applying the given formula

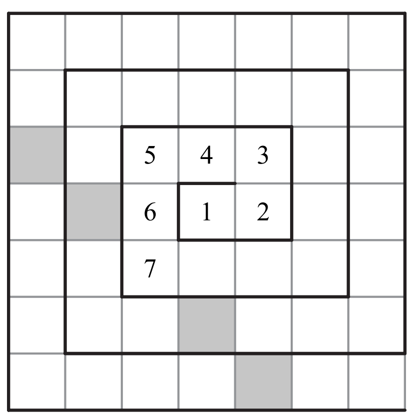
$$\text{wind chill} = \text{(air temperature)} - 0.7 \times \text{(wind speed)}$$

and substituting the temperature of 36°F and wind speed of 18 mph leads to the approximation

$$\begin{aligned} \text{wind chill} &= 36 - 0.7 \times 18 \\ &= 36 - 12.6 \\ &= 23.4, \end{aligned}$$

which is closest to choice **(B)** 23.

4. The numbers from 1 to 49 are arranged in a spiral pattern on a square grid, beginning at the center. The first few numbers have been entered into the grid below. Consider the four numbers that will appear in the shaded squares, on the same diagonal as the number 7. How many of these four numbers are prime?



(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

Answer (D): If the remaining numbers are entered into the grid using the spiral pattern, the shaded squares will correspond to the numbers 19, 23, 39, and 47. Of the four numbers, 39 is the only composite, so 3 of the numbers in the shaded squares are prime.

37	36	35	34	33	32	31
38	17	16	15	14	13	30
39	18	5	4	3	12	29
40	19	6	1	2	11	28
41	20	7	8	9	10	27
42	21	22	23	24	25	26
43	44	45	46	47	48	49

OR

It is possible to determine the number in any square without filling in the entire grid by using the square numbers as reference points. Note that when the positive integers are arranged in a spiral pattern on an $n \times n$ grid, the greatest number, n^2, will appear either in the upper left corner (if n is even) or in the lower right corner (if n is odd), as shown below for $n = 2$ and 3.

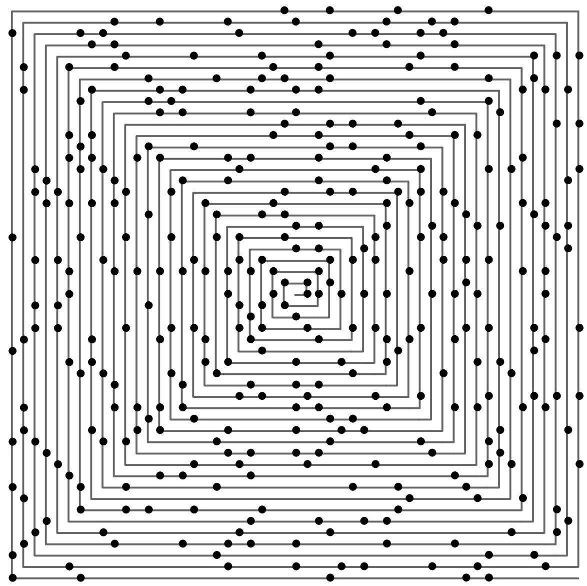
4	3
1	2

5	4	3
6	1	2
7	8	9

If the remaining square numbers (9, 16, 25, 36, and 49) are entered into the grid, one can determine that the shaded squares correspond to the numbers 19, 23, 39, and 47. Of the four numbers, 39 is the only composite, so 3 of the numbers in the shaded squares are prime.

	36					
		16				
39		5	4	3		
	19	6	1	2		
		7		9		
			23		25	
				47		49

Note: This *prime spiral*, or *Ulam spiral*, is named after Stanislaw Ulam (1909–1984), who discovered that when the positive integers are arranged in this pattern, the prime numbers often fall along diagonal lines. These diagonal patterns can be seen in the figure below, which shows the prime numbers between 1 and 2500 as black dots.

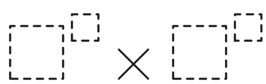


5. A lake contains 250 trout, along with a variety of other fish. When a marine biologist catches and releases a sample of 180 fish from the lake, 30 are identified as trout. Assume that the ratio of trout to the total number of fish is the same in both the sample and the lake. How many fish are there in the lake?

   (A) 1250 **(B)** 1500 **(C)** 1750 **(D)** 1800 **(E)** 2000

   Answer (B): In the sample, the trout make up $\frac{30}{180} = \frac{1}{6}$ of the number of fish caught. The trout-to-fish ratio is the same in both the sample and the lake, so the 250 trout in the lake must make up $\frac{1}{6}$ of the total. Therefore there are $6 \cdot 250 = 1500$ fish in the lake.

6. The digits 2, 0, 2, and 3 are placed in the expression below, one digit per box. What is the maximum possible value of the expression?



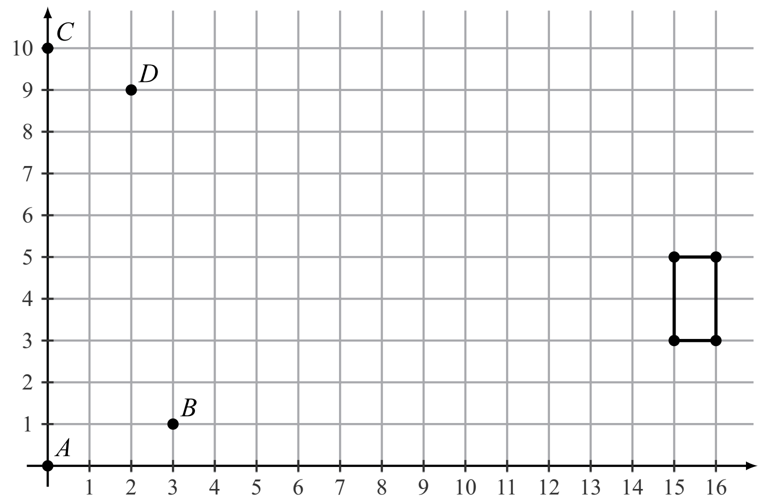
   (A) 0 **(B)** 8 **(C)** 9 **(D)** 16 **(E)** 18

   Answer (C): Consider the placement of the digit 0. If 0 is inserted into one of the larger boxes, then the resulting product will be 0. The product will have a greater value if the digit 0 is used as an exponent, leading to one of these three expressions:

$$3^0 \times 2^2 = 1 \cdot 4 = 4,$$
$$2^0 \times 2^3 = 1 \cdot 8 = 8,$$
$$2^0 \times 3^2 = 1 \cdot 9 = 9.$$

