CUPM Curriculum Guide Program Area Study Group: Physics

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If you want to learn about nature, to appreciate nature, it is necessary to understand the language that she speaks in.
—Richard Feynman [8]

1 Introduction/description of the area

This report focuses on the mathematical needs of students majoring in physics, students minoring in physics, physics students that double-major or minor in mathematics, as well as the mathematical needs of students of other disciplines who take physics courses. Indeed, many of the things that would benefit physics majors would benefit these larger audiences as well.

Physics students need to recognize and to use mathematics as a tool to assist their thinking, not just to solve problems. Mathematics serves to organize their conceptual knowledge, as well as help them look for patterns and limiting cases. The ability to map real world situations into the mathematics and thereby model those situations is also important for these students. Making a mathematical model forces them to clarify assumptions and specify relationships in order to incorporate them in unambiguous mathematical terms in the model. Intermediate and advanced physics courses regularly use abstract mathematics as a tool for interpretation and model building. Majors who are unable to use this tool effectively typically encounter difficulties as early as sophomore level physics courses. In the following sections we elaborate on the following outcomes we recommend for these students.

1. Students should be able to understand a logical argument.
2. Students should be able to construct a logical argument.
3. Students should be able to communicate mathematical arguments through the appropriate use of natural language and mathematical language, both in writing and in speaking.
4. Students should be able to translate between mathematics and the physical world, drawing mathematical conclusions and evaluating their physical implications.
5. Students working on a problem should be able to describe what they are doing, why they are doing it, how accomplishing their task will help them solve the problem, and when their solution is complete.
6. Students should be able to calculate correctly, cleanly and efficiently.

Mathematics faculty and physics faculty can work together to help physics students develop these broad skills. We encourage conversations at your institution between faculty in mathematics and those in physics to determine the needs of your students, the goals of your programs, and the best way in which you might implement the recommendations and suggestions from this report that make sense for you.

2 Cognitive goals for the mathematical training of students in this area

In the sections that follow, we acknowledge that mathematicians and physicists sometimes use the same words and symbols with different meanings. For example, given the expression $T(x, y)$, a mathematican would likely see T as a function of the two variables x and y, whereas a physicist would likely see $T(x, y)$ as the physical quantity T (temperature, say) expressed in rectangular coordinates — leading to two quite different interpretations of $T(r, \theta)$ (see [6] and [7]). And "Stokes' Theorem" to a mathematician shows that the formula for curl has something to do with circulation and area, whereas to a physicist that relationship is taken as primary, and the "theorem" is used to derive the formula (see [9]).

Similarly, a mathematician might regard "modeling" as the process of creating a model, where a physicist might regard the model as something given whose purpose is to emphasize key underlying ideas. This can result in miscommunication across the mathematics and physics communities, but even more, it can lead to confusion among students who are being introduced to the two disciplines simultaneously. More intentional discussions of these differences have the potential to benefit interdisciplinary cooperation and collaboration, as well as foster pedagogical awareness to assist students in their struggles to align the conventions and practices of the two communities. We also acknowledge that many connections exist between and among the topics we have placed under different headers for ease of reference.

Modeling: By modeling, we mean using mathematics to represent and simulate the behavior of real-world or other systems. Mathematical models can allow us to make sense of and predict phenomena of interest to a vast array of fields and disciplines. Every model should have some basis in reality. A major goal for students of physics is to learn to translate between mathematics and the physical world and, as an expert, to blend their mathematical and physical knowledge. (See [12].) This ability to translate and construct meaning relies on several of the skills described below.

The word "modeling" means many things in many contexts. Two meanings that are particularly relevant here are *conceptual* and *testable*. In conceptual modeling, the focus is on creating a model that helps the student understand a mathematical structure and how it might relate to a physical system. In testable modeling, the focus is on creating a mathematical model that attempts to describe specific experimental data.

One example of conceptual modeling is describing the position of a mass suspended from a spring using Hooke's law to generate a linear ODE. While this model only gives a correct description of the physical system over a small range of displacements for short times, it is conceptually valuable. Studying the mathematical solutions of this equation leads to a deeper understanding of how mathematics describes oscillatory behavior, and these insights are valuable in a wide range of applications from small oscillations of multivariable systems to quantum field theory.

Ideally testable modeling should satisfy certain conditions:

- The model should have as few parameters as possible;
- Variables represented in the model should be measurable so that it is possible to collect experimental data;
- The domain of each of the parameters and the variables should be specified (positive integers, real numbers, etc.);
- The model's predictions should be reasonably accurate and give a good fit to experimental data, and
- The model should improve our understanding of the real world situation.

Translating and generating representations: Students should be able to create, interpret and use multiple representations of mathematical ideas and to recognize coherence across representations. They should be able to use multiple representations to find patterns, make generalizations, and observe global features as well as local details. Students should also have the metacognitive skill of being able to think about the generation and use of multiple representations and regard this skill as mathematically valuable (see [4]). (These ideas are present in some calculus curricula: for example, to represent a function graphically, numerically, algebraically, verbally.)

Strategic thinking: Students should learn to step back and take an overview of their efforts to solve a problem. They should evaluate the tools they are using, where their calculations lead, and how their line of attack will bring them closer to a solution. In more advanced physics classes, the complexity of problems increases and the number of different mathematical tools required to solve a single problem grows. In these circumstances, the ability to think strategically becomes crucial. For example, before diving into an extensive and complicated series of computations, a student needs to strategize about which computations are appropriate. Students' ability to use multiple representations of a mathematical idea contributes substantially to their capacity for this kind of strategic thinking. (See [15].)

