2015 CUPM Curriculum Guide to Majors in the Mathematical Sciences

Carol S. Schumacher and Martha J. Siegel, Co-Chairs
Paul Zorn, Editor

MAA
MATHEMATICAL ASSOCIATION OF AMERICA

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Introduction

The Mathematical Association of America (MAA), through its Committee on the Undergraduate Program in Mathematics (CUPM), presents this *2015 CUPM Curriculum Guide to Majors in the Mathematical Sciences*. CUPM interprets its task broadly, both in affirming the breadth of possible offerings to undergraduate students majoring in mathematics and in understanding "mathematics" broadly to encompass what we call the mathematical sciences.

The roots of CUPM stretch back to the establishment of the Association in 1915. For many years CUPM has led MAA's effort to guide the undergraduate curriculum in mathematics. In the October 2013 issue of the *American Mathematical Monthly*, Alan Tucker traces the history and development of the mathematics major in the US; this article is included in a rich electronic archive of CUPM's work (www.maa.org/cupm). While the history of CUPM informs this Guide, we have tried to add information useful to today's mathematics departments as they continue to shape vital and effective undergraduate curricula.

Curricular change occurs slowly. *Reshaping College Mathematics* (MAA Notes #13, 1989), the 1991 CUPM report, *The Undergraduate Major in the Mathematical Sciences,* and *Confronting the Core Curriculum* (1998 MAA Notes # 45) all contain advice to the mathematical community with many of the ideas presented here. We urge faculty to read the 2001 CUPM Discussion Papers about Mathematics and the Mathematical Sciences in 2010: What Should Students Know? to gauge how well their programs measure up to those expectations. From reports at focus groups and from data collected by CBMS surveys, we have found that departments have expanded their scope to more of the mathematical sciences, incorporating more statistics and probability, computing, discrete methods and operations research. Indeed, we see that many mathematical sciences programs are becoming more applied and more interdisciplinary. At the same time, publications of the National Academies of Science have served to inform the mathematical community and the broader public about the ubiquity of mathematics.

The previous *CUPM Curriculum Guide* was published in 2004; it was followed by several related publications. Among them we call particular attention to the work of the CUPM Subcommittee on Curriculum Renewal Across the First Two Years (CRAFTY) on the Curriculum Foundations Project, which emphasized the use of mathematics in other disciplines, and informs this *Guide*.

Recommendations in this *Guide* reflect CUPM's reaffirmation of the principles in the 2004 *Guide*. Those principles, approved by MAA's Board of Governors in 2003 and reapproved in August 2014, can be found online. The 2004 *Guide* addressed the full range of mathematics offerings, including general education, service, and major courses. This *Guide* does not systematically address these non-major courses. This Guide focuses specifically on the design of mathematics majors, addressing the curricular demands of the wide—and widening—variety of mathematics programs now found across the nation. That diversity often leads to minors, concentrations, double majors, and interdisciplinary majors, as well as full majors in new and developing mathematically rich fields. The purpose of this *Guide* is to help departments adapt their undergraduate curricula to this changing landscape while maintaining the essential components of the traditional mathematics major.

Many forces can be expected to drive curricular change. The adoption of the Common Core State Standards by many U.S. states, for instance, will significantly change the preparation of incoming college students and will necessitate changes in the preparation of pre-service teachers. Two-year colleges have seen rising enrollments; many of their graduates transfer to four-year institutions. The publications of the American Mathematics Association of Two-Year Colleges (AMATYC) and the work of MAA's own Committee on Two-Year Colleges and its Committee on Curriculum Renewal Across the First Two Years (CRAFTY) have provided guidance over the years on the mathematics of the first two years and on developmental courses in the subject. Recent publications, such as that of the INGenIOuS Project, urge mathematics departments to prepare students for the modern workforce. Transforming Post-Secondary Education in Mathematics (TPSE), a project sponsored jointly by the Carnegie Corporation of New York and the Alfred P. Sloan Foundation, urges constructive change in mathematics education at community colleges, 4-year colleges, and research universities, especially in the first two years.

CUPM urges departments to think creatively about their introductory courses. Departments might consider revisions to their developmental programs that could attract underprepared students to take more mathematics, perhaps even leading to a minor in the subject. Such programs as Quantway, Statway, and the newly developed Mathway are examples of such mainstreaming and just-in-time courses for those students. Students who have Advanced Placement credit in mathematics should be offered interesting courses that explore the nature of modern mathematics and its applications. Students are attracted to mathematics for its beauty, its utility, and its intellectual merits. National efforts should be made to develop materials for challenging and interesting courses that include bridges and alternate pathways to the major.

New uses for mathematics in many disciplines have emerged in the decade since the 2004 *Guide*. The reports *The Mathematical Sciences in* 2025 and *Fueling Innovation and Discovery: The Mathematical Sciences in the 21st Century*, both publications of the Board of Mathematical Sciences and Their Applications of the National Academies, explore new areas of the physical, biological, and geophysical sciences; and of engineering, computing, and social sciences in which mathematics will play an essential part. The 2012 PCAST report, *Engage to Excel*, described strategies for producing one million additional degrees in STEM fields. This *Guide* is a partial response to the call for increasing the number of STEM degrees. The *2015 Curriculum Guide* focuses on undergraduate programs that are mathematically intensive and offers recommendations on the mathematics major and on mathematical sciences programs that tend to be interdisciplinary.

As work progressed on this *Guide*, the education committees of both the American Statistical Association (ASA) and the Society for Industrial and Applied Mathematics (SIAM) were preparing their own recommendations for undergraduate degrees in statistics and applied mathematics, respectively. CUPM has benefited from their work and hopes for continued consultation and cooperation among our societies.

Preparation for this *Guide*

Work on this *Guide* began around 2009, when CUPM began sponsoring focus group discussions to assess the community's general need for a new curriculum guide and, in particular, to gauge interest in addressing the variety of possible undergraduate paths to a major in mathematics. CUPM also began, through surveys and focus groups, to identify essential components of interdisciplinary majors that might be described as majors in a mathematical science. Broad agreement seems to exist on "core" elements of both a traditional mathematics program and a broader mathematical sciences major. We attempt to capture these elements in the Overview document that follows.

Although the *Guide* is a project of CUPM, a Steering Committee (members are listed at the end of this introduction) was appointed to refine the conceptual framework of the project and to manage its development. The Steering Committee helped in selecting the many small study groups contributing to this *Guide* and in improving their reports. It was agreed to produce both a relatively brief printed *2015 Curriculum*

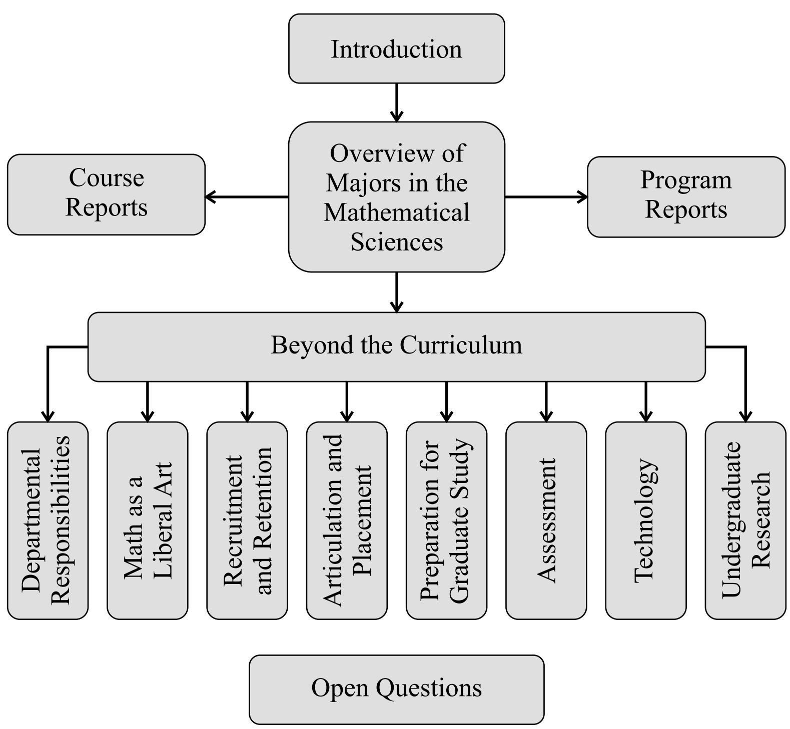
Guide to Majors in the Mathematical Sciences and a more extended online set of companion resources. The title reflects CUPM's intention to consider not only the "traditional" mathematics major but also programs that serve students who pursue (through majors, minors, concentrations, etc.) degrees across the range of mathematical sciences.

It was also agreed that this *Guide* would focus mainly on the curriculum itself, without aiming to discuss in detail the full set of related issues, including pedagogy, access, technology, articulation, placement, and diversity. Nevertheless, we acknowledge the critical importance of addressing these issues in delivering a curriculum effectively; for this reason we have included several brief reports on these matters in the section called Beyond the Curriculum. Where MAA committees have done recent work in crucial but non-curricular areas that affect teaching and learning, committee summaries are included as reports in that section. Where possible, we have cited models of effective programs, new courses, or best practices we recommend for investigation by departments aiming to improve in these areas.

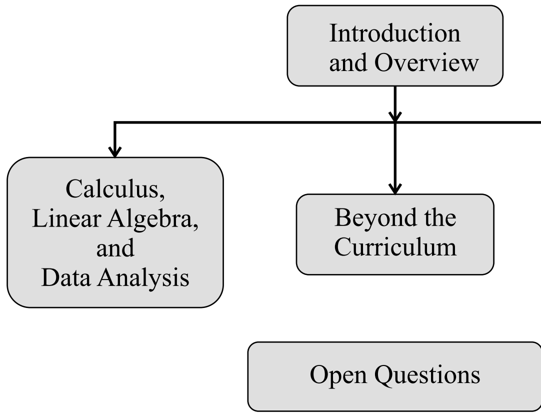
While this *Guide* pays only secondary attention to pedagogy, we fully acknowledge that the subject deserves its own serious attention and discussion in the community. MAA's Committee on the Teaching of Undergraduate Mathematics has launched its own study and will publish its own report on the growing body of knowledge on the pedagogy of collegiate mathematics education. Pedagogy and the curriculum are so closely related that the Cognitive Goals in this *Guide* were developed to reflect CUPM's firm belief that the major in any mathematical science should promote very special habits of mind. The intentional cultivation of these mathematical habits of mind takes precedence over a strict syllabus in all courses. The close connection between pedagogy and the curriculum is also reflected in the parallel cognitive and content recommendations in the Overview.

How to use this *Guide*

This *Guide* will be delivered in two forms: an abbreviated print version and a longer and dynamic online version. There are six parts to this report: Introduction, Overview of the Major, Course Area Group Reports, Program Area Group Reports, Beyond the Curriculum, and Open Questions. The diagram below describes their organization. The entire document and references will be available on maa.org/cupm.



The printed version includes the Introduction; Overview; the Calculus, Linear Algebra, and Data Analysis Course Area Group Reports; Preparation for Graduate Education; Beyond the Curriculum; and Open Questions.



Introduction and Overview

Everyone should read this Introduction and the Overview. The Overview contains the cognitive and content recommendations for all mathematical sciences major programs. CUPM regards these recommendations, approved by the MAA Board of Governors in August 2014, as the core of a mathematics major of any type.

The Overview was prepared after sustained consultation with MAA members and with the mathematical community at large. Several focus groups met at MathFest 2013 and offered constructive suggestions on a draft. At JMM 2014, CUPM, the Board of Governors and undergraduate program chairs from across the country contributed their views on the revised Overview. Member organizations of CBMS were also invited to comment on the Overview and on their choice of any documents in this *Guide*. The final form of these recommendations reflects the wisdom and experience of many contributors.

The Overview aims to help frame discussion for the design of any degree in the mathematical sciences. It is based on the premise that every department should develop and articulate a set of goals, and should intentionally craft a program that helps its students meet those goals. We recommend that departments discuss both cognitive and content goals and how they relate to each other and to the department's programs. We emphasize that there are many ways to craft a major: we see one type of program as treating mathematics as a liberal art, another concentrating on applied mathematics in general, another preparing students for professional work (such as teacher education or actuarial science), or preparing students for graduate school. We believe that our recommended cognitive and content goals form the framework of every one of these programs.

Every department and every institution has a unique mission. Curricular planning in any department should therefore be informed by recommendations from the mathematical community as well as by local knowledge of students, faculty, resources, administration, and, where they apply, governmental constraints and support. We emphasize that in order to attain its cognitive goals, a department must design not only courses but also specific exercises and activities that are constructed explicitly to advance students to the next cognitive level.

Every department should have a regular schedule of program review and self-study. The assessment of a department's progress should be regular and pre-planned, with data collected and analyzed systematically. Carefully facilitated departmental retreats for small departments (or for subgroups of large departments) work well for designing curricula to meet established goals. MAA's Committee on Departmental Review has prepared materials to assist departments with self-studies and periodic reviews.

CUPM has attempted to integrate recommendations for the traditional mathematics major program that often strove to prepare all students for graduate study with recommendations for a major that encourages students who love mathematics to apply their abilities to those fields in which mathematics plays an increasingly important role. To that end we have reports on many program areas and the many courses that support them.

Course Area Study Groups (CASGs)

Focus groups held as early as 2009 revealed that the names of common courses such as Real Analysis and Linear Algebra mean different things to different people. Discussions sometimes foundered over terms that, although "locally well defined," were understood differently among participants. This persistent miscommunication led to the formation of CASGs.

CASGs were asked to present several reasonable versions of courses, each with its natural student audience and, where possible, a few references. Several standing MAA committees, such as the Committee on Mathematics Across the Disciplines (MAD) and the ASA-MAA Joint Committee on Statistics, made substantial contributions to course area reports. We urged CASGs to consider cognitive goals along with content, and we welcomed diversity of viewpoint and discouraged votes to "prefer" one type of course over another.

Each CASG had three to six members, each with expertise in the area. Some CASG members are not MAA members; some are not mathematicians, though each group member has considerable experience with the subject matter. To get a broad view of courses we invited professors, users, and researchers in these areas. Some members are textbook authors, familiar with the range and level of courses offered in a given area in a wide variety of schools.

Program Area Study Groups (PASGs)

PASGs were invited to describe programs in a variety of mathematical sciences as well as in applied areas, usually within undergraduate departments. Several directors of graduate programs were asked for their perspectives on what makes good undergraduate preparation for graduate study in the field. A typical PASG included four to seven members, representing different types of schools and departments. Many members had initiated or run a major track, a full major, a minor, or a small concentration in the given area. Some non-mathematicians also served on PASGs, representing their disciplines. In many cases PASGs have offered suggestions for a full-fledged joint major involving mathematics and another discipline. In those areas we have included recommendations for adaptations for those departments with limited resources or fewer faculty and students.

One of the most important responsibilities of many mathematics departments is the preparation of teachers. This *Guide* features the recommendations of MAA's Committee on the Education of Teachers (COMET) for a professional program for the preparation of pre-service middle school and secondary school teachers of mathematics. Implementation of the program that is more of a professional track than the traditional liberal arts track is carefully discussed. The report marks the most significant changes in COMET's recommendations for the mathematical education of pre-service middle and high school teachers in 25 years.