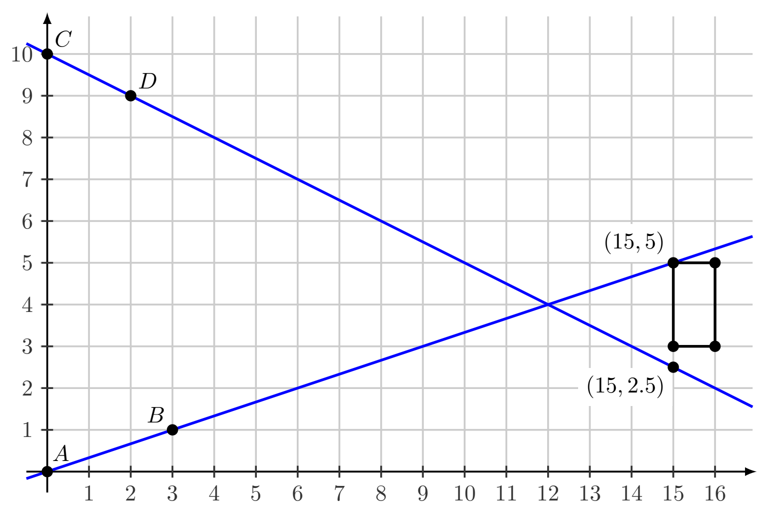
   Therefore the maximum possible value of the given expression is 9.

7. A rectangle, with sides parallel to the x-axis and y-axis, has opposite vertices located at $(15, 3)$ and $(16, 5)$. A line is drawn through points $A(0, 0)$ and $B(3, 1)$. Another line is drawn through points $C(0, 10)$ and $D(2, 9)$. How many points on the rectangle lie on at least one of the two lines?



(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

Answer (B): Line AB has a slope of $\frac{1}{3}$, moving up 1 unit for every 3 units to the right. It intersects the upper-left corner of the rectangle at $(15, 5)$. Line CD has a slope of $-\frac{1}{2}$, moving down 1 unit for every 2 units to the right. It passes through the points $(15, 2.5)$ and $(16, 2)$, below the rectangle. Therefore $(15, 5)$ is the only point on the rectangle that lies on the lines.



OR

Line AB has the equation $y = \frac{x}{3}$. It intersects the upper-left corner of the rectangle at $(15, 5)$. Line CD has the equation $y = -\frac{x}{2} + 10$, which passes through the point $(15, 2.5)$, below the rectangle. Therefore $(15, 5)$ is the only point on the rectangle that lies on the lines.

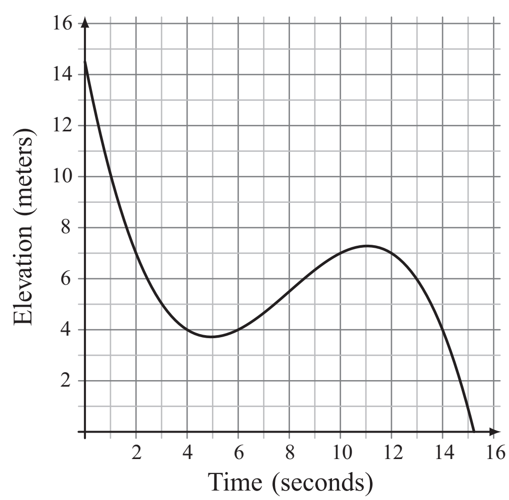
8. Lola, Lolo, Tiya, and Tiyo participated in a ping pong tournament. Each player competed against each of the other three players exactly twice. Shown below are the win-loss records for the players. The numbers 1 and 0 represent a win or loss, respectively. For example, Lola won five matches and lost the fourth match. What was Tiyo's win-loss record?

Player	Result
Lola	`111011`
Lolo	`101010`
Tiya	`010100`
Tiyo	`??????`

(A) `000101` **(B)** `001001` **(C)** `010000` **(D)** `010101` **(E)** `011000`

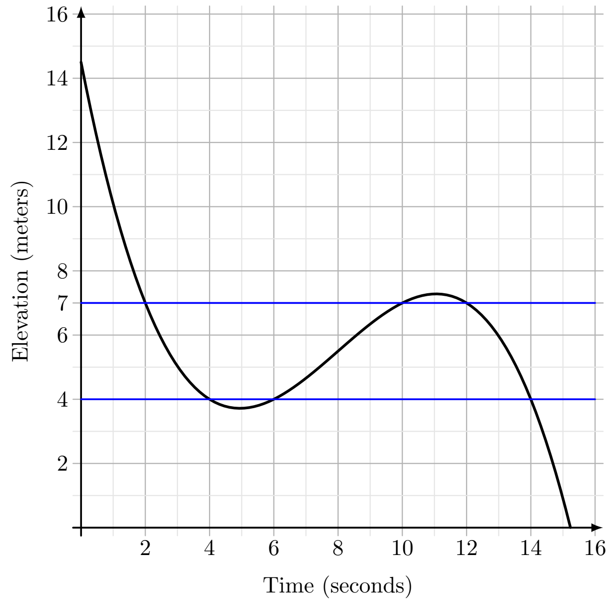
Answer (A): There are exactly 2 wins and 2 losses in each round. Summing the scores for each round for Lola, Lolo, and Tiya gives `222121`. Therefore Tiyo must have won the fourth and sixth games for a record of `000101`.

9. Malaika is skiing on a mountain. The graph below shows her elevation, in meters, above the base of the mountain as she skis along a trail. In total, how many seconds does she spend at an elevation between 4 and 7 meters?



(A) 6 **(B)** 8 **(C)** 10 **(D)** 12 **(E)** 14

Answer (B): Draw horizontal lines corresponding to the elevations of 4 meters and 7 meters.



The lines show that Malaika's elevation was between 4 meters and 7 meters over the following time intervals.

Start time	End time	Duration
2	4	2 sec
6	10	4 sec
12	14	2 sec

Therefore Malaika spent $2 + 4 + 2 = 8$ seconds between the elevations of 4 and 7 meters.

10. Harold made a plum pie to take on a picnic. He was able to eat only $\frac{1}{4}$ of the pie, and he left the rest for his friends. A moose came by and ate $\frac{1}{3}$ of what Harold left behind. After that, a porcupine ate $\frac{1}{3}$ of what the moose left behind. How much of the original pie still remained after the porcupine left?