Knowing when mathematical methods apply: In physics classes, students sometimes don't recognize that they have the mathematical tools to handle the task at hand. From an expert's perspective, it can be difficult to understand why students fail to apply the ideas they have learned in their mathematics courses to solving physics problems. This has been traditionally referred to as the problem of knowledge transfer.

Recent developments in educational research suggest that using the metaphor of transfer itself may be misleading. The problem may be better framed as one of knowledge activation. Knowledge of complex ideas in both mathematics and physics may not function as complete entities that simply transfer, but as complex systems of knowledge and ideas that need to be activated, sometimes independently and sometimes in combination, often differently across different contexts. Only through experiences in multiple contexts do students learn to recognize what mathematical ideas may be relevant, and how combinations of those ideas can be used to frame and structure physical situations. Because of this, it can be beneficial for students to see applications of the mathematics they are learning to physics and other contexts early and often. Instructors may find value in assisting and encouraging students to zoom out from the mathematical details to highlight the larger structure of both the mathematics and the physical contexts to which it can be applied. Students need time and experience to learn to recognize that what appear to be very different situations can be thought of as sharing a common structure. (See, for example, [5] and [11].)

Algebraic manipulative skills: Building up a mathematical structure from axioms and definitions or solving a complicated physical problem both require similar reasoning skills. Whether we call the process proof or derivation, it involves obtaining logical consequences from prior assertions. Success depends on choosing a strategy, carrying it out through a number of steps of logical deductions, and having a method to check the work for errors.

To carry out these steps students need a variety of skills. They need to be able to use mathematical expressions such as equations and inequalities with variables and parameters to express relationships. They also need to be able to use such expressions to formulate and analyze limiting cases. Students must be comfortable manipulating algebraic equations involving parameters, variables, and literal terms before substituting in numerical values. Competence in computation is necessary but must always be balanced by the strategic thinking skills described above. (See [1].)

Qualitative reasoning skills: Students feel more comfortable manipulating mathematical expressions with numbers instead of symbolic parameters, as such expressions are familiar from prior mathematics classes. Students therefore often rush to replace parameters with numbers, but this can easily result in students missing important qualitative aspects of the expression. For example, replacing a with .03 in the differential equation $P'(t) = aP$ leads to the conclusion that only exponential growth occurs, whereas the differential equation may show exponential decay or even have a constant solution.

Students should be able to manipulate expressions with multiple parameters in order to reason qualitatively about a particular problem or a family of problems. They should be able to determine the effect of increasing or decreasing a parameter, of approaching the boundary of a domain, or of moving toward a limiting case. The ability to create and use the diverse representations alluded to above and technical facility with expressions involving parameters are often essential for qualitative analysis of this kind.

There are many opportunities for giving students practice working with parameters in mathematics courses at all levels. Even in pre-calculus, we can use parameters when we describe shifting and stretching graphs of functions. Using software opens the door to many possibilities. For example, students can explore how the behavior of a dynamical system changes when the parameters are modified. Applets and other software that supports exploration and demonstration are often readily available online.

Proof: One role of proof is to convince both ourselves and others that the idea under consideration is, in fact, true. A proof of a theorem typically involves a chain of reasoning that starts with a given hypothesis and ends with a desired conclusion. Each link in the chain is forged by careful justification, which may involve an appeal to an axiom, definition, or previously proven result. A student who can break down a proof into these individual steps and provide arguments for their validity will be better able to develop or work through a complicated derivation in a physics course.

Another important role of proof is to gain insight. In teaching proofs, we should give priority to those that foster understanding. The classic proof by mathematical induction, for example, that the sum S(n) of the first n positive integers is $n(n+1)/2$ is a correct proof. It may, however, be confusing to the student who has not mastered induction or isn't really sure why it is a valid proof technique. More importantly, it sheds no light on how one might have discovered that $S(n) = 1 + 2 + 3 + \;\; + (n-1) + n = n(n+1)/2$.

Consider, however, the approach often attributed to Gauss when asked as a schoolboy to sum the terms of an arithmetic progression: write S(n) in two ways, forward and backward:

$$\begin{aligned} S(n) &= 1 + 2 + 3 + \cdots + (n-2) + (n-1) + n \\ S(n) &= n + (n-1) + (n-2) + \cdots + 3 + 2 + 1 \end{aligned}$$

and then add vertically to obtain

$$2S(n) = (n+1) + (n+1) + \cdots + (n+1)[\; n \text{ terms }] = n(n+1).$$

The second proof not only shows why the result is true. It also provides a technique for summing other arithmetic progressions. Most importantly, it teaches the power of looking at a problem in more than one way.

When we teach proofs we should be explicit in using them to help students think about the relevant concepts. Counterexamples are also important to help students see where the hypotheses of a theorem come from and why they are necessary. The fact that a continuous function on a closed interval achieves a maximum and a minimum on that interval can be nicely motivated if students are asked to supply counterexamples where one or more of these hypotheses fails to hold.

In some areas of modern physics, where physical intuition is not available, the mathematical argument may take the place of physical intuition. In particular, physics students need to grasp that reality doesn't always match one's intuition—what appears obvious is not always true. There are cases when it is necessary to trust the solid mathematical reasoning, even though the conclusion seems counterintuitive, or even nonsensical.

However, students can make mistakes, even when they are computationally skilled, and it can be very difficult for them to discover the source or location of their errors. It can be difficult to interpret physics into mathematics and vice versa (sign errors, set-up errors, etc., are common). Sometimes intuition aids in detecting and correcting these errors.