PASG reports reflect the work of many entities. MAA SIGMAAs (special interest groups) were invited to contribute to reports in their areas. MAA's Committee on the Mathematical Education of Teachers (COMET) and the ASA-MAA Joint Committee on Statistics were largely responsible for the reports on teacher education and statistics, respectively. We also relied heavily on members of the Society of Actuaries, the ASA, the Association for Computing Machinery, the American Chemical Society, the Society for

Mathematical Biology, the American Mathematical Society, the National Council of Teachers of Mathematics, and SIAM. We thank our many colleagues in other STEM disciplines for their assistance.

Beyond the Curriculum

Mathematics is more than curriculum. It is also a human activity. This *Guide* is principally about curricular and cognitive elements in the teaching and learning of mathematics, but we acknowledge that many social factors also play important roles in mathematics classrooms. Whenever we design curricula or single courses, we should consider students' backgrounds and what they bring into our courses or programs—as well as what we would like them to learn. Reports on building community and culture as well as ideas about recruitment and retention of majors were written by mathematicians with well-known, successful programs, especially for underrepresented groups.

Most of the other "Beyond the Curriculum" reports in this *Guide* are summaries from various MAA committees. The MAA's Committees on Assessment, Articulation and Placement, Research by Undergraduates, and Teaching of Undergraduate Mathematics all contributed to this report. A special focus group on Technology and the Mathematics Curriculum held at Mathfest 2014 led to the formation of a working group on that subject. While national professional norms and accreditation standards, federal guidelines, state laws, institutional requirements, and student interests and abilities influence the curricular process, the *department* is the unit that serves as the primary leader in effecting actual change in the curriculum and beyond. The Departmental Responsibilities section points to the major elements necessary for a department's systematic approach to carrying out its mission.

Online resources

Reports in the *Guide* point to many online documents and resources. We think of the report and its recommendations as evolving with time. CUPM expects these resources to be helpful now, but the committee plans to have them regularly updated by CASG- and PASG-type interest groups, and by the community at large. CUPM and the Steering Committee have suggested that MAA form online communities in both course and program areas, building toward rich, current resources of ideas, course modules, software, and other curricular materials.

CUPM in the future

CUPM is fully aware that in a changing mathematical world any static curriculum guide is necessarily incomplete and soon out of date. The current committee strongly recommends that CUPM, its subcommittees, or appropriate *ad hoc* task forces be charged with maintaining a dynamic document based on this *Guide*. Both the Steering Committee and the current members of CUPM suggest that CUPM be charged to update this document every year, adding or revising sections on a rolling basis. Materials from and examples of effective programs that have implemented recommendations in CASG and PASG reports will be especially useful to CUPM. CUPM or its designates should solicit, review, update, and share such contributions with the MAA membership. Compiling such "models that work" is an important extension of this *Guide.*

A word about Calculus

CUPM cannot conceive of a mathematics major without some part of its curriculum devoted to ideas of calculus. That said, CUPM is troubled both by the level of students' pre-calculus preparation and by the permanent exodus from mathematics of many good students who receive college credit for high school or AP calculus or are discouraged by their experience in college calculus classes. We know anecdotally that many mathematics majors begin college work with either Calculus II or Multivariable Calculus, but our present knowledge is scanty as regards what calculus our major students encounter, and when.

CUPM began this project at the same time that David Bressoud and colleagues were collecting data for the MAA research project on Calculus I, *Characteristics of Successful Programs in College Calculus (CSPCC).* The CASG report on Calculus draws on some information from the CSPCC study, and a separate report on the CSPCC study appears in the Calculus CASG report. Together, these reports present some promising recent initiatives in and live questions about calculus within the major curriculum. Over time we hope for additional research, careful studies of Calculus II and Calculus III that both mirror and extend the *CSPCC*, more examples of innovative courses, and more "proofs" of curricular success.

Thanks

When this project was conceived, Carol Schumacher (Kenyon College) chaired CUPM. She deserves much of the credit for the initiation and development of this project, which she now co-chairs with Martha Siegel, the current chair of CUPM. Michael Pearson and Linda Braddy of the MAA have given their unwavering support. Thanks also go to the members of the Steering Committee: Betsy Yanik (Emporia State University), David Bressoud (Macalester College), Jenna Carpenter (Louisiana Tech University), Michael Starbird (University of Texas at Austin), Alan Tucker (Stony Brook University), and Harriet Pollatsek (Mount Holyoke College). In addition, Beth Burroughs (University of Montana) and Joseph Malkevitch (York University of the City of New York), chairs of COMET and MAD, respectively, gave generously of their time and their wisdom. We were also fortunate to have Paul Zorn (St. Olaf College) and Harriet Pollatsek (Mount Holyoke College) as editors.

Funding for the *2015 Curriculum Guide* was provided to the MAA by the National Science Foundation (DUE-1228636). Additional support from the Educational Advancement Foundation (EAF), established by Harry Lucas, is gratefully acknowledged. EAF support allowed the Steering Committee to meet and discuss its ideas with a thoughtful group of mathematicians at several of the annual Legacy of R. L. Moore conferences.

To everyone named above and to the many volunteer professionals who contributed work and wisdom to course, program, and other reports, we offer our sincere thanks. Members of the community spent many hours researching and writing their group reports. Member organizations of the Conference Board on Mathematical Sciences (CBMS) made significant contributions in reviewing the *Guide.* Our course and program groups were gracious in responding to editorial comments and made valuable and timely revisions.

Comments, suggestions, and corrections should be directed to

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Overview of Majors in the Mathematical Sciences

As mathematics grows in depth and applicability, effective mathematical education becomes more important than ever. At the same time, this broader scope of our discipline works against a single shared understanding of the phrase "mathematics major." The mathematical preparation of beginning college students may also change if the Common Core State Standards are implemented as envisioned.

Against this background it makes little sense to prescribe in detail a single static list of "essential" requirements for a major in the mathematical sciences; many reasonable possibilities exist. Nevertheless, majors in the mathematical sciences do share some fundamental elements. The *2015 CUPM Curriculum Guide to Majors in the Mathematical Sciences* offers widely considered and broadly conceived national perspectives both on these fundamental elements and on the variety of possible majors they can support. Recommendations (and some language) in this document build on goals laid out in the 1990 and the 2004 *CUPM Curriculum Guides*.

At its August meeting in 2014, the Board of Governors of the MAA approved the Cognitive and Content Recommendations shown below in italics and also reaffirmed the principles of the *CUPM Curriculum Guide 2004*.

Designing a Major in the Mathematical Sciences

Major programs in the mathematical sciences should present the beauty, fun, and power of mathematics. They should be designed so that all students come to see mathematics as an engaging field, rich in beauty, with powerful applications to other subjects and contemporary open questions. Each department should create and maintain a community that welcomes and supports all students, including those from groups that have been traditionally underrepresented in mathematics.

How can this *Guide* help departments design and maintain a robust major program? Individual course design and overall program structure must be considered together. We offer a general set of cognitive and content goals to aid this discussion. A successful major offers a program of courses to gradually and intentionally leads students from basic to advanced levels of critical and analytical thinking, while encouraging creativity and excitement about mathematics.

Although this *Guide* makes recommendations about a great variety of programs that center on new and developing applications of mathematics, departments should continue to offer an option for a major in mathematics that is perhaps more traditional in scope and purpose. Some departments call this a "pure" mathematics track, some call it the preparation for doctoral programs in mathematics. Like all programs in the mathematical sciences, these rely on educational processes that lead to mathematical maturity and cultivate appreciation for mathematics. While they should incorporate some applications of the theory, they are focused on the theoretical nature of the subject. In all cases, undergraduate major programs in mathematics/mathematical sciences should have common elements. The cognitive and content goals reflect those common threads.

Cognitive Goals

Every mathematical sciences major should be designed to help students acquire "mathematical habits of mind." Students should develop the ability and inclination to use precise language, reason carefully, solve problems effectively, and use mathematics to advance arguments and increase understanding. These cognitive goals are not achieved in a single assignment or course; they must be approached within the context of each student's mathematical maturation throughout his or her undergraduate years. A well-constructed curriculum supports students in learning concepts, acquiring skills, and achieving cognitive goals. In the following paragraphs we describe several cognitive goals in more detail.

Cognitive Recommendation 1: *Students should develop effective thinking and communication skills.*

Major programs should include activities designed to promote students' progress in learning to

- state problems carefully, articulate assumptions, understand the importance of precise definition, and reason logically to conclusions;
- identify and model essential features of a complex situation, modify models as necessary for tractability, and draw useful conclusions;
- deduce general principles from particular instances;
- use and compare analytical, visual, and numerical perspectives in exploring mathematics;
- assess the correctness of solutions, create and explore examples, carry out mathematical experiments, and devise and test conjectures;
- recognize and make mathematically rigorous arguments;
- read mathematics with understanding;
- communicate mathematical ideas clearly and coherently both verbally and in writing to audiences of varying mathematical sophistication;
- approach mathematical problems with curiosity and creativity and persist in the face of difficulties;
- work creatively and self-sufficiently with mathematics.

Cognitive Recommendation 2: *Students should learn to link applications and theory.*

Mathematics students should encounter a range of contemporary applications that motivate and illustrate the ideas they are studying, become aware of connections to other areas (both in and out of the mathematical sciences), and learn to apply mathematical ideas to problems in those areas. Students should come to see mathematical theory as useful and enlightening in both pure and applied contexts.

Cognitive Recommendation 3: *Students should learn to use technological tools.*

Mathematical sciences major programs should teach students to use technology effectively, both as a tool for solving problems and as an aid to exploring mathematical ideas. Use of technology should occur with increasing sophistication throughout a major curriculum.

Cognitive Recommendation 4: *Students should develop mathematical independence and experience open-ended inquiry.*

A mathematical sciences major should be structured to move students beyond the carefully choreographed mathematical experiences of the classroom. A major curriculum should gradually prepare students to pursue open-ended questions and to speak and write about mathematics with increasing depth and sophistication.

Content Goals

The possible variety of major programs makes "core" content difficult to specify. Nevertheless, we do recommend that all programs in the mathematical sciences include several common elements. These recommendations were informed by broad consultation, including reports from study groups considering courses and programs, and discussions with directors of both Ph.D. and professional master's programs. These content goals connote mathematical subjects and ideas that can appear in a curriculum in various ways---not necessarily in specific courses with the same names.

Content Recommendation 1: *Mathematical sciences major programs should include concepts and methods from calculus and linear algebra.*

Every major in the mathematical sciences draws on ideas and methods from calculus and linear algebra. Calculus is the foundation of the study of continuous processes, and thus a gateway to the physical, biological, and social sciences. Instruction in calculus should include consideration of both single- and multivariable approaches. Linear algebra, the study of multivariate linear systems and transformations, is essential preparation for advanced work in the sciences, statistics, and computing. Linear algebra also introduces students to discrete mathematics, algorithmic thinking, a modicum of abstraction, moderate sophistication in notation, and simple proofs. Combining ideas from linear algebra and calculus helps students develop facility with visualization, see connections among mathematical areas, and appreciate the power of abstract thinking.

Content Recommendation 2: *Students majoring in the mathematical sciences should learn to read, understand, analyze, and produce proofs at increasing depth as they progress through a major.*

Proofs are indispensable to the practice and culture of mathematics. Students in all mathematics courses, whether or not for majors, should encounter elements of mathematical argument, precision, and justification. All mathematical science majors should learn to read, understand, analyze, and produce proofs, at increasing depth as they progress through a major. Individual departments may foster this development as institutionally appropriate: through dedicated "transition" courses or by distributing this content among several courses.

Content Recommendation 3: *Mathematical sciences major programs should include concepts and methods from data analysis, computing, and mathematical modeling.*

Working mathematicians often face quantitative problems to which analytic methods do not apply. Solutions often require data analysis, complex mathematical models, simulation, and tools from computational science. To meet these workplace expectations every mathematical sciences major should have, at a minimum:

- a command of data analysis and statistical inference at a level equivalent to that attained in an applied data analysis course;
- experience working with professional-level technological tools such as computer algebra systems, visualization software, and statistical packages;
- modest experience writing computer programs;
- experience tackling ill-posed real-world problems by building and analyzing appropriate deterministic and stochastic mathematical models.

Content Recommendation 4: *Mathematical sciences major programs should present key ideas and concepts from a variety of perspectives to demonstrate the breadth of mathematics.*

Programs should present key ideas from a variety of perspectives, employ a broad range of examples and applications to motivate and illustrate the material, promote awareness of connections to subjects both within and beyond the mathematical sciences, and strengthen each student's ability to apply the course material to these subjects. Programs should introduce historical and contemporary topics and applications, highlighting the vitality and importance of modern mathematics, and the contributions of diverse cultures.

Content Recommendation 5: *Students majoring in the mathematical sciences should experience mathematics from the perspective of another discipline.*

Applications of mathematics to other fields continue to evolve and expand. Mathematics students should encounter substantive applications throughout the curriculum. When possible, these applications should include perspectives of non-mathematicians who use mathematics to clarify or extend their own subject.

Content Recommendation 6: *Mathematical sciences major programs should present key ideas from complementary points of view: continuous and discrete; algebraic and geometric; deterministic and stochastic; exact and approximate.*

Students acquire mathematical depth and perspective by encountering and employing a variety of mathematical viewpoints:

- **Continuous and discrete:** Continuous mathematics—calculus, analysis, and differential equations—has long been central to mathematics major curricula. In recent decades, however, advances in computer science, operations research, mathematical modeling, and data analysis have dramatically increased the importance of discrete mathematics. Techniques from discrete mathematics—difference equations, recursive methods, combinatorial arguments, graph-theoretic models—prepare students to encounter the rich interplay between continuous and discrete approaches in future study and careers. Discrete mathematics should therefore take its place alongside continuous mathematics as a crucial element of undergraduate study.

- **Algebraic and geometric:** Algebra and algebraic structures—vector spaces, groups, rings, fields—are fundamental to mathematics and should be included in every undergraduate mathematics curriculum. Geometry and visualization are different ways of thinking and provide an equally important perspective. A geometry course is a useful part of any student's major program and is essential for future high school teachers. But geometric viewpoints should appear beyond geometry courses. Geometric reasoning and visualization complement algebraic thinking in linear algebra and multivariable calculus, and remain important in more advanced courses such as real analysis and differential equations.

- **Deterministic and stochastic:** New applications of mathematics to the biological and social sciences make understanding and modeling randomness and random phenomena increasingly important in undergraduate mathematics. Students should see the need for and master basic methods associated with elementary discrete- and continuous-time stochastic models. Deterministic models can fruitfully represent large-scale behavior, but random variation is often better seen through the stochastic lens. Stochastic applications appear naturally in statistics courses, but can also be introduced in courses such as elementary probability, matrix algebra, or graph theory. They can be explored further in courses such as differential equations or mathematical models.

- **Exact and approximate:** Students should be able to find, use, and evaluate approximations, understanding that exact answers, although desirable, are often unavailable or impractical. Students should be introduced to numerical methods, discrete approximations of continuous phenomena (and vice versa), and the general role of approximation in solving problems.

Balancing a variety of mathematical perspectives need not mean devoting separate courses to each. A more feasible strategy is to structure the curriculum and requirements intentionally to include these themes over the course of a student's major. For instance, modern computing tools—which themselves combine graphical, numerical, and algebraic resources—can facilitate presenting these viewpoints in many undergraduate courses.