(A) $\frac{1}{12}$ **(B)** $\frac{1}{6}$ **(C)** $\frac{1}{4}$ **(D)** $\frac{1}{3}$ **(E)** $\frac{5}{12}$

Answer (D): Harold ate $\frac{1}{4}$ of the pie, so he left behind $\frac{3}{4}$ of the pie. The moose ate $\frac{1}{3}$ of the remainder, so it ate $\frac{1}{3} \cdot \frac{3}{4} = \frac{1}{4}$ of the pie, leaving behind $\frac{3}{4} - \frac{1}{4} = \frac{1}{2}$ of the pie. Finally, the porcupine ate $\frac{1}{3}$ of the remainder, so it ate $\frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$ of the pie, leaving behind $\frac{1}{2} - \frac{1}{6} = \frac{1}{3}$ of the original pie.

OR

Harold left behind $\frac{3}{4}$ of the pie, the moose left behind $\frac{2}{3}$ of the remainder, and the porcupine left behind $\frac{2}{3}$ of that remainder. So the total fraction of the original pie that remained after the porcupine left was

$$\frac{3}{4} \cdot \frac{2}{3} \cdot \frac{2}{3} = \frac{3 \cdot 4}{4 \cdot 3 \cdot 3} = \frac{1}{3}.$$

OR

Suppose the plum pie is cut into 12 equal slices. Harold ate $\frac{12}{4} = 3$ slices, leaving 9 slices. Then the moose ate $\frac{9}{3} = 3$ slices, leaving 6 slices. Then the porcupine ate $\frac{6}{3} = 2$ slices, leaving 4 slices. Thus $\frac{4}{12} = \frac{1}{3}$ of the original pie remained after the porcupine left.

11. NASA's Perseverance Rover was launched on July 30, 2020. After traveling 292,526,838 miles, it landed on Mars in Jezero Crater about 6.5 months later. Which of the following is closest to the Rover's average interplanetary speed in miles per hour?

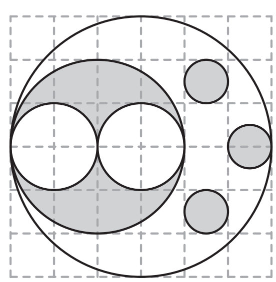
(A) 6,000 **(B)** 12,000 **(C)** 60,000 **(D)** 120,000 **(E)** 600,000

Answer (C): The average speed of the Rover can be calculated by dividing the total distance traveled by the number of hours in flight. The Rover covered almost 300,000,000 miles over 6.5 months. Assuming about 30 days in a month, the trip lasted about $6.5 \times 30 = 195$ days, which is almost 200 days. Therefore the Rover's average interplanetary speed was approximately

$$\frac{300{,}000{,}000 \text{ miles}}{200 \text{ days}} \cdot \frac{1 \text{ day}}{24 \text{ hours}} \approx \frac{300{,}000{,}000 \text{ miles}}{5000 \text{ hours}} = 60{,}000 \text{ mph}.$$

Note: Input data was taken from the NASA website https://mars.nasa.gov/mars2020/.

12. The figure below shows a large white circle with a number of smaller white and shaded circles in its interior. What fraction of the interior of the large white circle is shaded?



(A) $\frac{1}{4}$ **(B)** $\frac{11}{36}$ **(C)** $\frac{1}{3}$ **(D)** $\frac{19}{36}$ **(E)** $\frac{5}{9}$

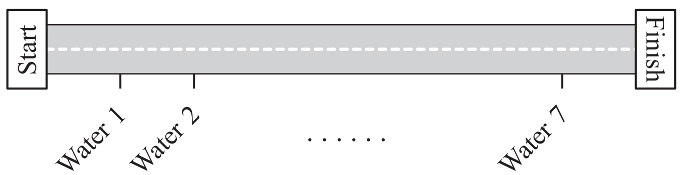
Answer (B): The large white circle has a radius of 3 units, so its area is $3^2 \cdot \pi = 9\pi$ square units.

There are three small shaded circles, each of which has a radius of $\frac{1}{2}$ unit. This means each small shaded circle has an area of $\left(\frac{1}{2}\right)^2 \pi = \frac{1}{4}\pi$ square units. Together, they contribute $\frac{3}{4}\pi$ square units of shaded area.

Finally, the larger shaded circle has a radius of 2 units, so its area is 4π square units, however it also contains two white circles. Each of those white circles has a radius of 1 unit, so the white circles occupy $2 \cdot 1^2 \cdot \pi = 2\pi$ square units.

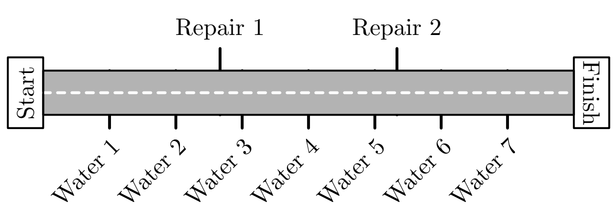
In total the shaded area is $(4\pi - 2\pi) + \frac{3}{4}\pi = 2\pi + \frac{3}{4}\pi = \frac{11}{4}\pi$ square units. Therefore the fraction of the large white circle that is shaded is $\frac{11}{4}\pi \div 9\pi = \frac{11}{36}$.

13. Along the route of a bicycle race, 7 water stations are evenly spaced between the start and finish lines, as shown in the figure below. There are also 2 repair stations evenly spaced between the start and finish lines. The 3rd water station is located 2 miles after the 1st repair station. How long is the race in miles?



(A) 8 **(B)** 16 **(C)** 24 **(D)** 48 **(E)** 96

Answer (D): The 2 repair stations are equally spaced along the route, so they divide the total race distance into thirds. Similarly, the 7 water stations are equally spaced, so they divide the distance into eighths. The 1st repair station is $\frac{1}{3}$ of the way from start to finish, and the 3rd water station is 2 miles further, $\frac{3}{8}$ of the distance along the way. The difference between $\frac{3}{8}$ and $\frac{1}{3}$ is $\frac{9}{24} - \frac{8}{24} = \frac{1}{24}$, and because $\frac{1}{24}$ of the total distance is 2 miles, the race must be 48 miles long.



OR

Let the total distance be x miles. Then the water stations are spaced $\frac{x}{8}$ miles apart and the repair stations are $\frac{x}{3}$ miles apart. It is given that the 3rd water station is located 2 miles after the 1st repair station, so

$$\frac{3x}{8} = \frac{x}{3} + 2.$$

Multiplying both sides of the equation by 24 and simplifying gives

$$\begin{aligned} 9x &= 8x + 48 \\ x &= 48. \end{aligned}$$

Therefore the length of the race is 48 miles.