Research has shown that what might initially be understood as students' misconceptions (whether about mathematics or about physics) can often arise from ideas and understandings that might serve them well in other contexts or under different conditions (see [16] and [10]). For example, students may correctly intuit that flipping a fair coin over a large number of trials tends to even out the relative frequencies of heads and tails; but, often counter-intuitively, the likelihood of getting exactly 50% of each is increased by flipping fewer times. It is useful for faculty to try to recognize students' productive thinking (even when misapplied) and to reconcile misconceptions with reality rather than trying to undo or ignore them. In both mathematics and physics, a balance must be maintained between honing and developing our intuitions, understanding the scope and limits of their applicability, and learning to hold them all with a healthy skepticism.

One way to encourage students to learn from their misconceptions is through the use of clicker questions such as ConcepTests, originally developed by Eric Mazur for introductory physics courses (see [13] and [3]), but later expanded into calculus courses (see [14]) and elsewhere.

Communication skills, oral and written: Students should be able to use appropriate combinations of natural and mathematical language and mathematical notation to communicate their ideas, both with others "like them" and with those from different backgrounds. By communicating ideas, we understand mathematical ideas to be rooted in a context—in this case, the physical world. A mathematics course which does not have students regularly communicating mathematics is like a foreign language class where students never have to speak the language.

Students need the ability to use mathematical symbols and representational forms in ways that represent their use across the mathematical and physics communities. The increasing use of online homework systems may diminish students' opportunities to write out their thinking and communicate it to another person. Students need many opportunities to practice written and oral communication. Some online courseware (for example, Mastering Physics, WebAssign, and the development version of WeBWork) does allow students to write and have their writing evaluated, but current technology still makes the coherent development and simultaneous presentation of written language, mathematical notation, and graphical representations cumbersome for many ordinary student assignments. There are other mechanisms (peer editing, short assignments, group assignments) for making the evaluation of student writing more efficient.

Beyond learning to communicate their ideas, students should learn to have a mathematical conversation with someone else. The person or group being addressed may agree, not agree, find an error, ask a helpful question, or offer a suggestion. This kind of interplay of ideas is how science is done, and learning to communicate in this way is an important skill for a scientist. Working in groups helps students develop these skills. Eventually, they may learn to ask themselves questions and critically evaluate their own thinking.

Appendix on examples: We referred in this section to the complexities students face in more advanced physics courses. In an appendix we include three detailed examples of problems from such courses. For each example, there is a problem statement, a solution, and discussion of the mathematical challenges students face in understanding and solving the problem.

3 Mathematical content needed by students in this area

The mathematician studies what can be proved rigorously. The physicist does too,
but may often proceed by approximation, analogy, and numerical conjecture:
all of these activities require a strong mathematical foundation.
University of Maryland mathematics website
(double major for physics)

We argue that if students have a sound command of basic mathematical ideas and a good picture of overall mathematical structure (not just computational pieces), it helps them understand the concepts and structures in physics. Teachers of courses for mathematics majors typically have as an explicit goal giving their students this coherent view of the mathematical structures, not just of the calculations. Physics courses make good use of this mix of types of knowledge. A mathematics course that focuses almost solely on the calculations in the name of efficiency may be less likely to help students recognize that the mathematics produces coherent structures for certain types of objects (derivatives, functions, vectors), and that these structures are powerful. Exploring a variety of vector spaces, for example, may help sow the seeds to recognize that wherever modeling includes operations like addition and scalar multiplication, there may be a vector space lurking around; then the full power of linear algebra can be applied.

We can also turn this point around. It takes students a while to learn to apply their computational skills (which tends to be what they take away from their mathematics class). Can we expect that students will develop real understanding in a mathematics course if they are not applying the mathematical ideas in at least one other setting? If students see applications in their mathematics classes, they will begin to recognize the value of looking at the specific problem and at the underlying structure both in the domain of application and in the mathematics. The appropriate balance of mathematics and applications is, of course, dependent on the particular student population. There are good opportunities for this use of structure in applications in many of the specific courses mentioned below.

We offer the following tables (Table 1 and Table 2: Desired Mathematical Competencies for Physics Majors) instead of a list of courses. The labels in the tables don't always match courses taught in college. The competencies listed under Trigonometry, Geometry and Algebra might be gained in high school (and would be good preparation for an introductory non-calculus-based physics course). Those listed under Vector Analysis might be gained in Linear Algebra or Multivariable or Vector Calculus. Those listed under Complex Variables might be gained in Ordinary Differential Equations. Also, some topics on this list might be taught in a course offered in a science department (such as the bra and ket notation or Fourier series and transforms or partial differential equations).

Especially at the introductory level, mathematicians may look at the table entries and say, "yes, of course, we cover those topics." However, we want to make three points about these tables. First, we all know that what we "cover" in a term and what students learn are not necessarily the same. Just because a topic is included in a course, we cannot simply expect that our students had sufficient time or experience to learn it in a way that makes it useful to them in other classes or situations. Second, these topics are singled out because of their importance for physics students, and the lists say something about desired priorities in mathematics courses. Third, what physicists and mathematicians understand by some of these topics can be surprisingly different, which leads to our most important recommendation: *These tables should be a basis for discussion among mathematicians and physicists.*