Content Recommendation 7: *Mathematical sciences major programs should require the study of at least one mathematical area in depth, with a sequence of upper-level courses.*

Mathematics grows through the construction of abstract theories from definitions, examples, and theorems. Students learn to cope with such complexity by grappling with clusters of related ideas, in depth and over an extended period. Every mathematics major student should encounter at least one area in depth, drawing on ideas and tools from previous courses and making connections among them. Departments can meet this goal by requiring either two related courses or a year-long sequence at the upper level. This goal prescribes neither a particular area of study nor whether the material be mainly theoretical or abstract; possibilities include Probability and Mathematical Statistics, Real Analysis I/II, and Abstract Algebra I/II.

Content Recommendation 8: *Students majoring in the mathematical sciences should work, independently or in a small group, on a substantial mathematical project that involves techniques and concepts beyond the typical content of a single course*.

Every major student should have a "high impact" experience that requires substantial work in mathematics outside the carefully scripted confines of ordinary course work. Students should present their results in written and oral forms. Institutions can provide this opportunity in various ways: undergraduate research experiences, courses driven by inquiry or open-ended problem solving, capstone courses, internships or jobs with a substantial mathematical component, etc.

Content Recommendation 9: *Mathematical sciences major programs should offer their students an orientation to careers in mathematics.*

Helping students explore professional options can improve recruitment and retention of majors. Because degree programs vary, departments should tailor these messages in locally appropriate ways. Departments can employ both "static" information—posters, links on department websites, career brochures, career information from professional societies—and more dynamic strategies: inviting external speakers (including alumni); short, self-contained, career-focused orientations or assignments within courses; and encouraging and supporting students' attendance at regional and national professional meetings.

Review and Renewal

A healthy mathematical sciences program should incorporate intentional evolution and continual improvement. Every mathematical sciences department should have and follow a strategic plan that acknowledges local conditions and resources, but is also informed by recommendations from the greater mathematical community. The process of planning and renewal should be guided by consultation both within the department and with outside stakeholders at the institution. Departments should assess their progress in meeting cognitive and content goals through systematic collection and evaluation of evidence. Two documents from the MAA Committee on Department Review, *Guidelines for Undertaking a Self Study in the Mathematical Sciences* and *Guidelines for Serving as a Consultant in the Mathematical Sciences,* provide valuable guidance to a department engaged in review and renewal.

A suggested framework for continuous improvement

Initially, departments should select a mix of long-term (five years, say) goals and short-term (one or two years) objectives and choose specific strategies for meeting long- and short-term objectives. Then, departments should regularly

1. collect data on success in meeting departmental goals (e.g., placement, persistence, success in downstream courses, ability to communicate mathematics);
2. assess whether departmental teaching and pedagogy effectively support departmental goals;

3. review progress toward goals and objectives set in previous years;
4. identify concrete actions to improve outcomes in one or two areas;
5. reaffirm, modify, or add departmental goals.

Suggested questions to guide review and renewal

1. To what extent do the general cognitive and content goals suggested by the 2015 *CUPM Curriculum Guide* align with your department's goals?
2. To what extent do Course and/or Program recommendations in this *Guide* align with your department's particular goals?
3. Among your department's goals, which are *not* addressed by this *Guide*? Why are these goals important to your program?
4. Do you find any of CUPM's recommendations less important or infeasible for your department? If so, why?
5. Which of your departmental goals are not being met very well? Be as specific as possible.

Course Area and Program Area Reports

This 2015 *CUPM Curriculum Guide* includes both a relatively short printed version and a more extensive online version. The former is a proper subset of the latter. As described in the Introduction, a substantial portion of the work on the guide was done by groups of experts that were assembled to provide information and advice about common courses in the mathematics curriculum and about specialized applied or interdisciplinary programs in the mathematical sciences.

All **Course Area Study Group (CASG)** reports appear online. The print version of the *Guide* includes only the three CASG reports: Calculus, Linear Algebra, and Applied Statistics and Data Analysis. These are the three content areas that are specifically recommended as required for every mathematical sciences major program. Additional CASG reports are as follows: Abstract Algebra, Capstone Courses, Complex Analysis, Ordinary Differential Equations, Discrete Structures, Geometry, History of Mathematics, Mathematical Modeling, Number Theory, Numerical Analysis and Numerical Methods, Operations Research, Partial Differential Equations, Probability and Stochastic Processes, Real Analysis and Advanced Calculus, Transitions to Proof, and Topology.

CASG members were chosen for their involvement in research and teaching of their particular areas. For some CASGs authors of textbooks were chosen for their knowledge of the variety and unique characteristics of courses being offered in the area. The Groups were asked to consider cognitive as well as content goals in their recommendations. And, in order to increase the usefulness of the recommendations, they were encouraged to give several different descriptions of reasonable courses in their area.

All **Program Area Study Group (PASG)** reports appear online. The PASG reports capture both specialized applied mathematics and interdisciplinary work that has a mathematical focus. The set of reports aims to reflect the many and various routes by which a student might gain such mathematical experience. Possibilities include an integrated interdisciplinary program, a second and distinct major program along with mathematics, a minor complementing a mathematics major, or simply a collection of electives to enrich a traditional mathematics major. We do not claim to have covered all possibilities, but have tried to describe a variety of options for working with colleagues in complementary disciplines to improve programs that rely strongly on mathematics. PASG reports address Actuarial Science, Applied Mathematics, Chemistry, Computing and Computational Science, Engineering, Environmental Science and Climate, Financial Mathematics, Mathematical Biology, Middle and High School Teacher Preparation, Operations Research, Physics, Social and Behavioral Sciences, and Statistics.

Future plans for MAA and for CUPM include online communities in which course and program areas represented here, and others suggested by the community, can be discussed. CUPM envisions each online community as a place to share ideas for useful problems, projects, and software for each of these areas. Several such communities exist already, some as SIGMAAs of the MAA; they are referenced in their respective CASG and PASG reports. Online CASG and PASG reports will evolve over time.

In some cases CASGs have listed possible textbooks for a given course. These should not be read as endorsements by the MAA, the CUPM, or the Steering Committee. In most cases textbooks were listed to illustrate the coverage, level, or style that the group considered appropriate to a particular version of the course being described. Specific schools and departments at which particular programs have been implemented are also sometimes mentioned in PASG reports. These, too, should be read as examples, not as "official" endorsements. Institutions may be cited because their programs may prove instructive to users of this *Guide* as they design and launch new programs of their own.

The Calculus Sequence

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Calculus dominates much of the undergraduate mathematics landscape. In the 2010 fall term, 720,000 students enrolled in single or multivariable calculus in US colleges and universities. It is a gateway course for most students heading into programs in science or engineering. At many institutions, Calculus I is considered the first college-level course in mathematics. At the same time, it is slipping into the high school curriculum. Over 700,000 students studied calculus in high school in 2013–14. Of the 300,000 students enrolled in college mainstream Calculus I each fall, over 50% have completed a course, often a full year, of calculus in high school. At our research universities, the figure is over 70%.[1]

Because of their centrality, calculus courses are under enormous pressure. Enrollments are large, failure rates are high (the average DFW rate in Calculus I is 27%), and many partner disciplines complain that too many students who have completed their study of calculus struggle to use that knowledge. Few colleges or universities have the resources to cut class size or undertake significant restructuring of the course. In those fortunate places where a dean or provost has promised targeted funding to improve calculus instruction, it is seldom clear how those funds would be best spent.

While there is widespread concern and much local experimentation, implementing improvements to such a well-established and fundamental sequence of courses is a slow process. This report deals separately with what is happening in curriculum, instruction, and institutional support, while presenting options and communicating what we know about best practices. It includes some of the findings from MAA's national study of Calculus I, *Characterisitics of Successful Programs in College Calculus* (CSPCC).[2] This report does not include a list of textbooks, partly because there are so many. We do, however, direct the reader toward some of the less traditional textbooks for Calculus.

I. Curricular Options

What is the current thinking about the most important ideas of the Calculus sequence?

The curriculum that is followed from the first course in Calculus I through Several Variable Calculus has been remarkably stable ever since George Thomas produced the first edition of his *Calculus and Analytic*

[1] Data from the CBMS 2010 Statistical Abstract, the National Center for Education Statistics, and the MAA study *Characteristics of Successful Programs in College Calculus*.

[2] Pronounced CriSPiCC. Further information about the study available online. Funded by NSF #0910240. Opinions expressed do not necessarily reflect views of the Foundation.

Geometry in 1951. In the late 1980s and early '90s, the Calculus Reform Movement produced several curricula that front-loaded material on differential equations. This came from a recognition that Calculus is about the study and modeling of dynamical systems.[3]

The emphasis within Calculus has traditionally been on derivatives as slopes of tangent lines and integrals as areas—a very static interpretation that makes it difficult for many students to transfer these tools to dynamical situations. Current work on calculus recognizes the central importance of the concept of *covariation*, understanding how change in one of two or more linked variables is reflected in change to the other variable or variables [5].

Many mathematics educators now recognize that it is more useful to see the derivative as the instantaneous rate of change or as a measure of *sensitivity* of change in one variable to change in another, rather than emphasizing its role as a method of finding slopes. Instead of privileging the integral as area, the emphasis should be on measuring *accumulation*. In an echo of Apostol's *Calculus* [1], Patrick Thompson at Arizona State University has found that students grasp the idea of accumulation more readily than that of derivative as a limit of average rates of change, and has been teaching the course with integration coming before differentiation [8].

Recent discussion has also focused on how to define the integral and present the Fundamental Theorem of Calculus (FTC). The traditional approach is to define the definite integral as a limit of Riemann sums and then explain FTC as stating that integration and differentiation are inverse processes. Most students, however, never grasp the formal definition and understand integration as antidifferentiation, thus removing any meaning from FTC. A better approach is to explain FTC as stating the equivalence of two ways of understanding the definite integral: as the change in the value of an antiderivative or as the limit of a summation. This strategy also accords with the historical understanding of this theorem, which, until its name was shortened in the 1960s, was known as the Fundamental Theorem of Integral Calculus (see [2] for more on this history).

What is happening to develop calculus curricula for specific majors?

Only a tiny fraction (about 2%) of college Calculus I students intend to major in the mathematical sciences. Similar low figures—about 7% in each case—hold for the physical sciences and for computer science. Almost certainly, these very low percentages are a result of the fact that many students heading into the mathematical, physical, or computer sciences will start their college mathematics above the level of Calculus I. The point is that the clientele for this course has shifted dramatically over the past couple decades.

Biology majors now make up the largest single group of students taking mainstream Calculus I (30%). (Here "mainstream" describes any Calculus I course that satisfies a Calculus I prerequisite for any subsequent courses in mathematics; it includes honors sections.) For this reason many colleges and universities have created versions of the calculus sequence specifically tailored to their needs. Many of these courses begin with the study of differential equations and strongly emphasize uses of calculus in modeling dynamical systems. The MAA Notes volume *Undergraduate Mathematics for the Life Sciences: Models, Processes, and Directions* [7] provides a sample of curricula now in use. Macalester College's approach to calculus, introducing multivariable topics from the very start and emphasizing calculus as a tool for modeling, is among the courses described in this volume. At St. Olaf College, the "standard" Calculus I course was recently redesigned to emphasize and draw on biological applications.

Engineering is the second largest major (27%) represented among Calculus I students. Here, too, the traditional curriculum may ill suit students' needs, and variant curricula have been developed. West Point's *Core Math* has been in place for a quarter-century; it introduces students to calculus by starting with differ-

[3] This is reflected in the 2009 recommendations for pre-med requirements issued by the Association of American Medical Colleges and the Howard Hughes Medical Institute. Calculus is only referenced indirectly, as the ability to "quantify and interpret changes in dynamical systems." More information on these recommendation is available online .

ence and differential equations. Wright State University has created *Introductory Mathematics for Engineering Applications* as the first mathematics course for prospective engineers. It is taught within Wright State's College of Engineering and Computer Science and serves as both a review of precalculus and an introduction to calculus that is built around problems that arise in engineering.

What is being done to meet the needs of students who enter college with credit for Advanced Placement Calculus, but who may not be well served by being placed into a more advanced course?

Many universities have developed or adopted curricula designed to challenge students who enter with credit for Advanced Placement Calculus. Two innovative approaches are Pomona's *Approximately Calculus* by Shahriar Shahriari, which focuses on the use of calculus as a tool for approximation and brings in other topics, and the University of Pennsylvania's *Coursera* course, *Calculus: Single Variable* by Robert Ghrist. Following very much in the footsteps of Euler and Lagrange, Ghrist's course uses Taylor series from the very beginning. Other approaches introduce elements of real analysis. Spivak's *Calculus* is the traditional choice of textbook for such a course, but there are now other contenders including *Calculus Deconstructed* by Zbigniew Nitecki and *The Calculus Integral* by Brian Thomson. (See [3] for citations and reviews of the Spivak, Nitecki, Shahriari, and Thomson texts, as well as an historical approach by Cates.)

What curricular modifications are common beyond Calculus I?

Several colleges and universities have moved sequences and series out of Calculus II and replaced them with an introduction to multivariable calculus. A project sponsored by the National Science Foundation (NSF), *Resequencing Calculus,* is developing curricular materials for such a course. Grinnell and St. Olaf Colleges have been using this approach for many years. Sequences and series are still important, and there are variety of alternatives to where they first appear in the undergraduate curriculum: in a separate half course, as part of differential equations, in a transition-to-proof course, or delayed until students take real analysis.

The perennial question for Several Variable Calculus is how much vector calculus—especially Green's, Gauss's, and Stokes' theorems—to include. Few are satisfied with the most common solution: to cram these topics into the last few weeks of the course. Many colleges and universities choose to offer a separate course, often designated Advanced Calculus, that includes the topics of vector calculus.

II. Better Pedagogical Approaches to Calculus Instruction

Better pedagogy is advantageous for all mathematics courses, but especially crucial in the calculus sequence because of its pivotal role.

Which fundamental issues shape the choice of pedagogical approach?

Most students say that they want to understand mathematics, but few know what this means or how to achieve it.[4] Part of the problem lies with the traditional lecture format of instruction. Students have difficulty identifying the most important aspects of what they are seeing and hearing. Either they try to record everything, creating notes that are of little use, or they focus on what they imagine to be important, the template solutions. The same is true when students are "studying." They focus on what they know how to do and what they expect will be important on the examinations, learning template solutions. As an MAA study of final exams has revealed,[5] most instructors play to this tendency by giving examinations that require little beyond an ability to master a set of template solutions.

4 For an account of what happens in a typical calculus lecture, see David Bressoud, Student attitudes in first-semester calculus, MAA Focus, 14:6-7, 1994.

5 Tallman, M., Carlson, M. P., Bressoud, D., & Pearson, M. (in preparation). A characterization of calculus I final exams in U.S. colleges and universities. *Journal of Mathematical Behavior*.

What is *active learning*?

What is known to work in helping students learn how to learn is a broad category of pedagogical approaches that are classified as *active learning.* When done well, active learning forces students to engage with the ideas of calculus in a setting where they can be directed and encouraged by the instructor. This is easiest to accomplish in small classes, but it is possible to incorporate active learning into any class. Iowa State uses reading quizzes that are taken before class so that students arrive in class with a context into which to place the instructor's explanations. They also employ clickers and collaborative activities in their large lecture classes so that students are forced to think about what has been said. Group projects undertaken in the recitation sections force students to apply their knowledge to unfamiliar problems and situations. Cornell University has developed a large set of "Good Questions," thought-provoking questions directed at common student misconceptions that can spur active engagement, even in large classes.

The whole point of active learning, helping students to move beyond exclusively procedural knowledge, is lost unless students know that they will be assessed on a more extensive understanding of calculus. One of the common attributes of the most successful calculus programs identified in the MAA calculus study (see Section III) is that they had high expectations for what students could do, and they included a substantial proportion of non-template problems on their examinations.