OR

Let w represent the distance between water stations and r represent the distance between repair stations in miles. The stations are equally spaced along the road and the 3rd water station is located 2 miles past the 1st repair station, so

$$8w = 3r \quad \text{and} \quad 3w = r + 2.$$

Solving for r in the second equation gives $r = 3w - 2$ and substituting into the first equation yields

$$8w = 3(3w - 2) = 9w - 6,$$

thus

$$w = 6.$$

Therefore the length of the race is $8w = 8 \cdot 6 = 48$ miles.

14. Nicolas is planning to send a package to his friend Anton, who is a stamp collector. To pay for the postage, Nicolas would like to cover the package with a large number of stamps. Suppose he has a collection of 5-cent, 10-cent, and 25-cent stamps, with exactly 20 of each type. What is the greatest number of stamps Nicolas can use to make exactly \$7.10 in postage?

(Note: The amount \$7.10 corresponds to 7 dollars and 10 cents. One dollar is worth 100 cents.)

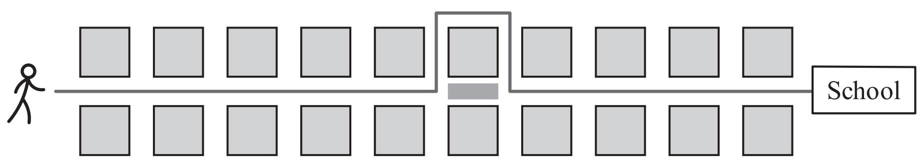
(A) 45 **(B)** 46 **(C)** 51 **(D)** 54 **(E)** 55

Answer (E): The greatest number of stamps can be found by maximizing the number of small-denomination stamps and minimizing the number of large-denomination stamps. If Nicolas uses all 20 of the 5-cent stamps and all 20 of the 10-cent stamps, their value would sum to $\$1.00 + \$2.00 = \$3.00$. To make \$7.10 in postage, Nicolas must include at least 17 of the 25-cent stamps, valued at \$4.25, and make up the difference with the smaller denominations. The difference of $\$7.10 - \$4.25 = \$2.85$ is not divisible by 10 cents, so Nicolas must choose an odd number of the 5-cent stamps. Selecting 19 of the 5-cent stamps will leave a remainder of $\$2.85 - \$0.95 = \$1.90$, which can be made with 19 of the 10-cent stamps. Therefore the greatest number of stamps can be attained by using 17 of the 25-cent stamps, 19 of the 10-cent stamps, and 19 of the 5-cent stamps, for a total of $17 + 19 + 19 = 55$ stamps.

OR

Nicolas's collection has $3 \cdot 20 = 60$ stamps in total, worth $\$8.00 = \$1.00 + \$2.00 + \$5.00$. To maximize the number of stamps that can make \$7.10, he can first minimize the number of stamps that can form the remainder of $\$8.00 - \$7.10 = \$0.90$, then subtract the minimum from the total number of stamps. The amount \$0.90 can be made with 5 stamps: 3 of the 25-cent stamps, 1 of the 10-cent stamps, and 1 of the 5-cent stamps. Subtracting this minimum of 5 stamps from the total leaves $60 - 5 = 55$ stamps, which is the greatest number that Nicolas can use to make \$7.10.

15. Viswam walks half a mile to get to school each day. His route consists of 10 city blocks of equal length and he takes one minute to walk each block. Today, after walking 5 blocks, Viswam discovers that he has to make a detour, walking 3 blocks of equal length instead of 1 block to reach the next corner. From the time he starts his detour, at what speed, in miles per hour, must Viswam walk in order to arrive at school at his usual time?



(A) 4 **(B)** 4.2 **(C)** 4.5 **(D)** 4.8 **(E)** 5

Answer (B): Viswam normally walks $\frac{1}{2}$ mile in 10 minutes, which equals $\frac{1}{6}$ of an hour, so his usual speed is $\frac{1}{2} \div \frac{1}{6} = 3$ miles per hour. When he starts his detour, he normally would have $10 - 5 = 5$ blocks to go. Because of the detour, he actually has $5 + (3 - 1) = 7$ blocks to go, so Viswam's speed must be $\frac{7}{5} \cdot 3 = \frac{21}{5} = 4.2$ miles per hour to arrive at school at his usual time.

OR

Viswam normally takes 10 minutes to get to school. His detour starts when he is halfway to school, so from that time he has 5 minutes $= \frac{1}{12}$ of an hour to get there. He must walk $5 + (3 - 1) = 7$ blocks, and because 10 blocks is $\frac{1}{2}$ mile, 7 blocks is $\frac{7}{10} \cdot \frac{1}{2} = \frac{7}{20}$ of a mile. Therefore, to arrive at school at his usual time, Viswam's speed must be $\frac{7}{20} \div \frac{1}{12} = \frac{84}{20} = 4.2$ miles per hour.

16. The letters P, Q, and R are entered into a 20×20 table according to the pattern shown below. How many Ps, Qs, and Rs will appear in the completed table?

⋮	⋮	⋮	⋮	⋮	⋰
Q	R	P	Q	R	⋯
P	Q	R	P	Q	⋯
R	P	Q	R	P	⋯
Q	R	P	Q	R	⋯
P	Q	R	P	Q	⋯

(A) 132 Ps, 134 Qs, 134 Rs

(B) 133 Ps, 133 Qs, 134 Rs

(C) 133 Ps, 134 Qs, 133 Rs

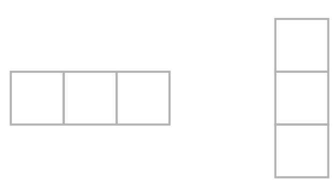
(D) 134 Ps, 132 Qs, 134 Rs

(E) 134 Ps, 133 Qs, 133 Rs

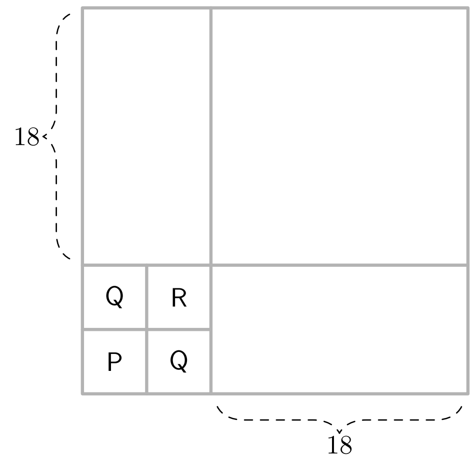
Answer (C): A copy of the third column can be inserted into the table as the 21st column to form a larger 20×21 table. Each row of the new table has 7 Ps, 7 Qs, and 7 Rs. Because there are 20 rows, the new table has a total of 140 each of the Ps, Qs, and Rs. As there are 7 Ps, 6 Qs, and 7 Rs in the 21st column, subtracting each from the total of 140 leaves 133 Ps, 134 Qs, and 133 Rs in the original table.