Trigonometry Students should be able to: • Use the *unit circle definition of cosine and sine* to quickly determine the values of cosine and sine for common angles (in all quadrants). • Use the *unit circle definition of cosine and sine* to determine the coordinates of points on circles. • Use *triangle trigonometry* to relate the sides of a right triangle, not necessarily aligned to coordinate axes. • Look up and use *basic trigonometric identities* such as the double angle, half-angle, and addition formulas to simplify trigonometric expressions.	**Geometry** Students should be able to: • Use *geometric facts* such as the equality of opposing or opposite angles, or the angle sum of triangles, to relate angles in given diagrams. • *Draw* figures based on a written or symbolic description. • Choose and label *generic points* in figures, and use them in theoretical arguments.
Algebra Students should be able to: • Simplfy algebraic expressions by *factoring*, and recognize when to do so. • Solve algebraic expressions involving multiple *parameters* for particular variables. • Decide whether given information is sufficient to *determine a value* for a particular variable (and if it is, do so). • Fluently manipulate *exponents*, including negative and fractional exponents. • Rearrange complicated *multistory fractions involving symbolic expressions*. • *Rationalize denominators*. • Solve *systems of 2 equations in 2 unknowns* (and additional parameters) in several ways. • Solve *systems of 3 equations in 3 unknowns*. • Recognize that solving a system of equations corresponds to *intersection points of graphs of solution sets*.	**Multivariable/Vector Calculus** Students should be able to: • Visualize *curves and surfaces* in space. • Relate *multiple representations of functions of several variables*, such as graphs and level sets. • Evaluate *partial derivatives*. • Evaluate vector derivatives, such as the *gradient* of a function, and the *curl and divergence* of a vector field. • Use the relationship between level sets and the gradient to recognize the geometry of conservative vector fields. • Set up and evaluate *multiple integrals* in standard coordinate systems. • Express vectors in standard coordinate systems, using the associated adapted basis vectors. • Set up and evaluate line and surface integrals. • Use the relationship between flux and divergence to approximate one in terms of the other. • Use the relationship between curl and circulation to approximate one in terms of the other. • Use the Divergence Theorem to relate flux and divergence. • Use Stokes' Theorem to relate curl and circulation. • Convert multiple integrals from one coordinate system to another, using Jacobians or otherwise.
Single Variable Calculus Students should be able to: • Quickly and accurately *evaluate derivatives* of complicated expressions, including possible repeated use of chain rule. • Quickly and accurately *evaluate both indefinite and definite integrals* of elementary functions, including the use of simple substitutions. • Evaluate integrals using *integration by parts*. • Evaluate integrals using *trigonometric substitution*. • Regard derivatives as "ratios of small changes" (and vice versa). • Interpret integrals as *accumulation*, allowing integrals to be regarded as *"chopping and adding"* (and vice versa).	**Vector Analysis** Students should be able to: • Decide whether a given quantity should be represented as a *scalar or vector*. • Use the *dot product* to determine the components of vectors along other vectors. • Use vectors to relate the sides and angles of triangles (*triangle trigonometry using vectors*). • Use the *cross product* to determine directed area. • Convert fluently between representations of vectors in terms of *components* and in terms of *direction and magnitude*.

Table 1: Desired Mathematical Competencies for Physics Majors

Power Series Students should be able to:

- *Multiply series* together, collecting like terms.
- Use *radii of convergence* to expand power series about the appropriate point for a given situation.
- Use truncated power series to *approximate a function* near a point.
- Use more terms of a series to obtain a better approximation over larger interval, while realizing that this doesn't change the radius of convergence.
- Compare the *relative behavior* of common functions such as e^u, $1/u$, $1/u^2$, $\log u$, both as $u \to 0$ and $u \to \infty$.
- Recognize and manipulate expansions involving *geometric series*.

Fourier Series and Transforms Students should be able to:
Fourier series and transforms do not have a reliable place in the curriculum. In physics, Fourier series represent the archetype of eigenfunction expansions. Fourier transforms use continuous integrals rather than sums.

- Recognize and manipulate Fourier series expressed in both exponential and trigonometric form.
- Recognize and use Fourier transforms.

Linear Algebra Students should be able to:

- Quickly and accurately perform *basic matrix manipulations*, including the computation of determinants.
- Quickly and accurately find *eigenvalues and eigenvectors*, including cases with complex eigenvalues.
- Recognize *hermitian matrices*, and be aware of their properties (such as having real eigenvalues).
- Recognize a wide variety of *linear operators*, including derivatives, matrix multiplication, the dot product, etc.
- Recognize and manipulate *linear combinations* of vectors.
- Recognize a wide variety of *vector spaces*, including not only row and column vectors, but also both matrices and elementary function spaces.
- Fluently move between many representations for vectors, including arrows, columns, rows, abstract symbols, bras/kets, functions, and Fourier series.
- Interpret and identify a *basis*, including orthonormal bases, in many different contexts, and be able to use a basis to expand elements of a given vector space.
- Expand vectors in terms of an *eigenbasis*, and recognize when doing so is useful in problem solving.

Complex Variables Students should be able to:
Again, complex variable theory does not have a standard place in the curriculum. All physics students need fluency with complex numbers, but only the most advanced students need topics from complex analysis, such as analyticity or contour integration.

- Fluently evaluate expressions involving *complex arithmetic*.
- Compute the *complex conjugate* of complicated expressions.
- Distinguish between z^2 and $|z|^2$.
- Convert fluently between the *rectangular and polar (exponential) form* of complex numbers, including being able to recognize that $e^{2i\theta} + 1 = e^{i\theta}(2\cos\theta)$.
- Use *Euler's formula* to simplify complex expressions, including trigonometric identities.
- *"Rationalize" the denominator to eliminate i*; that is, recognize that
$$\frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{c^2+d^2}$$
even when a, b, c, d are complicated expressions.

Ordinary Differential Equations Students should be able to:

- Understand the meaning of basic vocabulary, such as the terms *1st-order*, *2nd-order*, *linear*, *nonlinear*, *homogeneous*, *inhomogeneous*.
- Solve *systems of linear differential* equations with constant coefficients.