What is Inquiry-Based Learning?

Inquiry-Based Learning[6] (IBL) goes a step beyond finding ways of keeping students intellectually engaged in what is happening in class. It also involves students in the process of building toward the big ideas. Over the past decade, the mathematics departments at the Universities of Michigan, Chicago, Texas at Austin, and California at Santa Barbara have engaged in a controlled experiment in the effectiveness of IBL. The Ethnography & Evaluation Research Group at the University of Colorado-Boulder has studied this experiment. Comparing sections taught with and without IBL, they found that IBL improved both the number of subsequent mathematics courses that were taken and performance in those courses [6].

What are the concerns with active learning approaches?

Active learning is not easy. Classroom activities need to be chosen carefully. Mini-lectures need to be thoughtfully prepared. Instructors need to develop experience in identifying where students are struggling and how best to modify instruction to meet their needs. Group projects can be counter-productive unless mechanisms are in place to ensure full participation by all students. Anything done in recitation sections must be closely integrated with what is happening in the classroom. MAA's national study of Calculus I, *Characteristics of Successful Programs in College Calculus*, revealed that, while active learning approaches do improve attitudes and retention, their effect is heavily influenced by the basic quality of the teaching. Active learning approaches can be counterproductive when instructors have not built a basic level of rapport and trust with the students.

Fortunately, there is a lot of expertise in the creation and facilitation of active-learning classrooms. The University of Michigan has built a quarter-century of experience in using these techniques and training new faculty in their use. In almost all cases, involvement of or input from faculty with expertise in Mathematics Education has been essential to improving calculus instruction.

What are the options for using online videos and comparable materials, especially for "flipped" classes?

There are two current NSF-sponsored studies of the effects of flipping classes in mathematics, one at Harvey Mudd and the other at the University of Hartford. Hartford's experiment is happening within their

[6] The Academy of Inquiry-Based Learning provides materials and workshops. The Academy also can connect instructors with mentors and has a limited supply of small grants to help get IBL programs started.

calculus courses and involves presenting lectures via online videos and spending class time in small-group problem solving, whole-class discussion, or lab investigations, with mini-lectures provided in class as needed to clarify content and procedures or highlight important conceptual ideas. In fall 2012, Hartford ran a pilot program with half of the classes flipped, half taught in a traditional manner. Preliminary analysis was so encouraging that all of the classes were flipped for spring 2013. Beginning fall 2013, with funding from NSF, they will begin a more extensive and carefully controlled study of the effectiveness of flipped classes. The University of Pennsylvania is also running a similar project under the auspices of the Association of American Universities Undergraduate STEM Initiative.

The University of Hartford is also experimenting with *iPad* sets and multiple projection units to promote more student-to-student discussion and collaborative problem solving. This department also has built databases of questions and curriculum materials that support active learning. Coordination and constant monitoring lie at the root of what they are able to accomplish.

Those who are not prepared to flip their class can still direct students toward online resources that partially complement and partially replace in-class lectures, as well as providing remediation. Such resources include the Khan Academy videos, MIT OpenCourseware, University of Hartford Material and Paul's Online Math Notes.

What do we know about the use of online homework systems?

One of the tools that can help promote active learning is an online homework system such as *WeBWorK*. This may seem counter-intuitive since what these systems assess most effectively are responses to short-answer template problems. But the use of active learning does not mean that the ability to solve such problems is no longer an important part of learning calculus. Active learning is a strategy for going *beyond* developing proficiency with such problems. Online homework systems enable students to develop this proficiency at their own speed. Most importantly, the feedback they provide immediately informs students and the instructor of what has and has not been understood. At the University of Hartford, this enables instructors to focus class time on addressing student misunderstandings and difficulties. Online homework systems also free the instructor to spend time assessing student ability to tackle deeper and more challenging problems, including application of calculus to unfamiliar contexts, interpretation of answers, and explanation of the reasoning behind a solution.

The AMS Homework Software Survey revealed strongly positive student and instructor responses to the use of these tools and no evidence that it was in any way inferior to hand grading. Today these systems are being used primarily at large universities, simply to handle the homework grading that otherwise would not happen (60% of instructors at research universities and 42% of those at masters universities use online grading), but it is spreading to undergraduate colleges and two-year colleges (27% at undergraduate colleges, 25% at two-year colleges).[7]

What are other common uses of technology?

Many of the active learning techniques described in this section involve technology in the sense of using online resources. Technology in the form of Computer Algebra Systems (CAS) such as *Mathematica*, raise inevitable questions about how much purely procedural knowledge is needed. Here the picture is not clear, and perhaps surprising. MAA's CSPCC calculus study found that while graphing calculators are commonly used, computers as tools for calculating are not. The study also found no measurable impact either from banning or from requiring the use of CAS.

7 Percentages are from the Fall 2010 CSPCC study.

III. Lessons from the MAA Study of Calculus I

MAA's study of *Characteristics of Successful Programs in College Calculus* (CSPCC) consisted of a series of surveys sent to a stratified random sample of colleges and universities in fall 2010 and completed by 213 chairs or calculus coordinators, 502 instructors, and over 14,000 students. Instructors and students were surveyed at both the start and end of the term. Identification of "successful programs" was based on how student affective characteristics changed: confidence, enjoyment of mathematics, intention to continue the study of mathematics, and intention to continue in a major that required at least one additional term of Calculus.

Here we summarize and illustrate some of the insights into best practices gathered from that study and as well as case study visits to a wide variety of calculus programs. The characteristics of successful programs that have been identified fall under three broad categories: Coordination, Monitoring, and Active Learning. Active Learning has been discussed in Section II. This section will consider the other two categories. These two are directed toward issues that go beyond what happens in the individual classroom and involve the entire department or institution.

What is the role of coordination in building a successful calculus program?

Faculty prize their independence, and most will hold tenaciously to their freedom to teach their class the way they want to teach it. This can result in some very innovative and often successful approaches to teaching. The problem is that it is then very difficult to leverage these improvements, to spread them beyond the individual. In a small college or university or with experienced faculty, one can not only allow but encourage such individualization, provided there is some coordination. This could be as simple as periodic observations of each other's classes combined with regular meetings of those who are teaching calculus to share their materials, approaches, and difficulties. When such sharing is built into departmental expectations, it facilitates the dissemination and further development of good ideas. It also can help prevent a class from going off the rails without appearing to target a particular instructor for special attention.

What does coordination look like at larger universities?

At larger institutions, coordination is critical. The most successful programs have a course Coordinator. The Coordinator holds regular meetings in which calculus instructors talk about course pacing and coverage, develop midterm and final exams, and discuss teaching and student difficulties. In addition, the most successful programs have common examinations. In some cases, the homework assignments are coordinated.

It is important to have a Coordinator who is respected by the mathematics faculty and is invested in the program (rather than serving for a semester or a single year). Having a commitment that extends beyond a single year facilitates interaction with other departments and university offices and helps to establish guidelines for handling special situations. At the University of Michigan, these course Coordinators work with a departmental oversight committee that lends credence to the program and assures adherence to the standards and goals of the department.

Part of coordination includes common exams. This is often politically difficult, but it is important. Common exams lead to consistent expectations for instructors in subsequent courses. They inhibit students from selecting the "easy" instructor over one who may have higher expectations, and they allow the department (especially in a department that may have a lot of adjunct faculty or even temporary full-time faculty teaching in the program) to maintain standards and thereby reduce student complaints.

Coordination also includes training and mentoring. This is particularly important for graduate students whether they are taking on a recitation section or teaching their own class, but mentoring and some access to training are also important for adjunct faculty, new faculty, and even experienced faculty who may not have taught calculus for many years. Iowa State's Center for Excellence in Undergraduate Mathematics Education runs seminars for new graduate students as well as all teaching assistants in which they discuss teaching issues, read and discuss case studies in teaching, and discuss problems encountered by assistants.

At the University of Michigan, there is a presumption that all calculus instructors—whether graduate students, adjunct faculty, or new regular faculty—need to know what is expected and to be given guidance and feedback on how they are doing. Instructor training includes pre-semester meetings, weekly meetings, and classroom observations. Mentoring includes availability of coordinators, follow-up visits, feedback, and an openness to new ideas while not allowing an instructor to stray too far from the goals of the department for the course.

What is the role of monitoring in building a successful calculus program?

Monitoring includes attention to local data, issues of placement, and use of Learning Centers. One of the most important attributes of successful calculus programs is attention to what is happening in these classes. This extends from overall monitoring of the success rates of students in these classes to attention to targeted subpopulations including women and first generation college students. It means following the performance of individual students so that interventions can occur before it is too late. One of the tools for monitoring the effectiveness of the calculus program is the Calculus Concept Inventory [**5**], created by Jerry Epstein. This tool has undergone testing and validation and has been used at the University of Michigan. Iowa State has plans to use it to assess its calculus program.

The importance of continual monitoring of the calculus program was illustrated in MAA's *Models that Work* **[9].** The most successful programs are those that are constantly looking to improve their effectiveness, occasionally through major overhauls of what they are doing when it becomes clear that what they are doing is not effective, mostly through continuing small improvements. This requires the regular and consistent collection of information about what actually is happening.

What does monitoring look like at different types of institutions?

In the most successful programs, someone in the department routinely collects and analyzes data in order to inform and assess program changes. It is essential that departments take on this work themselves. While effective departments almost always work with the Institutional Research Office, they do not rely on this office to know what to collect or how to analyze what is collected. Some of the most useful data include pass rates, grade distributions, persistence, placement accuracy, and success in subsequent courses.

It is not enough to collect data. The department must be prepared to adjust or, sometimes, even radically change what they are doing to counter clearly identified problems. When the mathematics faculty at Macalester College discovered that almost none of the students in Calculus I continued on to Calculus II because they were Biology or Economics majors for whom only one semester of calculus was required and that the students who did take Calculus II—primarily physical science and mathematics majors—were entering with credit for Calculus I, they restructured the first year of calculus. The first semester was turned into a course on dynamical systems that could stand on its own and provide the insights into calculus as tool for modeling that would be useful to Biology and Economics students in their major discipline. At St. Olaf, the Calculus I class has evolved to place much more emphasis on applications, especially those relevant to the life sciences.

It is especially important to monitor the effectiveness of placement programs. The University of Illinois at Urbana-Champaign and Iowa State University have dramatically improved their success rates in Calculus by introducing new placement procedures. In both cases, they have moved to adaptive online testing with opportunities to retake placement exams. This helps to distinguish between those students who knew but have forgotten a critical piece of mathematics and are capable of coming back up to speed on it and those students who never understood it and will be severely disadvantaged if they try to start Calculus without it. Good placement procedures do more than restrict access to Calculus I. They accurately identify which students can succeed in the course, and they help those who are not ready understand where their deficiencies lie.

One of the resources available from MAA is the Calculus Concept Readiness examination. This placement test assesses student understanding of the conceptual underpinnings of calculus.[8] Iowa State is among the universities using materials developed by Marilyn Carlson and her team for the Precalculus Concept Assessment.[9]

What are some of the ways in which students are supported by departments and institutions?

In addition to placing students correctly, it is important to provide support services, especially early in the critical first term of Calculus when habits are established. Problems that are identified early in the term can be addressed in a timely manner through tutors, Learning Centers, or special courses. The University of Michigan offers a half-term, self-paced Precalculus course (Math 110) for those students who start in Calculus I and discover after the first exam that their algebra and precalculus skills are not adequate. This greatly increases the likelihood that misplacement into Calculus I will not delay completion of Calculus I by more than one semester.

Moravian College was one of the first to offer a Calculus I course stretched over two terms so that it can include review of precalculus topics as they arise within the calculus course. Worcester Polytechnic Institute also offers such an option. This is a successful alternative to inserting a precalculus class because students are constantly encountering new and more challenging material while getting the support they need.

A hallmark of the most successful programs is the attention paid to the Learning Center. Oklahoma State is one of the universities making a major investment in upgrading their "Mathematics Learning Success Center." The Learning Center must provide a welcoming environment that students know about and use. An effective center includes training programs for the tutors who will work with students encountering difficulties in Calculus. It is particularly important for the Learning Center to have a strong connection to the calculus instructors. This includes an established mechanism for identifying early in the term those students who are struggling and supplying them with academic support. Swarthmore College has an Academic Support Coordinator for Mathematics, with an office in the Mathematics Department, whose job it is to work with faculty to identify students who need additional support and then to see that they get it. The University of Hartford offers "vampire tutoring," staffing their learning center from 10pm to midnight, the peak time for students to work on their homework. The key here is adjusting the schedule so that it is convenient for the students, not necessarily the staff.

Conclusions and reflections for the future

From this survey of the landscape of college calculus, we see that this sequence of courses is poised for significant changes driven by several factors:

- The students who, in the past, were well served by the traditional calculus course no longer study calculus in college. They jump directly to more advanced courses.
- The primary audience for Calculus I has shifted to biology majors and others for whom the appropriate emphasis is the investigation of dynamical systems with much greater reliance on computation.
- We know more about how students learn and about the importance of creating an active learning environment in which students are forced to wrestle with the mathematics and build a robust personal understanding.

[8] Additional information available online.

[9] Additional information available online.

- The availability of online resources provides opportunities for instructors to spend less time lecturing and more time in active learning, directly engaging students with the critical concepts of calculus.
- With greater attention to the institutional bottom line, there is less tolerance of high failure rates as well as of students who go on without the knowledge or skills needed to succeed in subsequent courses. Placement, support, and effective instruction are more important than ever.
- Many very bright and talented mathematicians and mathematics educators already realize these points and are developing a variety of tools, curricula, and pedagogical approaches to meet the needs these courses serve. At the same time, we are getting better at assessing the effectiveness of these innovations. We now have a much clearer grasp of what works and why it works.
- There is support for reform of undergraduate mathematics instruction coming from influential actors: the White House Office of Science and Technology Policy, the National Academies, and the Association of American Universities among many others. The American Mathematical Society has joined MAA in recognizing the need for more effective undergraduate instruction, and a variety of joint efforts are now underway.

We also now realize that just having a better way of doing something and publicizing it is not sufficient for widespread adoption. We are now beginning to understand obstacles to institutional change and what it takes to facilitate this change.

In 2014, MAA received a grant from NSF, *Progress through Calculus*, #1420389, that will enable it to dig into this process of improving the effectiveness of the entire single variable calculus sequence, to investigate and evaluate departmental efforts to provide better courses, and to establish supportive networks of departments that seek to implement change.

This is an exciting time to be involved in the teaching of calculus. The only certainty is that twenty years from now calculus instruction will be very different. Getting to that place where these courses do a far better job of meeting the needs of our students is an effort that will require our very best minds.

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Applied Statistics and Data Analysis

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I. The Course

CUPM recommends, and the MAA Board of Governors agrees, that every student majoring in the mathematical sciences take an introductory course in Applied Statistics, with a clear focus on data analysis. We recommend that this course be taken during the first two years of the undergraduate program and that it be focused squarely on applied data analysis. This is a course quite distinct from the usual upper-level sequence in probability and mathematical statistics that is offered as an elective in most undergraduate mathematics programs and also quite distinct from the low-level procedural course or quantitative literacy course taught at many institutions.

II. Student Audience

Although the course may serve a much broader audience, the audience we focus on in this report includes all students majoring in the mathematical sciences, including programs in mathematics, applied mathematics, mathematics education, operations research, actuarial science, and combined mathematics majors (combined with economics or biology, for example). We believe that an applied data analysis course, taken relatively early in the program, is a necessary component in all of these mathematical sciences programs.