OR

Consider covering the board with 1×3 tiles called *triominoes*. A key realization is that any such triomino will cover exactly one P, one Q, and one R (in some order).



Divide the 20×20 square into four regions—an 18×18 square, a 2×18 rectangle, an 18×2 rectangle, and a 2×2 square—as shown below (not to scale):



The two rectangles and the 18×18 square each can be covered by non-overlapping triominoes because 18 is divisible by 3. Hence these three regions contain equal numbers of Ps, Qs, and Rs. Observe that the lower left 2×2 corner of the board contains one P, two Qs, and one R. Therefore the entire board will contain one more Q than it does Ps and Rs.

The board has $20 \times 20 = 400$ cells, and $400 \div 3$ gives a quotient of 133 with a remainder of 1. That remainder accounts for the extra Q cell. Therefore there are 133 Ps, 134 Qs, and 133 Rs in the completed table.

OR

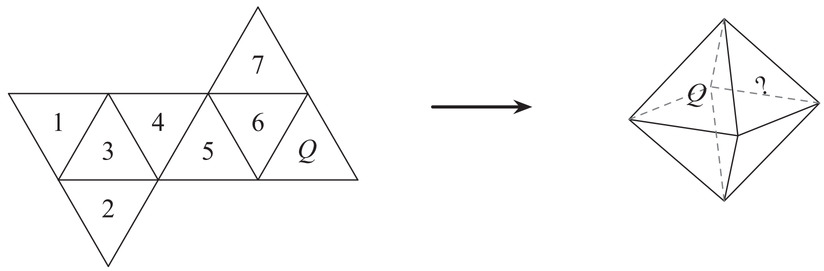
The square can be broken into diagonals, each of which contains all Ps, all Qs, or all Rs. Starting in the lower left corner, there is a diagonal of 1 P, then a diagonal of 2 Qs, then 3 Rs, 4 Ps, 5 Qs, and so on. The longest diagonal contain 20 letters, after which the diagonal lengths decrease down to 1.

Let p represent the number of Ps, q represent the number of Qs, and r represent the number of Rs. Then p, q, and r can be computed by summing the diagonal lengths:

$$\begin{aligned} p &= 1 + 4 + 7 + 10 + 13 + 16 + \mathbf{19} + 18 + 15 + 12 + 9 + 6 + 3, \\ q &= 2 + 5 + 8 + 11 + 14 + 17 + \mathbf{20} + 17 + 14 + 11 + 8 + 5 + 2, \\ r &= 3 + 6 + 9 + 12 + 15 + 18 + \mathbf{19} + 16 + 13 + 10 + 7 + 4 + 1. \end{aligned}$$

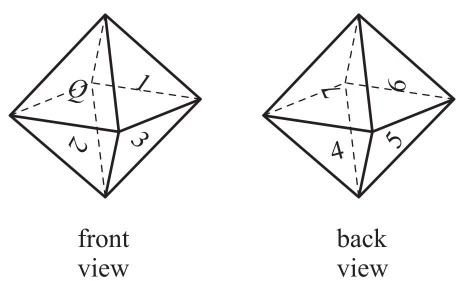
The total $p+q+r$ must equal $20 \cdot 20 = 400$. Observe that p and r have the same summands, so their values are equal. Also observe that each of the first 6 summands of q is exactly 1 more than the corresponding summand of p, while each of the last 6 summands of q is exactly 1 less than the corresponding summand of p. The difference between the middle summands of q and p is $20 - 19 = 1$, so the value of q is exactly 1 more than the value of p. Therefore $p + q + r = p + (p + 1) + p = 3p + 1 = 400$, so $p = 133$. The table contains 133 Ps, 134 Qs, and 133 Rs.

17. A *regular octahedron* has eight equilateral triangle faces with four faces meeting at each vertex. Jun will make the regular octahedron shown on the right by folding the piece of paper shown on the left. Which numbered face will end up to the right of Q?

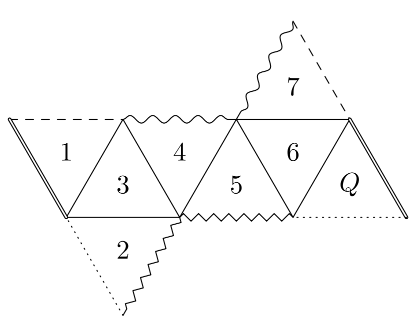


(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

Answer (A): The left figure shows that the four faces in the top half of the octahedron must be Q, 6, 7, and the unknown face. Note that each vertex of the octahedron belongs to exactly four faces. The figure shows that the four faces 2, 3, 4, and 5 all meet at a vertex. Since this vertex cannot be part of Q, 6, 7, or the unknown face, it is the vertex at the bottom of the octahedron and the faces 2, 3, 4, and 5 are the four faces in the bottom half of the octahedron. By elimination, the remaining face 1 is in the top half of the octahedron and must end up to the right of Q. The front and back views of the folded octahedron are shown below.



OR



The top edge of face 4 will meet the left edge of face 7, the top edge of face 1 will meet the right edge of face 7, and the left edge of face 1 will meet the right edge of Q, so face 1 will end up to the right of Q.

To complete the octahedron, the right edge of face 2 will meet the bottom edge of face 5, and the left edge of face 2 will meet the bottom edge of Q.

18. Greta Grasshopper sits on a long line of lily pads in a pond. From any lily pad, Greta can jump 5 pads to the right or 3 pads to the left. What is the fewest number of jumps Greta must make to reach the lily pad located 2023 pads to the right of her starting position?

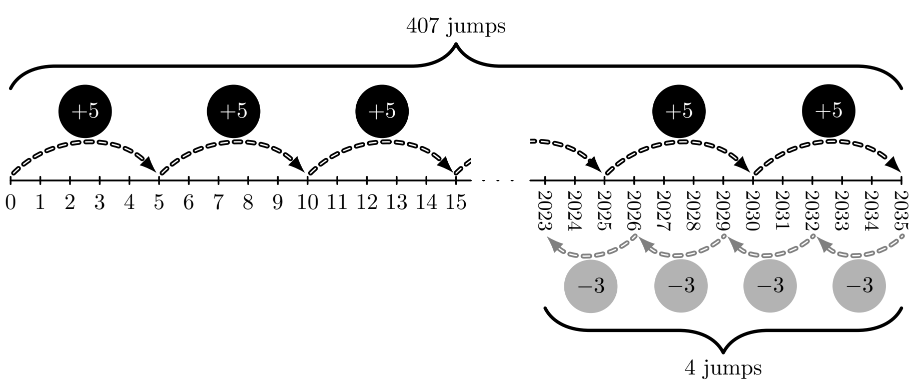
(A) 405 **(B)** 407 **(C)** 409 **(D)** 411 **(E)** 413

Answer (D): Assume the lily pads are labeled as if they are on a number line, with Greta Grasshopper starting on pad 0 and ending on pad 2023.