Partial Differential Equations Students should be able to:
This is advanced content that often appears in a separate "math methods" course taught by physicists.

- Understand the meaning of basic vocabulary, such as the terms *elliptic*, *parabolic*, *hyperbolic*.
- Solve PDEs using *separation of variables*.
- Recognize and manipulate standard *eigenfunction expansions* involving rectangular, cylindrical, and spherical coordinates.

Table 2: Desired Mathematical Competencies for Physics Majors, continued

4 Technology and computation

Students studying physics need some experience with software (Maple, Mathematica, Matlab, Sage, etc.), and they need to know how to program in some language. They might acquire this knowledge either in a specific course (in mathematics or in physics) or in a lab course, or they might pick it up on their own. Students need at least minimum competency in programming in order to develop familiarity with common constructs. One way to facilitate this is to require students to develop some familiarity with at least two different languages. Indeed, some constructive methods have become much more interesting and valuable because of the existence of computer tools. These methods are useful because they are computational. The fact that many physicists will be doing computations using technology should inform the choice of topics, methods of proof and examples in mathematics classes.

Using technology we can produce marvelous visualizations, even of analytic examples, that can be displayed quickly and easily, enrich class discussion, and permit assigning students rich problems for investigation. A good example is in ODEs, where visualization technology enables the description and analysis of both long-term and local parametric behavior.

However, there are also dangers in utilizing the capabilities of technology. For example, in multivariable calculus it becomes easier to describe bizarre surfaces, so there can be a tendency to focus on those instead of on the more standard highly symmetric surfaces. Physics students should learn how to exploit the symmetry of these standard situations in solving problems. There can also be a danger in using only a brute force approach because it hides some of the beauty. On the other hand, many courses that do not include technological capabilities focus too strongly on symmetric (but unrealistic) examples that can be solved analytically. This hides the value and power of the mathematical tools for more general situations. A balance is needed here.

The fact that a working physicist will have Wolfram Alpha on her desktop does mean something. This reality should inform our choices about mathematics course content and our expectations of what students can do. But we have to use the technology thoughtfully. For a related example, calculators can be used effectively in the elementary grades to help students learn number sense, but often that is not how they are used. We all know students who rely on this technology when they should not. Use of symbolic manipulators should be matched with an understanding of the underlying concepts and the limitations with the tool. (See [2].)

5 Pedagogy

As this is written, the MAA Committee on the Teaching of Undergraduate Mathematics (CTUM) is at work on a companion to this document that focuses on pedagogy. It will appear a year or two after the CUPM report. For this reason, we keep our comments here quite brief.

An instructor's decisions about the learning and content goals for a course must inform and shape his or her decisions about pedagogy. And it goes both ways: what our students can achieve depends on our pedagogical choices.

There are many pedagogical strategies, and we owe it to our students to inform ourselves of the options and learn to expand our teaching repertoire. It is important for the classroom pedagogy we choose to provide sufficient feedback to the instructor on how the student is interpreting and understanding the material. There are many options, old-fashioned and modern, to meet this goal. Examples include using clickers, having students write on white boards, including time for student questions and discussion, having students write "minute papers", etc. The recommendations in several earlier sections of this report imply the use of pedagogical strategies that create opportunities for students to communicate in writing and speaking, to solve and discuss modeling-type exercises, and to articulate and evaluate their analysis of a problem and their choice of mathematical methods.

6 Resources

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   PPLATO (Promoting Physics Learning And Teaching Opportunities) is a UK initiative to establish a set of computer-based resources to support the teaching of physics and physics-related mathematics topics. It began in 2004, based on the existing FLAP (Flexible Learning Approach to Physics) material written by the Open University in the mid-1990s.

2. The International History, Philosophy and Science Teaching Group has issued a call for papers (due July 2014) for a special issue of the journal *Science & Education* on "The Interplay of Physics and Mathematics," to be edited by Ricardo Karam of the University of Hamburg.

Sample programs/courses

- *Oregon State University*
  Paradigms in Physics, `http://physics.oregonstate.edu/portfolioswiki`. Complete redesign of physics major around unifying mathematical themes (such as spherical symmetry). Continuously funded by NSF since 1997. Primarily affects physics majors. However, the current research focus is on student mastery of partial derivatives across multiple disciplines.

  Vector Calculus Bridge Project, `http://www.math.oregonstate.edu/bridge`. Coherent geometric redesign of (multivariable and) vector calculus, one of whose goals is a seamless transition to junior-level physics (electromagnetism) courses. Funded independently by NSF from 2001-2007, and ever since as part of the Paradigms in Physics project. Includes freely available online second-year calculus text.

- *University of Maryland College Park* Applied mathematics track: Calc I, II, III, Linear Algebra, ODE, Introduction to proof + 8 more: introduction to probability, statistics 4xx, applied statistics, AMSC 460 or 466, another applied statistics , plus one of . . . (list of course titles on web page didn't include all), plus a programming course or equivalent and a 3-course sequence in area of application.

  Double major track: Beyond the mathematics required for the physics major (Calc I, II, III, Linear algebra, ODE), suggests: Complex variables for engineers, introduction to nonlinear dynamics and chaos, two-semester advanced calculus sequence, abstract algebra ("less crucial" than analysis), introductions to ODE and PDE, introduction to Fourier analysis, differential geometry, probability, courses in computational and applied mathematics.