III. History and Background

We are guided by our CUPM predecessors in this process. Every Curriculum Guide for at least the past 30 years has demonstrated to us the need to increase understanding of statistics in students majoring in the mathematical sciences and to do so in a course clearly focused on data analysis.

- **From the 2004 *Curriculum Guide***: "The CUPM Guide 2004 supports the 1991 CUPM recommendation that every mathematical sciences major should study statistics or probability with an emphasis on data analysis."
- **From the 1991 *Curriculum Guide***: "Every mathematical sciences major should include at least one semester of study of probability and statistics … The major focus of this course should be on data and on the skills and mathematical tools motivated by problems of collecting and analyzing data."
- **From the 1981 *Curriculum Guide***: "The Statistics Subpanel believes that an introductory course in probability and statistics should concentrate on data and on skills and mathematical tools motivated by the problems of collecting and analyzing data."

We reiterate and endorse, as strongly as we can, the recommendation that every mathematical sciences major include a course in applied statistics, focused on data analysis.

IV. Current Status

The MAA Curriculum Guides have been recommending for more than 30 years, and with increasing emphasis, that every student majoring in the mathematical sciences take a course in statistical data analysis. How are we doing at meeting this recommendation? In an effort to discover the answer to that question, we did some of our own data collection and analysis. The results we found were quite discouraging and even worse than we had expected.

In our sample of fifty-five undergraduate programs in mathematics, selected from a wide variety of different types of schools, only 4 (about 7%) currently require a course in applied statistics. Furthermore, only 12 (about 22%) even *allow* a course in applied statistics (which is distinct from the upper-level probability and mathematical statistics courses) to count toward the major program. Even the 22% estimate is probably quite high, since many of these electives in "applied statistics" appear to be weighted more heavily toward probability rather than data analysis. Despite the strong recommendations to the contrary, we are making very little progress in producing mathematics graduates with a sound knowledge of statistical data analysis.

The situation in undergraduate programs in Applied Mathematics and in Mathematics Education appears to be slightly better but still not good. In our relatively small sample of these programs, we found that about 50% of programs in Applied Mathematics allow an applied statistics course to count toward the major, while about 63% of Mathematics Education programs do so. The results in Mathematics Education are a particular concern, since prospective secondary math teachers need a thorough preparation to teach the modern data collection and interpretation content of the Common Core standards. All students going on to K–12 math education should be required to take at least one and preferably two modern data-driven courses in applied statistics. See Appendix C.

We are living in a world in which the ability to analyze data is increasingly important, across almost all disciplines. Graduates of undergraduate programs in the mathematical sciences go on to a wide range of careers in education and business, graduate and professional programs in a wide range of areas, and doctoral programs across the range of mathematical sciences. In every single case, our graduates would be well-served with solid knowledge and skills of statistical data analysis. Indeed, we believe such a course would serve the vast majority of our students far better than one additional theoretical math elective.

The obvious question, of course, is why departments continue to discourage students from taking such a course as a part of the major program. The second obvious question to ask is what can we do to change this pattern.

We provide several possible answers to the first question. One is largely historical: Introductory statistics courses at many institutions have been viewed as low-level service courses with not enough mathematical content to warrant credit for a student majoring in mathematics. One is based on resources: There are not enough faculty members to provide such a course for majors, and there are not enough statisticians around to teach it. One is based on the tendency of mathematicians to view all courses through the lens of theoretical mathematics and to therefore evaluate the courses on how they might prepare students for doctoral programs in pure mathematics. Answers to the second question on what we can do to change the pattern are harder to come by. We address some of these challenges in Appendix B.

V. Description of the Recommended Course

Our Statistics Area Study Group is fortunate to have a widely respected set of guidelines from which to start. The GAISE Guidelines (Guidelines for Assessment and Instruction in Statistics Education) were written in 2005 and endorsed by the American Statistical Association. They have since also been endorsed by the American Mathematical Association of Two-Year Colleges. The GAISE Guidelines include "Goals for Students in an Introductory Course: What it Means to be Statistically Educated" as well as a list of six specific recommendations to help students attain the learning goals. These goals and recommendations are available online and are included in Appendix A of this report. The GAISE Guidelines are currently being updated and the revised guidelines are expected to be available in 2015. We unanimously support the goals and recommendations of the GAISE report, and these guidelines strongly inform our work.

For students majoring in the mathematical sciences, we recommend a course focused on applied data analysis and driven by real data. The course should stress conceptual understanding, foster active learning, and introduce students to statistical technology. The focus should be on the effective collection and analysis of data, along with appropriate interpretation and communication of results.

Just as Mathematics Departments routinely do with calculus courses, such a course in Applied Statistics can serve a wide audience. Also as with calculus, some institutions will have one level of the course while other larger institutions might have different courses for different audiences. In every case, however, the focus should be on understanding effective data analysis rather than on the underlying mathematical theory. The concepts involved in statistical inference are notoriously challenging for students to master, and a course focusing on these concepts provides an intellectually rigorous course, even without teaching these concepts from a theoretical mathematics perspective. There are significant differences between statistical and mathematical thinking (see, for example, Cobb & Moore, "Mathematics, Statistics, and Teaching", *The American Mathematical Monthly*, November 1997) and this course should focus explicitly on statistical thinking.

Even though we have used the singular "course" in this section, we believe that many different courses could achieve the goal of introducing mathematics students to effective data analysis. We provide syllabi for some such courses in Appendix D.

Cognitive Goals

Applied Statistics is an outstanding course for helping students meet the cognitive goals set out in this *Guide*. Specifically, in the process of working with real data, students have to read with understanding, recognize patterns, identify essential features of a complex situation, and apply appropriate methodologies. All of these enhance critical thinking skills. Communication skills are also emphasized in such a course, as students learn to effectively interpret and justify their conclusions. Learning to use technology intelligently as an effective tool is an integral part of a good data analysis course.

Applied Statistics is also an outstanding subject to promote awareness of connections to other subjects. A strong course in applied statistics will illustrate applications to a wide variety of different areas and the importance of interacting across disciplines. Such a course can also serve to enhance student perceptions of the vitality and importance of mathematics and statistics in the modern world.

Additional cognitive goals of an applied statistics course include dealing with randomness and uncertainty, understanding the distinction between exact answers and models/approximations, and working with data visualization.

An introductory course in Applied Statistics should be taught using all the current best thinking about how people learn. Classes should be interactive with regular active participation by students. Statistics lends itself well to student projects, to experiential learning, and to team explorations, and we strongly encourage the use of these interactive pedagogies in statistics classes.

Mathematical Outcomes

In addition to the outcomes listed in the cognitive goals, we offer the following goals for an introductory course in Applied Statistics:

- An understanding of the process by which statistical investigations are performed, from formulating questions to collecting data, then analyzing data and drawing inferences, and finally interpreting results and communicating conclusions.
- A solid conceptual understanding of the key concepts of statistical inference: estimation with intervals and testing for significance.
- The ability to perform statistical inference procedures, using traditional methods and/or modern resampling and permutation methods.
- Experience using technology to explore statistical concepts and to analyze data graphically, numerically, and inferentially.
- An understanding of the importance of data collection, the ability to recognize limitations in data collection methods, and an awareness of the role that data collection plays in determining the scope of conclusions to be drawn.
- The knowledge of deciding which statistical methods to use in which situations and the ability to check necessary conditions for those methods to be valid.
- Extensive experience with interpreting results of statistical analyses and communicating conclusions effectively, all in the context of the research question at hand.
- An awareness of the power and scope of statistical thinking for addressing research questions in a variety of scientific disciplines and in everyday life.

Prerequisites

Basic proficiency in algebra is all that is required, combined with a bit of analytical maturity. Some data analysis courses could have calculus as a prerequisite.

Sample Syllabi

Sample syllabi and course outlines are provided in Appendix D. We also include in Appendix C a recommended two-course sequence for future mathematics teachers, shared with us by the authors of the MET2 report (Math Education of Teachers). With the increased emphasis on statistics in the Common Core State Standards in Mathematics, and the dramatic and record-breaking rise in students enrolling in AP Statistics courses, this recommendation for future mathematics teachers is particularly important.

VI. What About Other Courses in Statistics?

Most statistical analyses involve the analysis and modeling of relationships between many variables. While a first course in applied statistics is likely to focus mainly on univariate and bivariate methods of data analysis, the course can serve as a bridge to and introduction of data analysis situations involving many variables. Statistics is much more than can be covered in just this one-semester course! There is much to say about other courses in statistics for students majoring in mathematical sciences, including discussions of excellent second-level courses in applied statistics, the theoretical probability and mathematical statistics sequence, and other options for introductory courses for other majors and at other levels. These other courses, however, are both more accepted already within mathematics programs and less controversial as courses for math majors. For these reasons, we have opted to focus on the most important change we think undergraduate mathematics programs need to make as we prepare 21st century mathematics majors: to provide and actively encourage all students majoring in the mathematical sciences to include at least one course in applied data analysis.

Appendix A: GAISE Guidelines

The Guidelines for Assessment and Instruction in Statistics Education (GAISE) were developed in 2010 and the full details are available online.

Recommendations

1. Emphasize statistical literacy and develop statistical thinking.
2. Use real data.
3. Stress conceptual understanding, rather than mere knowledge of procedures.
4. Foster active learning in the classroom.
5. Use technology for developing concepts and analyzing data.
6. Use assessments to improve and evaluate student learning.

Goals

Students should believe and understand why:

- Data beat anecdotes
- Variability is natural, predictable, and quantifiable
- Random sampling allows results of surveys and experiments to be extended to the population from which the sample was taken
- Random assignment in comparative experiments allows cause-and-effect conclusions to be drawn
- Association is not causation
- Statistical significance does not necessarily imply practical importance, especially for studies with large sample sizes
- Finding no statistically significant difference or relationship does not necessarily mean there is no difference or no relationship in the population, especially for studies with small sample

Students should recognize:

- Common sources of bias in surveys and experiments
- How to determine the population to which the results of statistical inference can be extended, if any, based on how the data were collected
- How to determine when a cause-and-effect inference can be drawn from an association based on how the data were collected (e.g., the design of the study)
- That words such as "normal," "random," and "correlation" have specific meanings in statistics that may differ from common usage

Students should understand the parts of the process through which statistics works to answer questions, namely:

- How to obtain or generate data
- How to graph the data as a first step in analyzing data, and how to know when that's enough to answer the question of interest
- How to interpret numerical summaries and graphical displays of data—both to answer questions and to check conditions (to use statistical procedures correctly)
- How to make appropriate use of statistical inference

- How to communicate the results of a statistical analysis

Students should understand the basic ideas of statistical inference, including:

- The concept of a sampling distribution and how it applies to making statistical inferences based on samples of data (including the idea of standard error)
- The concept of statistical significance, including significance levels and p-values
- The concept of confidence interval, including the interpretation of confidence level and margin of error

Finally, students should know:

- How to interpret statistical results in context
- How to critique news stories and journal articles that include statistical information, including identifying what's missing in the presentation and the flaws in the studies or methods used to generate the information
- When to call for help from a statistician

Appendix B: Challenges

Statisticians

This is a bit of a Catch-22. In order to attract more quantitatively-inclined students into statistics, we need to expose more of them to the subject earlier in their college careers. However, in order to offer these courses, we need to recruit more statisticians as faculty in Mathematics Departments. We don't have a ready solution for this one, but a recent ASA/MAA Joint Report, "Qualifications for Teaching an Introductory Statistics Course," offers some guidance.

Resources

How can departments, already stretched for resources, afford the resources to offer courses such as the one proposed? One solution is to reconfigure the current introductory statistics course offered at many schools. As a low-level probability and statistics course with a focus on procedures and formulas, the course is designed for non-math majors. By redesigning the course, however, with a focus on concepts, technology, and real data analysis, the course remains viable (and better) for the current audience while also becoming an appropriate course for students majoring in the mathematical sciences.

Connecting with Other Courses

Students should be encouraged to take an applied data analysis course at any time in the first two years, which could be before, concurrent with, or after calculus. Just as students take Calculus and then are exposed to the theoretical foundations in Advanced Calculus and Real Analysis, students taking this course could then be exposed to the theoretical underpinnings in the probability and mathematical statistics sequence (as well as, if resources allow, more advanced courses in applied data analysis.) We see no problem with interactions between this course and existing courses.

Attitude

One of the biggest challenges to the goal of having all mathematical sciences students complete a semester of data analysis is the belief of many mathematicians that such a course is not an appropriate course for a student majoring in mathematics. Some hold a strong belief that all true mathematics courses should follow a theorem and proof model. However, departments are starting to embrace the idea of offering a mathematical modeling course that asks students to deal with complex real situations and that is project-based with a heavy emphasis on communication skills. In the same way, we hope that departments will embrace the idea of having their students explore the field of statistics, so that they are prepared for a world full of data and are exposed to more of the richness of the mathematical sciences.

Appendix C: Recommendations for Mathematics Education of Teachers

The MET2 report (Math Education of Teachers) recommends that prospective high school teachers obtain a thorough preparation to teach the modern data collection and interpretation content of the Common Core standards. Chris Franklin and Dick Schaeffer recommend the following two-semester lower-level sequence for future teachers, shared with us by the authors of the MET2 report. The Statistics Education of Teachers (SET) report currently being produced by the American Statistical Association (ASA) will provide more details about the content of these two courses. The report should be available online through the ASA in early 2015. We strongly recommend that schools refer to this document. This would be an outstanding sequence for future teachers, for schools that can support a two-semester sequence.

a. One-semester Introductory Statistics course emphasizing data analysis:
   - formulation of statistical questions
   - exploration of univariate data sets and comparisons among multiple univariate data sets
   - introduction to the use of randomization in data production and inferential reasoning
   - inference for means and proportions and differences of means or proportions; notions of p-value and margin of error
   - introduction to probability from relative frequency perspective; additive and multiplicative rules, conditional probability and independence

b. One-semester Statistical Methods course
   - bivariate categorical data: two-way tables, association, chi-square test
   - bivariate measurement data: scatterplots, association, simple linear regression, correlation
   - exponential and quadratic models; transformations of data (logs, powers)
   - introduction to study design: surveys, experiments and observational studies
   - randomization procedures for data production and inference
   - introduction to one-way ANOVA

Appendix D: Sample Syllabus #1

Course Area: Statistics

Title of Course: Introductory Applied Statistics

Credit hours/semester: 3 or 4

Description of the target student audience: Similar to the range of students taking calculus. Students majoring in the mathematical sciences as well as those in a variety of other fields (which will vary depending on the institution)

Course Description: An introduction to applied data analysis, designed to enable students to effectively collect data, describe data, and make appropriate inferences from data. Students are expected to communicate effectively about statistical results and to use a statistical software package for data analysis.

Proposed prerequisites: Basic algebra

How the course fits into a program of study: The course should be taken during the first two years of an undergraduate program in the mathematical sciences. It can be taken concurrently with calculus or any other sophomore-level courses in mathematics. Those students wishing to continue on in statistics are urged to take the probability and mathematical statistics sequence as well as any additional courses in advanced data analysis offered at the institution.