First note that the order in which Greta makes her jumps does not affect her final destination. Only the number of jumps to the right or left matters. For example, jumping right, then left, then right, ends at pad $5 - 3 + 5 = 7$, which is the same destination as jumping right twice, then left.

Therefore, it can be assumed that Greta takes some number of jumps to the right, followed by some number of jumps to the left. The jumps to the right must place Greta on a multiple of 5 greater than 2023, at a distance from 2023 that is a multiple of 3.

The first multiple of 5 that is greater than 2023 is 2025, but its distance from 2023, which is 2, is not divisible by 3. The next multiple of 5 is 2030, but once again its distance from 2023, which is 7, is not divisible by 3. However, the next multiple of 5 is 2035 and its distance from 2023 is 12, which is divisible by 3. Therefore, the shortest path to 2023 requires $\frac{2035}{5} = 407$ jumps to the right, followed by $\frac{12}{3} = 4$ jumps to the left. This gives a minimum of $407 + 4 = 411$ jumps that Greta must make to reach the 2023rd pad.



OR

Let R and L represent the number of Greta's jumps to the right and left, respectively. The goal is to minimize the sum $R + L$ given the condition $5R - 3L = 2023$. First check $R = 405$, which is the least value of R that satisfies $5R \geq 2023$, placing Greta to the right of the 2023rd lily pad. That value of R does not lead to an integer value for L, and neither does $R = 406$. The next value, $R = 407$, corresponds to the integer value $L = 4$, so the fewest number of jumps Greta must make to reach the 2023rd pad is $R + L = 407 + 4 = 411$.

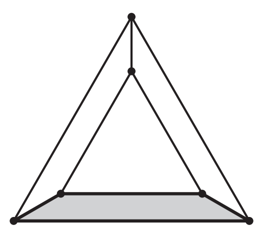
OR

To reach the 2023rd lily pad using the fewest number of jumps, Greta will make more right jumps than left jumps. Suppose each left jump is paired with a right jump, producing P pairs, leaving Q unpaired right jumps. Then the total number of jumps is $2P + Q$.

Each right jump will move 5 pads to the right and each left-right pair of jumps will move $5 - 3 = 2$ pads to the right. It follows that solving the equation $2P + 5Q = 2023$ using the least value of P will minimize the total number of jumps.

Given that the quantity $2023 - 2P = 5Q$ is a multiple of 5, the least possible value of P is 4. The corresponding value of Q is $\frac{2023-2\cdot 4}{5} = 403$. Therefore the fewest number of jumps Greta must make to reach the 2023rd pad is $2P + Q = 2 \cdot 4 + 403 = 411$.

19. An equilateral triangle is placed inside a larger equilateral triangle so that the region between them can be divided into three congruent trapezoids, as shown below. The side length of the inner triangle is $\frac{2}{3}$ the side length of the larger triangle. What is the ratio of the area of one trapezoid to the area of the inner triangle?



(A) 1 : 3 **(B)** 3 : 8 **(C)** 5 : 12 **(D)** 7 : 16 **(E)** 4 : 9

Answer (C): Let the area of the inner triangle equal 1. Because the ratio of the triangle side lengths is $2 : 3$, the ratio of their areas is $2^2 : 3^2 = 4 : 9$. It follows that the area of the outer triangle is $\frac{9}{4}$, and the area of the region between the triangles is $\frac{9}{4} - 1 = \frac{5}{4}$. Each trapezoid has area equal to one third of the region: $\frac{1}{3} \cdot \frac{5}{4} = \frac{5}{12}$. Thus the ratio of the area of one trapezoid to the area of the inner triangle is $\frac{5}{12} : 1$, which equals $5 : 12$.

OR

Let A be the area of the inner triangle. Because the ratio of the outer triangle side length to the inner triangle side length is $3 : 2$, the ratio of their areas is $3^2 : 2^2 = 9 : 4$. Thus the area of the outer triangle is $\frac{9}{4}A$. Let X be the area of a trapezoid. Then

$$A + 3X = \frac{9}{4}A,$$

which simplifies to

$$3X = \frac{5}{4}A,$$

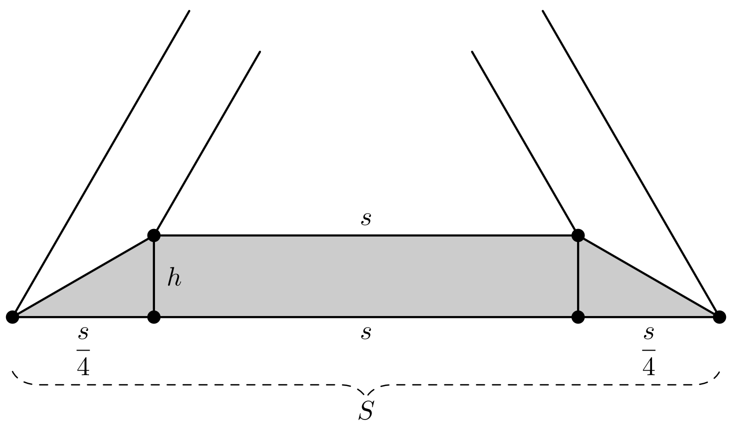
yielding the proportion

$$\frac{X}{A} = \frac{5}{12}.$$

Therefore the ratio of the area of one trapezoid to the area of the inner triangle is $X : A = 5 : 12$.

OR

Shown below is an enlarged view of the shaded trapezoid.



Let s and S denote the side lengths of the inner and outer triangles, respectively. The ratio of the two side lengths is $2 : 3$, so $S = \frac{3}{2}s$. The area A of the inner equilateral triangle equals half its base times its height.

$$A = \frac{s}{2} \cdot \frac{\sqrt{3}}{2}s = \frac{\sqrt{3}}{4}s^2$$

The area T of the shaded trapezoid equals its height h times the average of its bases.

$$T = h \cdot \frac{s+S}{2} = h \cdot \frac{s + \frac{3}{2}s}{2} = \frac{5}{4}hs$$

Note that the base angles of the trapezoid measure 30°. Perpendiculars drawn from the base vertices of the inner triangle to the base of the trapezoid will form $30-60-90°$ triangles. Each triangle has a base length of $\frac{1}{2}(S-s) = \frac{s}{4}$ and a height $h = \frac{s}{4\sqrt{3}}$. Thus the area of the trapezoid is

$$T = \frac{5}{4}hs = \frac{5}{4} \cdot \frac{s}{4\sqrt{3}} \cdot s = \frac{5}{16\sqrt{3}}s^2.$$

Dividing T by A gives

$$\frac{T}{A} = \frac{\frac{5}{16\sqrt{3}}s^2}{\frac{\sqrt{3}}{4}s^2} = \frac{5}{16\sqrt{3}} \cdot \frac{4}{\sqrt{3}} = \frac{5}{12}.$$

Therefore the ratio of the area of the trapezoid to the area of the inner triangle is $5 : 12$.