- *University of Michigan*: major in mathematical sciences must choose one of eight specializations, among them Mathematical Physics. It is designed to facilitate a concurrent major in physics including at least 2 physics courses from among a list of 7 and requiring math courses: Intermediate differential equations, Boundary value problems for PDE, Introduction to complex variables, Introduction to differential geometry, Introduction to numerical methods.

- *University of New Hampshire* Connecting Meaning and Mathematics, Dawn Meredith, Physics, http://pubpages.unh.edu/~dawnm/connectm&m.html. Includes guided inquiry tutorials and overviews. Most pieces written for an upper level undergraduate classical mechanics course but could yield useful examples or projects for a mathematics course.

- *University of Portland* Mathematical Methods for Science and Engineering. This course is offered under both mathematics and physics course numbers, but is taught by the physics department. Unlike similar courses typically taken by juniors or seniors, this course targets sophomores, is required by all physics majors, and is also taken by many mathematics majors and physics minors. According to the listed course outcomes, students completing this course will, among other things, be able to work with infinite series and complex numbers, use vector calculus, solve ordinary and partial differential equations, and solve eigenvalue problems. Contacts are Tamar More (more@up.edu) and Max Schlosshauer (schlossh@up.edu).

Combined calculus/calculus-based physics courses (lower division)

- *Princeton University* http://commons.princeton.edu/kellercenter/courses/emp.html. Offers an integrated engineering, physics and engineering experience for first year students. The primary goal of this integrated course is to give first-year engineering students a sense of the excitement of modern engineering and of the foundational roles that mathematics and physics play in various engineering disciplines

- *Technologico de Monterrey* (ITESM) Integrated Physics and Mathematics course for engineering students: A first experience. See the article by that name by A. Dominguez, G. Zavala and J. A. Alanis, Paper ID#7997 in *120th ASEE Annual Conference & Exposition*, June, 2013.

- *Union College*
  http://www.union.edu/academic_depts/physics_astronomy/phys_courses.php.
  Offers an integrated first year physics and calculus course. Course description: IMP 111, 112, 113: Integrated Math and Physics. In addition to the introductory sequence described above, we also offer a year-long introductory sequence combining introductory physics (the content of Physics 120 and 121) with introductory calculus (Math 113, Math 115, and Math 117). The class is team-taught by faculty members from Physics and Mathematics, integrating the two subject areas together, so that new mathematical tools and concepts are introduced just as they are needed to understand new areas of physics. The course is designed for science and engineering majors.

- *University of Guelph* http://www.physics.uoguelph.ca/single_course.php?idx=111. Integrated Mathematics and Physics. This is a foundational course for students in B.Sc. mathematical and physical sciences majors. The disciplines of Mathematics and Physics are taught in an integrated fashion that demonstrates how they support and enrich one another. Atomic structure, algebra and trigonometry, forces and Newtons laws, functions and graphing, differentiation, angular momentum and energy conservation, limits, integration, kinematics, simple harmonic motion, and special relativity are presented in a harmonized fashion to insure students have an improved understanding of these fundamentals. The course is intended to give a student grounding in topics in physics and calculus in a manner that uses the physics as an example to ground the calculus and provides the calculus needed for the topics in physics. This integration of the two topics is intended to make both sets of material easier to absorb.

- *University of Puget Sound*, Tacoma, WA Martin Jackson (martinj@ups.edu) Textbook: *Integrated Physics and Calculus*, Andrew Rex & Martin Jackson, Addison Wesley, 1999. (Course last offered in 2007.)

7 APPENDIX: Typical problems from upper level physics courses

We present here three examples that are fairly typical of junior-level coursework for physics majors, followed in each case by a brief discussion, emphasizing the challenges students have when applying their prior mathematical knowledge. Bear in mind that these examples arise shortly after students have learned the underlying mathematical concepts.

The electric field above a line segment

Problem

Find the electric field a distance z above the midpoint of a straight line segment of length $2L$ that carries a uniform line charge λ.

(*David J. Griffiths*, **Introduction to Electrodynamics**, 4th edition, Pearson (2013), 64–65.

Background

The electric field $\vec{\boldsymbol{E}}$ due to a point charge q at the origin is

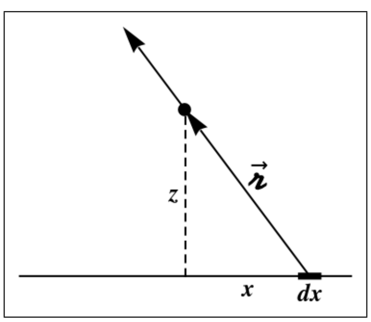
$$\vec{\boldsymbol{E}} = \frac{1}{4\pi\epsilon_0}\frac{q\,\hat{\boldsymbol{r}}}{r^2} \tag{1}$$

This leads through the superposition principle to the electric field of a linear charge distribution λ along a curve C being

$$\vec{\boldsymbol{E}} = \frac{1}{4\pi\epsilon_0}\int_C \frac{\lambda\,\hat{\boldsymbol{\mathscr{r}}}\,d\ell'}{|\vec{\boldsymbol{\mathscr{r}}}|^2} \tag{2}$$

where $\vec{\boldsymbol{E}}$ is regarded as a function of position $\vec{\boldsymbol{r}}$, and the integral is over the position $\vec{\boldsymbol{r}}'$ along the curve. Thus, λ depends on $\vec{\boldsymbol{r}}'$, and $d\ell' = |d\vec{\boldsymbol{r}}'|$.