Course Outline

- Data collection, including random sampling and design of experiments (2 weeks)
- Data description, including graphs and summary statistics for categorical and quantitative variables and relationships between variables (2 weeks)
- Introduction to the key ideas of estimation and testing, using modern resampling methods to build conceptual understanding (3 weeks)
- More on confidence intervals and hypothesis tests, using the normal and t distributions (3 weeks)
- Advanced tests, as time permits, such as chi-square tests, ANOVA, regression tests, multiple regression (3 weeks)

1. Students complete three data-analysis projects during the semester, each using a statistical software package and culminating in a written report (and, if possible, an oral report).
2. The course uses active learning, with regular in-class activities and group projects.
3. The course uses real data of interest to the students and emphasizes the connections of the subject to a wide variety of other fields.
4. The focus is on deep understanding of concepts such as variability of sample statistics, understanding random chance, estimation, and the meaning of the p-value, rather than on memorizing procedural methods.
5. Students are regularly presented with data in a real context, so that they experience the problem-solving and multiple approaches often necessary to move from a question of interest to reaching a conclusion.
6. Students gain extensive experience with effectively interpreting and communicating the results of data analysis.

Appendix D: Sample Syllabus #2

Course Area: Statistics

Title of Course: Introduction to Statistics

Credit hours/semester: 3 or 4

Description of the target student audience: Students majoring in mathematical sciences as well as those in other mathematically-related fields such as biology and economics

Course Description: Introduction to statistics for mathematically inclined students, focused on reasoning process of statistical investigations from asking question and collecting data to analyzing data and drawing inferences. Substantial use of statistical software.

Proposed prerequisites: Calculus I

How the course fits into a program of study: The course should be taken during the first two years of an undergraduate program in the mathematical sciences. Students wanting to continue in statistics should take a second course that introduces more advanced concepts and methods such as regression techniques and analysis of variance.

Course Outline

Unit 1: Analyzing single binary variable

- Simulation, null model, statistical significance, p-value, binomial probabilities
- Two-sided test, significance level, rejection region, test decision, types of error, power
- Normal probability model, normal probability calculations, z-score, test statistic
- Standard error, critical value z^*, confidence interval, sample size determination
- Sampling, sampling bias, simple random sampling, precision

Unit 2: Comparing two groups on binary variable

- Two-way table, conditional proportions, segmented bar graph
- Binomial simulation analysis for comparing two proportions
- Normal approximation, standard error, two-proportion z-test, z-interval
- Observational studies, confounding variables, randomized comparative experiment
- Simulating randomization test for assessing statistical significance with 2×2 tables
- Hypergeometric probabilities, Fisher's exact test, relative risk, odds ratio

Unit 3: Comparing two groups with quantitative response

- Simulating randomization test for comparing two groups with quantitative response
- Histogram, measures of center and variability, five-number summary, boxplot
- Two-sample t-test, t-interval for comparing means
- Randomization test for paired data, paired t-procedures, prediction interval

Course Principles

- Simulation-based inference is introduced throughout, prior to exact probability methods and theory-based techniques based on normal approximations.

- Students analyze genuine data from scientific research studies throughout.
- Students work through investigation activities to discover statistical concepts.
- Students complete course projects applying all aspects of statistical investigation process.

Linear Algebra

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Introduction

> *"I believe that linear algebra is **the** most important subject in college mathematics. Isaac Newton would not agree! But he isn't teaching mathematics in the 21st century*." — Gilbert Strang [5]

Strang is not alone in thinking that linear algebra has a prominent place in today's undergraduate mathematics curriculum. Linear algebra, calculus and data analysis are the only three specific content areas recommended for all mathematical science majors in this *Guide.* From the *Overview*:

> Linear algebra, the study of multivariate linear systems and transformations, is essential preparation for advanced work in the sciences, statistics, and computing. Linear algebra also introduces students to discrete mathematics, algorithmic thinking, a modicum of abstraction, moderate sophistication in notation, and simple proofs. Linear algebra … helps students develop facility with visualization, see connections among mathematical areas, and appreciate the power of abstract thinking.

Technology facilitates solving large systems of linear equations quickly and efficiently. Thus, first courses in linear algebra allow students (as the *Overview* puts it) to "link applications and theory" and also "to use technology effectively, both as a tool for solving problems and as an aid to exploring mathematical ideas." Linear algebra is also especially useful in helping students learn to "use and compare analytical, visual, and numerical perspectives in exploring mathematics."

Prerequisites

The most frequent prerequisite for a Linear Algebra course is Calculus II. But Linear Algebra is taught in so many ways, with such widely varying goals, that other prerequisites, such as Calculus I and Discrete Mathematics, are common. Other institutions require that students complete Multivariable Calculus before Linear Algebra—or *vice versa*. A few institutions require some specific background in a computer language or package designed to treat linear algebra computations and applications. In practice individual institutions must choose prerequisites that best fit their own goals, student populations, and local resources.

Who takes Linear Algebra?

Few subjects other than calculus can rival linear algebra in serving many areas of study and application and in attracting a wide variety of students. Mathematics majors certainly need linear algebra. So, increasingly, do students majoring in mathematics education, physics, chemistry, mathematical biology, computer science, various engineering fields, business, economics, and finance. In recent decades students have brought increasing diversity of background and interest to Linear Algebra classrooms. This sea change has important implications both for content and for pedagogy in Linear Algebra courses.

Computation and the evolving Linear Algebra curriculum

In 1997 Carl C. Cowen received the MAA's Deborah and Franklin Tepper Haimo Award for Distinguished College or University Teaching. In his Haimo lecture, titled *The Centrality of Linear Algebra in the Curriculum* [4], Cowen argued that because "no serious application of linear algebra happens without a computer," computation should be part of every beginning Linear Algebra course.

Linear Algebra courses as we now know them became common in the mathematics curriculum only in the late 1960s or early 70s. Cowen notes: "... [E]ngineers have known for more than a century that many problems could be modeled as systems of linear equations or as eigenvalue problems. But what would be the point? Even in the 50s few engineers could hope to solve a system of 100 equations in 100 unknowns; linear algebra was really irrelevant!" As *Matlab* and similar programs came into common usage, faculty in many disciplines began to advise an increasingly diverse population of students to take Linear Algebra. This development caused "many colleges and universities to change from courses dominated by proofs of theorems about abstract vector spaces to courses emphasizing matrix computations and the theory to support them." While the increasing applicability of linear algebra does not require that we stop teaching theory, Cowen argues that "it should encourage us to see the role of the theory in the subject as it is applied."

Beyond supporting the study of substantive applications, computing technology lets students and faculty focus on linear algebra concepts rather than calculations. With matrix manipulation software students can explore open-ended questions that motivate core principles; with computer visualization, students can connect geometric and algebraic aspects of the subject.

Technology can be incorporated in various ways in Linear Algebra courses. In lecture-based courses, technology can appear either interwoven into lectures or in separate lab times. In a lab-based course students might have access to computers during the class period; in courses that use online resources, course technology is readily available. We strongly recommend incorporating technology in one or more forms, while acknowledging that doing so can be labor intensive, especially in a first course iteration.

Core content

Linear Algebra courses are now offered in many "flavors" for many disciplines. Indeed, "Linear Algebra" now connotes more a broad category than a single course, and so it is difficult to make fully general recommendations. Still, the various flavors of Linear Algebra do share important features and explore essentially the same core topics (see further details on topics below).

Pedagogy

Every Linear Algebra course should develop the core topics in an instructional framework that promotes conceptual understanding of fundamental ideas. Regardless of the course's particular emphasis, assignments should help students build effective thinking skills, as recommended in the *Overview*: critical thinking, pattern recognition, formulating conjectures, writing, student engagement, creation of examples to illustrate a concept, technology, reading proofs, introduction to construction of proofs, and counterexamples. Problem solving arises in both computational (as in matrix computations) and theoretical (as in writing proofs) forms. All Linear Algebra courses should stress visualization and geometric interpretation of theoretical ideas in 2- and 3-dimensional spaces. Doing so highlights "algebraic and geometric" as "contrasting but complementary points of view," as suggested in the *Overview*. Course projects are especially useful in helping students develop written and oral communication skills.

Learning goals

Students who master the core content of a Linear Algebra course should attain some specific learning goals. They should be able to

- compute with and recognize properties of particular matrices;
- formulate, solve, apply, and interpret properties of linear systems;
- recognize and use basic properties of subspaces and vector spaces;
- determine a basis and the dimension of a finite-dimensional space;
- find the eigenvalues and eigenvectors of a matrix, and use them to represent a linear transformation;
- recognize and use equivalent forms to identify matrices and solve linear systems;
- read proofs with understanding;
- use definitions and theorems to prove basic results in core topics;
- recognize and use equivalent statements regarding invertible matrices, pivot positions, and solutions of homogeneous systems;
- decide whether a linear transformation is one-to-one or onto and how these questions are related to matrices.

Applications

Every Linear Algebra course should incorporate interesting applications, both to highlight the broad usefulness of linear algebra and to help students see the role of the theory in the subject as it is applied. Attractive applications may also entice students majoring in other disciplines to choose a minor or additional major in mathematics. While most instructors will choose topics they like or judge appropriate to their students' backgrounds, certain elementary applications and extensions frequently appear in texts and have the reputation of revealing fundamental ideas. Among these, in no particular order, are Markov chains, graph theory, correlation coefficients, cryptology, interpolation, long-term weather prediction, the Fibonacci sequence, difference equations, systems of linear differential equations, network analysis, linear least squares, graph theory, Leslie population models, the power method of approximating the dominant eigenvalue, linear programming, computer graphics, coding theory, spectral decomposition, principal component analysis, discrete and continuous dynamical systems, iterative solutions of linear systems, image processing, and traffic flow.

Core topics

Instructors may vary the given order, or omit topics not appropriate for a given audience.

- Matrices and linear systems: matrix operations and properties; special matrices; linear systems of equations; echelon forms and Gaussian elimination; matrix transformations (with geometric illustrations); matrix inverses; properties of vectors in R^n ; LU-decomposition
- Determinants: computation and properties of determinants, including relation to products, inverses, and transposes; geometric interpretations (Not everyone sees determinants as "core" material in a Linear Algebra course; see, e.g., [1].)
- Vector spaces: vectors; subspaces; linear independence; basis and dimension; row and column spaces; rank; rank-nullity theorem
- Linear transformations: matrix representations; change of basis; functional properties; kernel and range

- Eigenvalues and eigenvectors: definitions, examples, and properties; computational methods; diagonalization of matrices
- Inner products: norms; orthogonality; orthogonal bases; Gram-Schmidt orthogonalization (optional depending upon time and course goals)

Curricular variations on a theme: Sample syllabi

Some institutions offer several Linear Algebra courses, serving various goals. Here we list some possible examples.

Linear Algebra with Computer Lab

Goals: Provide a foundation for topics in linear algebra together with applications. Integrate the use of MATLAB software within course topics and applications. Teach students to read proofs effectively, and to construct elementary proofs involving matrices and linear combinations.

Labs: Once a week, two hours; using MATLAB. Computational lab activities promote understanding both of basic concepts from algebraic, symbolic, and geometric viewpoints, and of linear algebra applications.

Differential Equations with Linear Algebra

Goals: This fast-paced course, primarily in ordinary differential equations but emphasizing the use of linear algebra, has two main objectives: (1) to teach students how to solve linear differential equations and systems thereof; (2) to introduce students to linear algebra concepts, including the majority of core topics. Students also learn to read and construct elementary proofs involving matrices and linear combinations.

Theoretical Linear Algebra: A Second Course

Goals: This second course adopts a higher degree of abstraction than would a traditional first course. Students aim to achieve full understanding of the material, to prove key theorems, and to solve challenging problems.

Topics: Major topics, covered more deeply than in a first course, include vector spaces, linear transformations, determinants, eigenvalues and eigenvectors, canonical forms, inner product spaces, the finite-dimensional spectral theorem, singular value decomposition, and bilinear forms. Other topics can be chosen by the instructor.

Several Linear Algebra course varieties can be directed at future engineers. Computer use is essential.

Linear Algebra for Engineers

Topics include matrices, determinants, vector spaces, eigenvalues and eigenvectors, orthogonality and inner product spaces; applications include brief introductions to difference equations, Markov chains, and systems of linear ordinary differential equations.

Linear Algebra and Vector Analysis for Engineers

Topics include matrix theory, linear equations, Gaussian elimination, determinants, eigenvalue problems and first order systems of ordinary differential equations, vector field theory, theorems of Green, Stokes, and Gauss.

Linear Algebra with Application to Engineering Computations

Topics include basis, linear independence, column space, null space, rank, norms and condition numbers, projections, and matrix properties**.** Students learn to solve matrix-vector systems and consider direct and iterative solvers for non-singular linear systems of equations—their accuracy, convergence properties, and computational efficiency. The course covers under- and over-determined linear systems, and nonlinear systems of equations, as well as eigenvalues, eigenvectors, and singular values--their application to engineering problems.

Linear Algebra and Differential Equations for Engineers

Content areas include matrix theory, eigenvectors and eigenvalues, ordinary and partial differential equations. This course has three parts: matrix algebra (core type topics); systems of linear differential equations; PDEs and Fourier series.

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Preparation for Graduate Study

Many students majoring in the mathematical sciences go on to graduate study. Advice in this Guide about preparation for graduate study in particular program areas is mainly contained in the PASG reports available online. All undergraduate advisors should be aware of general requirements for "mainstream" graduate degrees. To this end, the print version of this Guide includes reports on preparation for professional science master's programs and programs leading to the PhD in mathematics.

Professional Science Master's Programs in the Mathematical Sciences

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What is a PSM?

At the end of the 1990s a new graduate degree came to life, as a response to strong workforce demands for STEM professionals. The two-year program leading to the Professional Science Master's (PSM) degree was conceived to supplement strong interdisciplinary knowledge in science and mathematics, with expertise in business areas such as communications and project management. The PSM degree is appropriate for those who aspire to careers in business, industry, or government within mathematical or scientific contexts. Typically, entry to such jobs requires expertise beyond that offered by a bachelor's degree. That PSM programs meet a need in US higher education is confirmed by the rapid expansion of these programs: over 300 now exist in a variety of disciplines.

PSM programs range from biotechnology and pharmaceuticals to nanotechnology and agriculture. All are based on the need of scientific, business, or government enterprises for employees with solid scientific and technical backgrounds that complement business skills and interests. The first mathematically-oriented PSM programs originated in Financial and Applied Mathematics (or in similar programs called Industrial Mathematics or Computational Mathematics), followed recently by Bioinformatics and Data Analytics programs, which have significant mathematics and statistics components. The National Professional Science Master's Association (NPSMA) coordinates and highlights activities of the various institutions offering PSM programs. These interdisciplinary programs are usually designed in cooperation with business, industry, and government to reflect their needs in specific technical areas. In general, PSM degree programs combine higher-level scientific and technical coursework with courses in business.

What do PSM programs seek to accomplish?

PSM programs vary widely in structure and areas of emphasis, but they share some general features and purposes.

Cognitive outcomes. Mathematics-focused PSM programs, regardless of area, promote several key attitudes, practices, and abilities.

- **Analytical thinking**: Students' strong theoretical mathematics and/or statistics knowledge supports deep understanding of important applied problems.

- **Modeling skills:** Students are comfortable using mathematics and statistics to model real-world problems.
- **Professional and leadership skills:** Students acquire expertise in mathematics and statistics; professional and leadership skills in communication, management, and team building; and the flexibility to adapt to corporate needs.
- **Collaborative skills:** Students learn to collaborate through team projects and internships.

Content outcomes. Content requirements differ among mathematics-rich PSM programs, but all programs apply mathematical and statistical knowledge, reasoning, and techniques to problems in their fields. Facility in these areas is developed through components common to all PSM programs:

- mathematics and statistics core courses;
- management and business courses;
- real-world projects and internships.