20. Two integers are inserted into the list $3, 3, 8, 11, 28$ to double its range. The mode and median remain unchanged. What is the maximum possible sum of the two additional numbers?

(A) 56 **(B)** 57 **(C)** 58 **(D)** 60 **(E)** 61

Answer (D): The smallest and largest numbers in the original list are 3 and 28, respectively, so the original range is $28 - 3 = 25$ and the doubled range is $2 \cdot 25 = 50$. If 3 is still the smallest number after the list is expanded, the largest number in the new list will be $3 + 50 = 53$.

Let a and b represent the two additional integers. For the mode of 3 to remain unchanged, neither a nor b can equal 8, 11, or 28, nor can a and b be equal in value. Suppose $a < b$. For the median of 8 to remain unchanged, a must be less than 8 and b must be greater than 8. It follows that the sum $a + b$ will be maximized if $a = 7$ and $b = 53$. The maximum possible sum of the two additional numbers is $7 + 53 = 60$.

21. Alina writes the numbers 1, 2, …, 9 on separate cards, one number per card. She wishes to divide the cards into 3 groups of 3 cards so that the sum of the numbers in each group will be the same. In how many ways can this be done?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

Answer (C): The sum of all 9 integers is $\frac{9 \cdot 10}{2} = 45$, so the sum of the integers in each group must be $\frac{45}{3} = 15$. Because 15 is odd, the number of odd integers in each group must be odd, so one group must contain three odd integers and the other two must contain one each. The only groups of three odd integers with a sum of 15 are $\{1, 5, 9\}$ and $\{3, 5, 7\}$.

If the group of three odd integers is $\{1, 5, 9\}$, then the number 3 must be combined with two even integers with a sum of 12, and the only possibility is $\{3, 4, 8\}$, leaving $\{2, 6, 7\}$ as the third group.

If the group of three odd integers is $\{3, 5, 7\}$, then the number 1 must be combined with two even integers with a sum of 14, and the only possibility is $\{1, 6, 8\}$, leaving $\{2, 4, 9\}$ as the third group.

Therefore the 9 cards can be divided into 3 groups with identical sums in exactly 2 ways.

OR

The sum of all nine integers is $\frac{9 \cdot 10}{2} = 45$, so the sum of the integers in each group must be $\frac{45}{3} = 15$. Observe that the numbers 7, 8, and 9 must all be in different groups, because if two of them were in the same group, the sum of that group would be too large. Symmetrically, the numbers 1, 2, and 3 must all be in different groups, because if two of them were in the same group, with any third number the sum of that group would be too small. Consider the group that contains the number 5. The two remaining numbers add up to 10, so the group must be $\{3, 5, 7\}$, $\{2, 5, 8\}$, or $\{1, 5, 9\}$. (Note that the number 5 cannot be grouped with 4 and 6 because each group must contain 7, 8, or 9.) If $\{3, 5, 7\}$ is one of the groups, then the other two groups must be $\{1, 6, 8\}$ and $\{2, 4, 9\}$. Similarly, if $\{1, 5, 9\}$ is one of the groups, then the other two groups must be $\{2, 6, 7\}$ and $\{3, 4, 8\}$. There is no way for $\{2, 5, 8\}$ to be one of the groups, because both of the ways to match 7 and 9 with 3 and 1 to make groups of sum 15 result in repeated numbers. Therefore the 9 cards can be divided into 3 groups with identical sums in exactly 2 ways.

22. In a sequence of positive integers, each term after the second is the product of the previous two terms. The sixth term in the sequence is 4000. What is the first term?

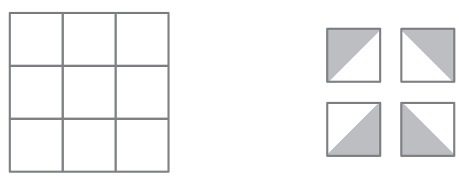
(A) 1 **(B)** 2 **(C)** 4 **(D)** 5 **(E)** 10

Answer (D): Denote the first term of the sequence by a and the second term by b. Then the first six terms are

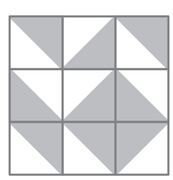
$$a,\ b,\ ab,\ ab^2,\ a^2b^3,\ a^3b^5.$$

The sixth term is $a^3b^5 = 4000 = 5^3 2^5$. It follows that $b = 2$ and the first term of the sequence is $a = 5$. The first six terms of the sequence are $5, 2, 10, 20, 200$, and 4000.

23. Each square in a 3×3 grid is randomly filled with one of the 4 gray-and-white tiles shown below on the right.



What is the probability that the tiling will contain a large gray diamond in one of the smaller 2×2 grids? Below is an example of such a tiling.



(A) $\dfrac{1}{1024}$ **(B)** $\dfrac{1}{256}$ **(C)** $\dfrac{1}{64}$ **(D)** $\dfrac{1}{16}$ **(E)** $\dfrac{1}{4}$