Solution



We have

$$\begin{gathered}
\vec{\boldsymbol{r}} = z\,\hat{\boldsymbol{z}} \qquad \vec{\boldsymbol{r}}' = x\,\hat{\boldsymbol{x}} \qquad d\ell' = dx \\
\vec{\boldsymbol{\mathscr{r}}} = \vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}' = z\,\hat{\boldsymbol{z}} - x\,\hat{\boldsymbol{x}} \qquad |\vec{\boldsymbol{\mathscr{r}}}| = \sqrt{z^2 + x^2} \\
\hat{\boldsymbol{\mathscr{r}}} = \frac{\vec{\boldsymbol{\mathscr{r}}}}{|\vec{\boldsymbol{\mathscr{r}}}|} = \frac{z\,\hat{\boldsymbol{z}} - x\,\hat{\boldsymbol{x}}}{\sqrt{z^2 + x^2}}
\end{gathered} \tag{3}$$

Thus,

$$\begin{aligned}
\vec{\boldsymbol{E}} &= \frac{1}{4\pi\epsilon_0}\int_{-L}^{L} \frac{\lambda}{z^2 + x^2}\frac{z\,\hat{\boldsymbol{z}} - x\,\hat{\boldsymbol{x}}}{\sqrt{z^2 + x^2}}dx \\
\vec{\boldsymbol{E}} &= \frac{\lambda}{4\pi\epsilon_0}\left[z\,\hat{\boldsymbol{z}}\int_{-L}^{L}\frac{1}{(z^2 + x^2)^{3/2}}\,dx - \hat{\boldsymbol{x}}\int_{-L}^{L}\frac{x}{(z^2 + x^2)^{3/2}}\,dx\right] \\
&= \frac{\lambda}{4\pi\epsilon_0}\left[z\,\hat{\boldsymbol{z}}\left(\frac{x}{z^2\sqrt{z^2 + x^2}}\right)\Bigg|_{-L}^{L} - \hat{\boldsymbol{x}}\left(-\frac{1}{\sqrt{z^2 + x^2}}\right)\Bigg|_{-L}^{L}\right] \\
&= \frac{1}{4\pi\epsilon_0}\frac{2\lambda L}{z\sqrt{z^2 + L^2}}\hat{\boldsymbol{z}}
\end{aligned} \tag{4}$$

For points far from the line ($z \gg L$),

$$E \cong \frac{1}{4\pi\epsilon_0} \frac{2\lambda L}{z^2} \tag{5}$$

This makes sense: From far away the line looks like a point charge $q = 2\lambda L$. In the limit $L \mapsto \infty$, on the other hand, we obtain the field of an infinite straight wire:

$$E = \frac{1}{4\pi\epsilon_0} \frac{2\lambda}{z} \tag{6}$$

Discussion

- Note that $\vec{\boldsymbol{r}} = z\,\hat{z}$ describes the vector $\vec{\boldsymbol{r}}$ in terms of its length z and its direction $\hat{z}$. (The hat in $\hat{z}$ denotes a unit vector.)
- One major confusion here is the need to distinguish between $\vec{\boldsymbol{r}}$, which gives the functional dependence of the final answer, and $\vec{\boldsymbol{r}}'$, which gives the functional dependence of the integrand.
- Equivalently, students need to distinguish between the dummy variable x, which should not appear in the final answer, and the position z, which is a constant during integration, but the only variable in the final answer.
- The problem statement in Griffiths includes a picture, showing x, z, a typical piece dx of the line segment, and the vectors $\vec{\boldsymbol{r}}'$ and $\vec{\boldsymbol{\mathscr{r}}}$. This picture is crucial to the first step of identifying the components of $\vec{\boldsymbol{r}}$ and $\vec{\boldsymbol{r}}'$.
- Notation such as $\vec{\boldsymbol{E}} = E\hat{\boldsymbol{E}}$, in which base letters such as E are reused for a vector ($\vec{\boldsymbol{E}}$), and the magnitude (E) and direction $\hat{\boldsymbol{E}}$ of the vector, is common in physics, but rare in mathematics.
- The integrals themselves are doable, but not easy; ditto for the algebraic simplifications in the last step.
- The casual discussion of limiting cases at the end is an automatic reality check for physicists, but not yet second-nature for students. Furthermore, checking those limiting cases amounts to taking the leading term of a power series, a competency physicists expect students to have already mastered.

Solutions of Schrödinger's Equation

Problem

Find the general solution of the Schrödinger equation for a time-independent Hamiltonian.

(*David H. McIntyre*, **Quantum Mechanics: A Paradigms Approach**, Pearson Addison-Wesley (2012), 68–70.

Background

The time evolution of a quantum system is determined by the Hamiltonian or total energy operator H through the Schrödinger equation

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = H|\psi(t)\rangle \tag{7}$$

The energy eigenvalue equation is

$$H|E_n\rangle = E_n|E_n\rangle \tag{8}$$

The eigenvectors of the Hamiltonian form a complete basis, so that

$$|\psi(t)\rangle = \sum_n c_n(t)|E_n\rangle \tag{9}$$

We assume that H, and hence $|E_n\rangle$ and E_n, are independent of t.