Mathematical content is used in different ways in different PSM programs. The following examples suggest some of this variety.

Industrial/applied and computational mathematics: Students learn to apply mathematical and computational tools to model problems from various science and engineering fields. Students learn to distill the underlying mathematical model from the stated problem, develop qualitative analysis of the solution, determine the acceptable approximation, and understand the importance of the data and accuracy needed. Students learn to find, interpret, and convey results in the language and context of the original problem, respecting real-world constraints that may dictate approximate, probabilistic, or numerical solutions.

Financial mathematics: Graduates use mathematical and computational tools to model pricing of financial derivatives, fixed income securities and their derivatives, portfolio valuation, credit markets, and risk management. Students learn about business and economics through courses regularly included in MBA programs. They also encounter computer programming, algorithms, database design and management, and the design of software systems.

Data analytics: Graduates must use mathematical, statistical, and computational tools effectively to analyze large data sets and apply results to fact-based decision-making. Graduates must translate a business problem into a data problem that can be solved, with results that inform a business solution. These steps involve the understanding both of business and of techniques for data mining, data visualization, machine learning, and distributed computing.

Bioinformatics, biotechnology, and genomics: Graduates must use sophisticated statistical tools along with mathematical and computational analysis to manage and interpret biological data. Students must learn to analyze and model complex biological phenomena, combining strong statistical reasoning and computational skills with understanding of biomedical aspects of the problem.

Recommended undergraduate preparation for mathematics-focused PSM programs

While there are many PSM programs in mathematics and related fields, each with its own admission requirements, there is a commonality of background and experience that can prepare a mathematics major for successful admission to and graduation from a PSM program. Some institutions offer a so-called BS/MS (4+1 years) program: high performing undergraduates who have completed the prerequisites for the PSM can take graduate courses while enrolled as undergraduates, and thus complete both undergraduate and PSM degrees in 5 years.

This background and experience can be obtained from course work and involvement in professional activities. Some undergraduate programs offer credit for experience working in the field. Some PSM programs (e.g., in Data Analytics or Bioinformatics) admit students who do not have a mathematics undergraduate degree. Because admissions criteria depend on program site, here we describe universal criteria only for mathematics majors; readers should consult specific institutional websites for other cases. NPSMA lists institutions and departments offering the degree, along with information on specific programs and their admission requirements.

Mathematics courses. All PSM programs require calculus (including multivariate), linear algebra, and probability and statistics. Some computing experience or familiarity with software packages is also expected. Experience with mathematical modeling is highly desirable. This may involve coursework in modeling, mathematics-intensive courses in other sciences, business, or economics, or out-of-class experience in an applied field. Requirements for additional courses vary with area of the PSM:

- **Industrial/applied and computational mathematics**: Differential Equations, Discrete Mathematics, and Numerical Methods. Courses in engineering or sciences, depending on the area of interest, including programing/computing courses.
- **Financial mathematics**: Differential Equations, Real Analysis, Discrete Mathematics, Numerical Methods. Courses in business (e.g., accounting), economics, or finance. Experience with basic computer programming (MATLAB, R, Java, C++, Python).
- **Data analytics**: At least two courses in statistical methodology/regression, experience with statistical/mathematical computing (SAS, SPSS, R, Minitab, Python, etc.).
- **Bioinformatics**: At least two courses in statistical methodology/regression, Numerical Methods, experience with statistical/mathematical computing (SAS, SPSS, R, Minitab, Python, etc.). Additional courses in biology (e.g., cell biology, genetics, and molecular biology), organic chemistry, and programming/computing.

Other experience. It is important for prospective PSM students to have completed courses in other science, engineering, and business disciplines. Alternatively, some students might acquire perspective on other disciplines through extracurricular experiences such as summer work, internships, or research opportunities. Such experiences both help students make informed career choices and contribute to their success in PSM programs. Previous experience with applications of mathematics and statistics to other fields and with problem solving in a real-world settings is also valuable.

Collaboration is crucial to training for careers in business, industry, or government; experience with teamwork on any project is an asset to PSM program applicants. Teamwork with people from diverse backgrounds helps students see things from different perspectives. Collaborative experiences can be arranged through undergraduate class projects, internships, or research projects with other students and faculty.

Clear communication in oral and written forms is essential; students should develop these skills as undergraduates both through writing courses and by preparing written class and project reports. Students can build oral presentation skills through mathematical presentations—especially to non-mathematical audiences. PSM programs develop these skills further by requiring at least one major team project; teams must present results to both technical and non-technical audiences.

Faculty advisors should inform themselves and their undergraduate students about master's programs that support careers requiring significant mathematical and statistical knowledge and skills. Online NPSMA resources can assist faculty and their advisees in understanding the relatively new PSM degree. This *Guide* offers general advice about undergraduate preparation for PSM programs, but faculty should advise students to research particular PSM programs as early as possible. Doing so will permit students to choose courses best suited to prepare them for admission to PSM programs of their choice. See also [1], the proceedings of an NSF-sponsored workshop on PSM programs in mathematical sciences, for more information on existing programs, curriculum, industrial experience, placement of graduates and future trends.

Reference

1. J. Carbonara and B. Vernescu (eds), *Creating Tomorrow's Mathematics Professionals*, COMAP, 2013.

Undergraduate Preparation for PhD Programs in Mathematics[1]

Estela Gavosto, *University of Kansas, chair*
Sylvia Bozeman, *Spelman College*
Stephen DeBacker, *University of Michigan*
Richard McGehee, *University of Minnesota*
Oggie Trifonov, *University of South Carolina*
Larry Washington, *University of Maryland*

For majors intending to pursue doctoral work in the mathematical sciences, faculty must communicate the fact that facility with the tools of "epsilon-delta" calculus and abstract linear algebra is fundamental, as is some knowledge of programming. Students should also be both comfortable with and experienced at writing and reading mathematical proofs.

Coursework Background

Though doctoral programs in pure mathematics vary widely, nearly all require their students to take graduate courses in analysis and algebra and to pass demanding qualifying exams based on these courses. These two areas are at the core of the continuous and discrete aspects of mathematics; cf. *Contrasting but complementary points of view* in the Overview to this *Guide*. Departments that prepare students for doctoral work in mathematics should ensure that their students arrive in graduate school prepared to take such courses. For example, the departmental website at the University of Illinois at Urbana–Champaign[2] lists basic topics with which they assume familiarity. Expectations for real analysis and abstract algebra are as follows:

Real analysis: Completeness properties of the real number system; basic topological properties of *n*-dimensional space; convergence of numerical sequences and series of functions; properties of continuous functions; and basic theorems concerning differentiation and Riemann integration.

Abstract algebra: Modular arithmetic, permutations, group theory through the isomorphism theorems, ring theory through the notions of prime and maximal ideals; additional topics such as unique factorization domains and classification of groups of small order.

Some doctoral programs require additional topics in real analysis and abstract algebra or specify that preparation for their programs requires an undergraduate analysis class at the level of Rudin's *Principles of*

[1] Recommendations (and some language) in this document build on goals laid out in the 1990 and the 2004 CUPM Curriculum Guides.

[2] www.math.illinois.edu/GraduateProgram/undergrad-prep.html. This reference should not be interpreted as an endorsement of these topics as the proper undergraduate preparation for a doctoral program in mathematics. It is simply an example of what one respected graduate program expects.

Mathematical Analysis and an undergraduate algebra class at the level of Herstein's *Topics in Algebra*; others will allow entering students to start with a preparatory course at the level of Rudin or Herstein. Many programs also ask that students see specific areas beyond abstract algebra and real analysis. Again for example, the departmental website at the University of Illinois at Urbana–Champaign states that incoming PhD students should be familiar with the areas of real analysis, abstract algebra, complex analysis, and abstract linear algebra: and at least two of the following six areas: geometry and topology, number theory, differential equations, logic, combinatorics, statistics and probability. See also the list of courses and books in section 2.1, "How to Prepare for Graduate School" of [1] and the section "The Typical Educational Path in the Mathematical Sciences Needs Adjustments" on future needs in mathematical training in [2].

Reading and Writing Proofs

Preparing students for doctoral work in mathematics requires more than exposing them to topics in real analysis and abstract algebra. Students need also to be able to read and critique proofs; cf. *Proofs* in the Overview to this *Guide*. This skill includes the ability to determine how assumptions are used and to find counterexamples when any of the hypotheses are weakened. Students need familiarity with the common techniques employed to prove results in analysis and in algebra, and they should be able to use these techniques to prove simple theorems they have not seen before. Above all, students heading into doctoral programs in mathematics need to be able to read rigorous textbooks written in the languages of analysis and algebra. This is a skill that departments should ensure that their students learn; cf. *Effective thinking skills* in the Overview to this *Guide*.

Additional Preparation

Participation in an REU (Research Experiences for Undergraduates) program can be a useful supplemental preparation that introduces students to the excitement and hard work involved in doing mathematical research. Such experience could be especially important for students from colleges where coursework opportunities are limited. Related beyond-the-classroom opportunities include other summer programs aimed at students considering doctoral study in mathematics, semester-long programs for visiting students (like the "Mathematics Advanced Study Semester" at Pennsylvania State University or the "Junior Program of the Center for Women in Mathematics" at Smith College) and semester abroad programs (like "Budapest Semesters in Mathematics" or "Math in Moscow"). Departments should be aware of these programs and direct promising students toward them.

Other experiences that challenge students beyond the usual coursework—undergraduate seminars, capstone courses, independent study or research within a student's department, or undergraduate thesis work—are also highly recommendable for graduate school preparation; cf. *High impact experiences* in the Overview to this *Guide*. Such experience may also encourage talented students who may not yet aspire to earn the doctorate degree.

Departments that cannot afford to provide the full preparation needed for the doctoral program to which a student aspires should consider providing alternative options, such as reading courses, on-line courses, and cross-registration at a nearby institution. These departments might also recommend a transitional fifth year of study, a post-baccalaureate program, or even transfer to another institution. Similar recommendations should be made for students who decide late in their undergraduate careers to pursue a doctoral program. For some students a master's degree in mathematics is an attractive option; it permits them to "try out" graduate school and to improve their preparation for doctoral study if they choose to continue. However, funding opportunities may be more limited for master's programs than for doctoral programs, and some doctoral programs do not admit students who are aiming only at master's degrees.

Advising Students

Some programs offer a wide variety of qualifying topics and students may have some choice of which qualifying examinations to take. Students should be advised to consider a range of graduate programs since departments have varying admission requirements and policies for allowing students to catch up. Departments should advise students accordingly, and appraise realistically how much preparatory work would be needed before taking a particular graduate course. Students should be urged to get to know some of their mathematics professors. Letters of recommendation play a huge role in graduate program admissions, and the strongest letters come from faculty who can give admissions committees an in-depth portrait of the candidate. Students should also be advised to take the subject GRE[3] in mathematics and to maintain a minimum GPA as these are admission requirements in many departments. See [3] for views on the use of GRE and GPA in graduate school admissions.

References

1. Steven G. Krantz. *A Mathematician's Survival Guide: Graduate School and Early Career Development.* American Mathematical Society, 2003. isbn: 9780821834558.
2. Committee on the Mathematical Sciences in 2025; Board on Mathematical Sciences, Their Applications; Division on Engineering, and Physical Sciences; National Research Council. *The Mathematical Sciences in 2025*. The National Academies Press, 2013. isbn: 9780309284578. www.nap.edu/openbook.php?record_id=15269.
3. Julie R. Posselt. "Toward Inclusive Excellence in Graduate Education: Constructing Merit and Diversity in PhD Admissions". *American Journal of Education* 120 (2014), pp. 481–514. doi: 10.1086/676910.

[3] This reference should not be interpreted as an endorsement by this report's authors of the practice of requiring standardized exams in graduate school admissions. It reflects the requirements in many selective doctoral programs.

Beyond the Curriculum

Departmental Responsibilities in Curricular Reform

Mathematics, technology, and pedagogy are all elements of the dynamically changing environment in which we teach. Regular curricular review and renewal are therefore essential to good departmental management. Departments should respond thoughtfully and deliberately to change. The need to consider a department's curriculum can be precipitated by program review, by accreditation requirements, or by internal factors: results of an assessment, students' need for an expanded major or minor, rising or falling enrollments, and the like.

Curriculum review and subsequent change can be especially difficult in mathematics because of the hierarchical nature of mathematics curricula. Changing one course often forces corresponding changes in prerequisites or in following courses. The leader in the process is normally the chair of the department, though individual faculty may initiate changes. In any case, all faculty in a department should be invited to participate, because most changes affect the program as a whole. Ideally, the department chair fosters an atmosphere in which change is seen as vital to healthy growth.

CUPM recommends that all departments review their curricula regularly. The principles presented in this *Guide*'s Overview are intended to inform this process. Successful curricular review requires both introspection and looking outward to the department's mission within the institution and to its place in the larger mathematical landscape.

Building an excellent program entails several responsibilities of mathematical sciences departments.

- *Provide a welcoming community to attract and retain a diverse student and faculty community.* Programs thrive when students and faculty enjoy a collegial working environment. Beyond explicit efforts to recruit and retain students (as elaborated in the section *Recruitment and Retention*), departments should ensure that faculty are readily available to students for visits, questions, and informal discussion of careers and mathematical interests. Faculty should feel free—and encouraged—to suggest new curricular approaches, new pedagogical techniques, or new uses of technology they believe will improve undergraduates' experience in mathematics. Thoughtful experiments by faculty should be encouraged, supported, and assessed with an open mind.

- *Be aware of changing national standards and recommendations.* The times they are a-changin'—rapidly. Department chairs and leaders should follow recommendations from national accreditation bodies and professional organizations. The MAA is one among many organizations that regularly announce new curricular recommendations. Every department of mathematical sciences should monitor reports from the MAA, the National Academies, CBMS, and our sibling organizations, including ASA, AMS, AMATYC, SIAM, and NCTM. Schools with actuarial programs should adhere to the advice coming from the Society of Actuaries and the Casualty Actuarial Society. The accreditation of education programs, formerly under NCATE or NCTM, is now merged under the

Council of Accreditation of Educator Programs (CAEP). Encouraging faculty to attend national and regional meetings, to participate in professional organizations, and to report back to departments will contribute to a department's awareness and vitality.

- *Collect data and evidence to inform decisions.* Departments are unique. National statistics may not reflect a given institution's student demographics, pressures to provide service courses, number and nature of faculty appointments, and resource constraints. Departments need reliable and comprehensive local data to facilitate preparation of reports and to inform departmental decisions. Important information includes the mathematical preparation of incoming students, retention rates, numbers of majors in each "track," whether and where students continue to graduate study, and graduates' typical career paths. Tracking how students fare after introduction of new placement exams, course content changes, or pedagogy experiments can also improve decision-making. Departmental staff or institutional research offices might gather such data, but faculty should interpret the results. Periodic alumni surveys can also help identify valuable academic experiences and suggest new ones. The *Assessment* section has more details.

- *Hold regular discussions about curriculum.* Departments should spend time as a whole or in special committees to evaluate the effectiveness of their curricula. The departmental discussion should start with an examination of enrollments and demand. Are placement methods, prerequisites, and programs working well for their students? Are the courses aligned with national guidelines? Are grades reasonably distributed? Are textbooks well-aligned with departmental goals for content and cognitive outcomes? Do upper-level courses lead to cognitive development that matches departmental goals? Should new courses or "tracks" be introduced? The departmental database, feedback from students, and speaking with stakeholders inside and outside academia can inform the discussion. Entire departments or subgroups in the department may find it helpful to dedicate a period of time, perhaps a retreat, to discuss these matters in a deliberate and concentrated way. Department chairs should schedule such a retreat periodically and structure the agenda to focus discussion on areas needing attention. Occasionally, an outside facilitator can help a department define its mission and address problematic issues.