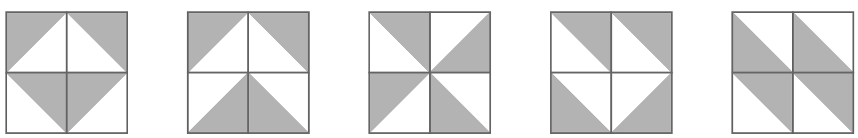
Answer (C): Observe that at most one large gray diamond can appear in a tiling. Because each of the 9 squares is filled with 1 of 4 possible tiles, a total of 4^9 tilings are possible. If a tiling has a 2×2 gray diamond, then there are 4 possible locations for the center of the diamond: any one of the 4 corners of the center square. Once that location is determined, each of the remaining 5 squares can be filled with any of the 4 possible tiles, for a total of $4 \cdot 4^5 = 4^6$ tilings. Therefore the probability that a tiling contains a 2×2 gray diamond is

$$\frac{4^6}{4^9} = \frac{1}{4^3} = \frac{1}{64}.$$

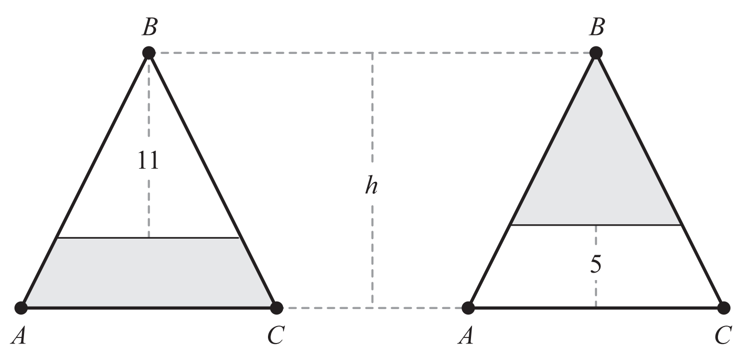
OR

If a tiling contains a 2×2 gray diamond, the diamond must include the center tile, which will have one corner that is all gray. The 3 tiles adjacent to that corner must be oriented to match the gray color of the center tile. There is no restriction on the orientations of the other 5 squares. Because each of the 3 adjacent tiles has a probability of $\frac{1}{4}$ of having the correct orientation, the probability that a tiling contains a large gray diamond is $\left(\frac{1}{4}\right)^3 = \frac{1}{64}$.

Note: These tiles are called *Truchet tiles*, named after Sébastien Truchet (1657–1729), who studied the possible patterns formed by these tiles. Below are some of the patterns that can be made on a 2×2 grid.



24. Isosceles triangle ABC has equal side lengths AB and BC. In the figures below, segments are drawn parallel to $\overline{AC}$ so that the shaded portions of $\triangle ABC$ have the same area. The heights of the two unshaded portions are 11 and 5 units, respectively. What is the height h of $\triangle ABC$?



(A) 14.6 **(B)** 14.8 **(C)** 15 **(D)** 15.2 **(E)** 15.4

Answer (A): Given two similar triangles, the ratio of their areas is equal to the square of the ratio of their corresponding heights. Thus, in the figure on the left, the ratio of the areas of the unshaded triangle and $\triangle ABC$ is $\frac{11^2}{h^2}$. It follows that the fraction of $\triangle ABC$ that is shaded is $1 - \frac{11^2}{h^2}$.

Similarly, in the figure on the right, the shaded triangle has height $h - 5$, and fills $\frac{(h-5)^2}{h^2}$ of $\triangle ABC$. The shaded portions in the two figures are equal in area, so

$$1 - \frac{11^2}{h^2} = \frac{(h-5)^2}{h^2}.$$

Multiplying both sides by h^2 and expanding the right side gives

$$h^2 - 11^2 = h^2 - 10h + 25,$$

which simplifies to

$$10h = 121 + 25 = 146$$
$$h = \frac{146}{10} = 14.6.$$

Therefore the height h of $\triangle ABC$ is 14.6.

OR

Let the base of $\triangle ABC$ be $b = AC$. Then the area of the triangle is $K = \frac{1}{2}bh$.

In the left figure, let the upper triangle have a base of b_1 and an area of $K_1 = \frac{1}{2} \cdot 11 \cdot b_1$. The upper triangle is similar to $\triangle ABC$ because their bases are parallel, so $\frac{11}{b_1} = \frac{h}{b}$ and $b_1 = 11\frac{b}{h}$. Thus $K_1 = \frac{11^2}{2}\frac{b}{h}$.

In the right figure, the height of the upper triangle is $h - 5$. Let its base be b_2 and its area be $K_2 = \frac{1}{2}b_2(h-5)$. The upper triangle is similar to $\triangle ABC$, so $\frac{h-5}{b_2} = \frac{h}{b}$ and $b_2 = (h-5)\frac{b}{h}$. Thus $K_2 = \frac{(h-5)^2}{2}\frac{b}{h}$.

The two shaded areas are equal.

$$K - K_1 = K_2$$
$$\frac{1}{2}bh - \frac{11^2}{2}\frac{b}{h} = \frac{(h-5)^2}{2}\frac{b}{h}$$

The equation above can be solved for h by first multiplying both sides of the equation by $\frac{2h}{b}$, eliminating b.

$$h^2 - 11^2 = (h-5)^2$$

Then expanding the expressions and simplifying gives

$$\begin{aligned} h^2 - 121 &= h^2 - 10h + 25 \\ 10h &= 146 \\ h &= 14.6. \end{aligned}$$

The height h of $\triangle ABC$ equals 14.6.

25. Fifteen integers $a_1, a_2, a_3, \ldots, a_{15}$ are arranged in order on a number line. The integers are equally spaced and have the property that

$$1 \le a_1 \le 10, \quad 13 \le a_2 \le 20, \quad \text{and} \quad 241 \le a_{15} \le 250.$$

What is the sum of the digits of a_{14} ?

(A) 8 **(B)** 9 **(C)** 10 **(D)** 11 **(E)** 12

Answer (A): Let d be the integer difference between adjacent numbers a_i and a_{i+1}. Then the difference between the 1st and 15th numbers is $a_{15} - a_1 = 14d$. Given the range of values for a_1 and a_{15}, the quantity $14d$ must lie in the following interval:

$$\begin{gathered} 241 - 10 \le 14d \le 250 - 1, \\ 231 \le 14d \le 249. \end{gathered}$$

The only multiple of 14 in this interval is $238 = 14 \cdot 17$. Thus the difference d equals 17 and the numbers a_2 and a_{15} can be expressed in terms of a_1 as follows:

$$\begin{aligned} a_2 &= a_1 + d = a_1 + 17, \\ a_{15} &= a_1 + 14d = a_1 + 238. \end{aligned}$$

Next observe that the greatest possible value of a_1 is bounded by the maximum value of a_2.

$$a_2 \le 20 \implies a_1 + 17 \le 20 \implies a_1 \le 3.$$

Similarly the least possible value of a_1 is bounded by the minimum value of a_{15}.

$$a_{15} \ge 241 \implies a_1 + 238 \ge 241 \implies a_1 \ge 3.$$

The only value of a_1 that satisfies both inequalities is $a_1 = 3$. Therefore

$$\begin{aligned} a_2 &= a_1 + 17 = 20, \\ a_{15} &= a_1 + 238 = 241, \end{aligned}$$

and the value of a_{14} is

$$a_{14} = a_{15} - d = 241 - 17 = 224.$$

Thus the sum of the digits of a_{14} is $2 + 2 + 4 = 8$.