Solution

Substitute the general state $\psi(t)$ above into the Schrödinger equation:

$$i\hbar\frac{d}{dt}\sum_n c_n(t)|E_n\rangle = H\sum_n c_n(t)|E_n\rangle \tag{10}$$

and use the energy eigenvalue equation to obtain

$$i\hbar\sum_n \frac{dc_n(t)}{dt}|E_n\rangle = \sum_n c_n(t)E_n|E_n\rangle \tag{11}$$

Each side of this equation is a sum over all the energy states of the system. To simplify this equation, we isolate single terms in these two sums by taking the inner product of the ket on each side with one particular ket $|E_k\rangle$. The orthonormality condition $\langle E_k|E_n\rangle = \delta_{kn}$ then collapses the sums:

$$\langle E_k|i\hbar\sum_n \frac{dc_n(t)}{dt}|E_n\rangle = \langle E_k|\sum_n c_n(t)E_n|E_n\rangle \tag{12}$$

$$i\hbar\sum_n \frac{dc_n(t)}{dt}\langle E_k|E_n\rangle = \sum_n c_n(t)E_n\langle E_k|E_n\rangle \tag{13}$$

$$i\hbar\sum_n \frac{dc_n(t)}{dt}\delta_{kn} = \sum_n c_n(t)E_n\delta_{kn} \tag{14}$$

$$i\hbar\frac{dc_k(t)}{dt} = c_k(t)E_k \tag{15}$$

We are left with a single differential equation for each of the possible energy states of the system $k = 1, 2, 3, ...$ This first-order differential equation can be rewritten as

$$\frac{dc_k(t)}{dt} = -i\frac{E_k}{\hbar}\,c_k(t) \tag{16}$$

whose solution is a complex exponential

$$c_k(t) = c_k(0)e^{-iE_k/\hbar} \tag{17}$$

We have denoted the initial condition as $c_k(0)$, but we denote it simply as c_k hereafter. Each coefficient in the energy basis expansion of the state obeys the *same* form of the time dependence, but with a *different* exponent due to the different energies. The time-dependent solution for the full state vector is summarized by saying that if the initial state of the system at time $t = 0$ is

$$|\psi(0)\rangle = \sum_n c_n|E_n\rangle \tag{18}$$

then the time evolution of this state under the time-independent Hamiltonian H is

$$|\psi(t)\rangle = \sum_n c_n e^{-iE_n/\hbar}|E_n\rangle \tag{19}$$

Discussion

- This computation is very abstract! The state $|\psi\rangle$ could be a column vector, a function, or an element of some other vector space.
- The use of bra/ket notation for vectors is becoming standard in quantum mechanics, but forces students to translate their knowledge of linear algebra into a new notation.
- Note the reuse of labels when writing the eigenstate with eigenvalue E_n as $|E_n\rangle$.
- The ability to pull operators through sums is "just" linearity, but not yet easy for students.
- The casual use of the Kronecker delta may be unfamiliar to students.
- An interesting feature of this presentation (and the book from which it comes) is the emphasis on using basis vectors to determine vector components. In the more familiar language of vector analysis, this is the statement that the x-component F_x of a vector field $\vec{\boldsymbol{F}}$ is given by

$$F_x = \hat{\boldsymbol{x}} \cdot \vec{\boldsymbol{F}} \tag{20}$$

  Thus, when determining the differential equation for $c_k(t)$, an explicit dot product is used, rather than simply equating the components of the vector. One could argue that this approach is unnecessary in this example, but it is crucial in situations where the given vector is *not* already expressed in terms of the desired basis.

Maxwell relations

Problem

Derive the Maxwell relation associated with internal energy.
(*Ashley H. Carter*, **Classical and Statistical Thermodynamics**, Prentice-Hall (2001), 134.

Background

For homogeneous systems, with a well-defined *temperature* T and *pressure* P, the first law of thermodynamics becomes the *fundamental thermodynamic relation*

$$dU = T\,dS - P\,dV \tag{21}$$

where V is the *volume* of the system, S its *entropy*, and U denotes the *internal energy*.

Solution

The internal energy U is a state variable whose differential is exact. Since the value of a mixed second partial derivative is independent of the order in which the differentiation is applied, we have

$$dU = T\,dS + (-P)\,dV = \left(\frac{\partial U}{\partial S}\right)_V dS + \left(\frac{\partial U}{\partial V}\right)_S dV \tag{22}$$

The exactness of dU immediately gives

$$\frac{\partial^2 U}{\partial V\,\partial S} = \left(\frac{\partial T}{\partial V}\right)_S = \frac{\partial^2 U}{\partial S\,\partial V} = -\left(\frac{\partial P}{\partial S}\right)_V \tag{23}$$

The equality of the first derivatives,

$$\left(\frac{\partial T}{\partial V}\right)_S = -\left(\frac{\partial P}{\partial S}\right)_V \tag{24}$$

is known as a *Maxwell relation*. Its utility lies in the fact that each of the partials is a state variable that can be integrated along any convenient reversible path to obtain differences in values of the fundamental state variables between given equilibrium states.

Discussion

- First of all, note the names of the variables: T, S, P, V, U; there's not an "x" in sight.
- It is not obvious which of these variables are dependent, and which are independent. A common feature of problems in thermodynamics is that equations express relationships among physical quantities, any of which can be regarded as independent in appropriate circumstances. The above exercise is typically followed by similar derivations starting from

$$dH = T\,dS + V\,dP \tag{25}$$

$$dA = -S\,dT - P\,dV \tag{26}$$

$$dG = -S\,dT + V\,dP \tag{27}$$

  where H is the *enthalpy*, A the *Helmholtz free energy*, and G the *Gibbs free energy*. Students find this changing role of the given quantities from one context to another to be quite confusing.
- This flexibility in the designation of independent variables leads naturally to the need to specify explicitly which quantities are being held fixed, as denoted above with subscripts. "The derivative of V with respect to P" is ambiguous, since

$$\left(\frac{\partial V}{\partial P}\right)_S \neq \left(\frac{\partial V}{\partial P}\right)_T \tag{28}$$

- Manipulations such as these require comfort with differentials, yet differentials tend to be used in calculus only for linear approximations, a very different concept.