- *Hold regular discussions about pedagogy.* Ongoing studies of teaching and learning in college-level mathematics suggest that changes in our pedagogical methods improve our students' learning and retention. Attention to evidence-based improvements in pedagogy should be part of the department's regular conversation. Research in mathematics instruction at the lower levels (K-12) shows that when teachers feel free to observe their colleagues and feel empowered to speak regularly with them about effective methods in the classroom, the quality of teaching improves. Such practices can improve teaching at the college level as well. Departments should encourage regular sharing of teaching experiences and ideas.

- *Schedule frequent conversations with partner disciplines within the institution.* Given the ubiquity of mathematics in the social, biological, and physical sciences, mathematics departments should schedule frequent and regular dialogue with other departments. These interdisciplinary conversations can benefit everyone and build good will for the mathematics department. Collaboration can be formal or informal, but it should be built into the system of governance of the mathematics department so that regular reports are made to the mathematics faculty. Comparing textbooks used in mathematics and in partner disciplines can be revealing for all parties. Such interdisciplinary meetings should also involve the education faculty, if the institution offers education degrees. Close cooperation requires scheduling meetings regularly, at least once a semester.

- *Build and maintain working relationships with mathematics departments in feeder (and receiver) schools to ensure smooth articulation.* The experience of students who move from high school to college, from two-year to four-year programs, and from undergraduate to graduate programs should be made as smooth as possible. In some large public institutions the fraction of two-year college transfers may be around 50%, and articulation may be mandated by the state. In any case, it is crucial that colleges and universities collaborate on an equal footing with their feeder secondary schools and community colleges. Such dialogue is especially important in times of significant change, such as adoption of the Common Core State Standards.

  In some states, including Maryland, an informal network, including mathematical specialists in the public school system and delegates from mathematics departments in public and private two- and four- year colleges and research institutions, meets at least once a semester to discuss issues of common concern. An atmosphere of mutual respect and shared interest has fostered excellent relationships and useful collaborations. As a result, many fully articulated programs have been negotiated between individual two-year and four-year schools. This means that students getting an Associate's degree in mathematics from the two-year school have a very high probability of graduating with a Bachelor's in mathematics from the four-year program in just two additional years. The group is co-chaired by two mathematicians: one from the flagship university and one from a community college.

  There are many examples across the states. The Illinois Articulation Initiative requires that changes in general education and transfer courses be coordinated jointly by the Illinois Mathematics Association of Community Colleges (IMACC) and the Illinois Section of the Mathematics Association of America (ISMAA). Visiting the Illinois Section page shows that such approvals appear on the agenda and are discussed at regular section meetings. See the 2013 Section meeting agenda.

  (For more information, see the section on Articulation in this *Guide*.)

- *Stay abreast of new technologies and review their place in the department's offerings.* Technology is evolving rapidly, and it raises complex pedagogical, mathematical, and resource-related questions. CUPM recommends that departments consult the section in this *Guide* on Technology and the Undergraduate Mathematics Curriculum.

- *Encourage faculty to engage undergraduates in mathematical research.* We recommend reading the chapter in this *Guide* on Undergraduate Research in Mathematics as a way to begin departmental discussion of its place in the curriculum. Faculty members who supervise undergraduate research should receive professional credit for doing so. Students seek out such experiences, and graduate schools and employers increasingly look for evidence of such activity. Although all programs should offer research experiences to undergraduates, their variety and extent will naturally depend on locally available resources.

- *Align the faculty reward structure with educational goals.* Faculty who want to experiment with curriculum or pedagogy often complain that such work is both time-consuming and unrewarded in the promotion, tenure, and merit processes. Failing to align rewards with department needs for renewal and reform leads to stagnation. A department that values faculty involvement in undergraduate research, interdisciplinary courses, experimental coursework, and new pedagogy should assure that suitable credit is awarded in annual reviews. Deans, department chairs, and colleagues should recognize that colleagues who risk doing innovative work deserve both encouragement and support in the planning and execution stages of projects and appropriate rewards when they come up for periodic review.

- *Observe a regular schedule of external reviews.* Many departments are required to undergo external reviews every 5–7 years. Since accreditation agencies and other outside reviewers ask for a lot of information when they visit, departments are compelled to do systematic self-studies. Such studies are a lot of work, but departments should approach reviews as chances to evaluate, renew, and re-energize their curricula. A continual process (perhaps yearly) of internal review should ease the burden of the external process. Two documents from the MAA Committee on Department Review, *Guidelines for Undertaking a Self Study in the Mathematical Sciences* and *Guidelines for Serving as a Consultant in the Mathematical Sciences* offer valuable guidance to departments engaged in review and renewal. The MAA Committee on the Profession publishes *Guidelines for Departments in the Mathematical Sciences* every ten or fifteen years. Those and the *CUPM Curriculum Guide* are frequently used to measure the success of undergraduate programs.

- *Work to integrate adjuncts, full-time contractual, and part-time faculty into the department, providing professional development and further training*. In many mathematics departments over half of all student-credit hours are generated by adjunct, part-time, and full-time contractual faculty, as well as teaching assistants. The most recent (2010) CBMS study shows that the percentage of undergraduate *sections* in mathematics departments of four-year colleges and universities taught by tenured, tenure-eligible or permanent faculty increased slightly between fall 2005 and fall 2010 from 48% to 49%, and from 47% to 49% in statistics departments. But this is still under 50%. In public two-year colleges, the percentage of mathematics and statistics sections taught by full-time faculty declined from 56% in fall 2005 to 54% in fall 2010. Section D of the Summary in the CBMS study gives more extensive data. Since the sections taught by the non-tenured or tenure-track (NTT) faculty are generally lower-level courses with larger enrollments, a significant number of undergraduates taking courses in mathematics are taught by this workforce.

  Innovative ideas come from many sources, and many grant-funded experiments are worth replicating. Nationally-funded, innovative ideas to improve the undergraduate curriculum are numerous. Nevertheless, few of them are scaled up enough to have a national impact. Moreover, much of this innovation focuses on the first two years of the curriculum. The large number of NTT faculty teaching these courses makes it important that there be an entirely new professional development effort directed toward them.

  Departments with many NTT faculty should work systematically to integrate this workforce into the department in a way that compensates them for their time in professional development and promotes dialogue with regular faculty about the course content, technology, and pedagogical methods they are being asked to employ. CUPM encourages departments to modernize their curricula in tandem with the professional development of the NTT teaching corps.

  MAA has offered NSF-supported Professional Enhancement Programs (PREP) (funded at least through summer 2015) for regular faculty for many years. CUPM recommends that departments consider supporting several NTT instructors in similar offerings. A recent grant to the MAA will provide a program of professional development to those who are responsible for helping new teaching assistants succeed in the classroom. CUPM urges a greater national effort to improve and support professional development.

- *Department chairs should participate in professional meetings and workshops specifically designed for chairs.* CUPM encourages department chairs and undergraduate coordinators to attend national meetings and PREP workshops, where meetings of chairs and coordinators are devoted to sharing information and providing support for the management of the full range of departmental responsibilities. Networks for mentoring new chairs should be facilitated through the professional societies. CUPM recommends that MAA initiate a kind of “Project Chairs” to include chairs of two-year schools as well as colleges and universities.

Departments have the primary responsibility to set the curriculum, to help other departments select appropriate mathematics courses for their own students, and to develop major programs that are thoughtfully and intentionally designed to prepare their students for success in subsequent courses and for careers in the mathematical sciences. The department chair should provide leadership that encourages collegiality, cooperation, and innovation in assessment of these programs and in their continual renewal.

Mathematics as a Liberal Art

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The value of a mathematical education to a future member of society goes well beyond the mathematics itself. The content and skills conveyed by mathematical training have for millennia been part of a classical education. Logic was among the three fundamental subjects of the *trivium*; arithmetic and geometry were among the four arts of the *quadrivium*. (The other two, music and astronomy, have mathematical connections themselves.) The Platonic ideal for logically persuasive reasoning has long been mathematical. For instance, in writing the Declaration of Independence, Thomas Jefferson looked to Euclid for the persuasive form of an argument. Mathematics is an essential tool in science, technology, business, and elsewhere. At its core, mathematics is a way of thinking that enriches all human activity, even where mathematical content does not explicitly appear. Majoring in mathematics is preparation for life, regardless of one's intended career.

Mathematics is an art—both in the aesthetic sense and in the sense of reasoned living. Many students choose a mathematics major not as career preparation but for its own sake, because they happen to be successful at it or are interested in it. Society benefits from college graduates who are generally educated in higher mathematics, whose lives and social activities are influenced by their understanding of mathematics and, through it, of interesting aspects of history and culture. Beyond career and employment issues, "pure" mathematics majors are parents, aunts, uncles, volunteers in schools, tutors, voters in elections, school board members. Pure mathematics courses, including those driven mainly by aesthetic concerns, can help prepare students to become valuable citizens, all of whose contributions are augmented by skills and habits of mind developed through learning mathematics.

The cognitive goals laid out in the Overview to this *Guide* aim to foster effective thinking that is useful for everyone. Thinking clearly, producing and following logical arguments, and learning to distinguish sound reasoning from malarkey are skills of universal value. Working with hypotheticals and investigating their consequences are powerful strategies for approaching problems of all sorts. Understanding the role of quantifiers in argument helps people see a complicated world with detail and precision. Learning to explore the unknown by examining it from several different points of view is an effective skill in any domain. Mathematics can teach students the value of productive persistence, learning from mistakes, and turning present errors into future successes. Students who major in mathematics become better writers, better speakers, better thinkers, and better members of society.

All benefits of mathematical training notwithstanding, it is painfully clear that few leaders in our society are well trained in mathematics. This deficiency has many causes, some of them beyond our control. But we mathematics educators share the blame. It is easy to encourage and cultivate students who excel at all things mathematical—those who seem destined for graduate school, and may remind us of ourselves. We should continue to encourage these students, but we should also work to attract and nurture a cadre of people who combine mathematical training with broader humanistic skills. We should encourage students who love mathematics, but may struggle with it. For these students, too, a mathematics major can be a great choice. They need to hear us acknowledge that the world needs more people who, like them, approach mathematics with love, not fear.

Curriculum and pedagogy

This *Guide* makes specific content suggestions for diverse student audiences, including those planning graduate study in pure mathematics, prospective secondary mathematics teachers, and those planning careers in mathematical areas as varied as actuarial science, statistics, or financial mathematics.

The content needs of students not planning to pursue mathematics professionally are less specific. The broad-based mathematical training suggested by the Overview document will serve these students well, and the core mathematical subjects traditionally described as "pure" are especially rich vehicles for developing thinking skills that are rare and of special value to the student and to society.

One of our profession's important challenges is to make the mathematical experience successfully foster these uplifting steps toward better thinking. Mathematics can help transform students into better thinkers—but only if we allow and help students develop and practice those thinking skills. Mimicking procedures is not enough. Students must experience for themselves what it means to understand mathematical ideas deeply. They must feel the exhilaration of mathematical insight. They must learn and appreciate productive struggle. They must learn to articulate mathematical ideas clearly, orally and in writing. Students who develop their thinking skills through mathematical investigation will become the clear-thinking lawyers, business people, artists, leaders, and citizens that our world so urgently needs.

Recruitment and Retention

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Introduction

President Obama, Education Secretary Arne Duncan, the National Academies of Science and others have made producing more STEM graduates a top national priority. While one need not be a mathematics major to enter a STEM field, most of the STEM disciplines require a solid background in the mathematical sciences. Recruitment and retention of students who major in the mathematical sciences is thus a national priority.

Mathematical sciences departments need to become "pumps, not filters" in the STEM pipeline. There is some good news. The November 2013 U.S. Department of Education report *STEM Attrition: College Students' Paths into and out of STEM Fields* finds that attrition in other areas (health science and education) exceeds that in STEM fields. In addition, more students graduate with a major in mathematics than express that intention when they enter our colleges and universities.[1] Among all the major STEM disciplines, moreover, only mathematics picks up majors. Nevertheless, the challenge to mathematics departments is substantial: to cultivate and excite students who come to us with a keen interest in mathematics, and to adequately prepare students who arrive with interest in a related discipline that relies on collegiate mathematics.

Affirming diversity

We emphasize the importance of intentionally welcoming and including diverse populations to the major. Department-sponsored activities like seminars and guest speakers, contests, picnics, and field trips help build community. Study groups create bonds between students while helping them learn. Many of these activities, especially when encouraged for students early in their college careers, may also attract students to mathematics by highlighting the rewards of choosing mathematics as a first or even second major. Alumni engaged in interdisciplinary careers are particularly helpful as mentors or speakers.

In classrooms we aim to create communities in which diversity is respected. We try to develop a classroom culture that is inclusive, motivates students towards excellence, encourages them to support each other, and promotes persistence on difficult tasks. For students with deficits in learning or performing, we ensure that resources are provided. We pay attention to the cognitive and the affective (attitudes, beliefs, values, and motivations) dimensions of learning, because both influence what students will learn by the end of a course. Whether students enroll as part of a program in the mathematical sciences or to fulfill a general

[1] *Intended* majors are entering full-time freshmen at 4-year institutions who list Math or Stat as their first choice for an intended major. (See the American Freshman Survey) . *Actual* majors (as measured by National Center for Education Statistics) is the number of students for whom Math or Stat was their primary or first listed major.

requirement, what they retain from a course may contribute to their life-long view of mathematics, and thus to the public image of our discipline. Some of our students will become parents, teachers, and political leaders; all of them will become citizens and decision-makers.

Who recruits whom?

Not all mathematics faculty see it as their responsibility to recruit mathematics majors. Some departments are reluctant to recruit students who perform below A-level in mathematics. We reject these views. Professors who find their discipline exciting should communicate that excitement to students. Almost any student, regardless of major, can benefit from taking mathematics. Every student who enjoys the subject and is reasonably good at it should be encouraged and invited to continue.

Different students, different strategies

Intentional outreach can greatly increase the number of students taking advanced courses in mathematics. Letters of welcome from a mathematics department can encourage incoming students with strong mathematical backgrounds to enhance their careers—regardless of major—by adding mathematics courses to their programs. Students who succeed in a first course in mathematics often decide to continue. Mathematics departments should actively invite students who score A's and B's in a given course to continue. Many students become math majors "by induction": taking a first course, doing well, continuing to the next course, and so on.

Beginning college and university students present many different stages of mathematical preparation and maturity. To meet each at her or his level is a challenge. Talented students who arrive with advanced credits in mathematics (e.g., through Advanced Placement or International Baccalaureate programs) may need encouragement to take *any* mathematics in college. Retaining these students in the discipline requires interesting and challenging courses. At the same time, we should not assume that students who arrive with less mathematical experience are incapable—or unworthy—of completing a mathematics major program. We need to design mathematics curricula that serve a wide range of students, with courses that meet students where they are, advancing them to the next level and beyond.

This curricular challenge has spawned various strategies, some more effective than others. For students who begin below precalculus we have offered remedial courses. Precalculus courses themselves may receive college credit. Many institutions offer "stretch" or "supplemental instruction" courses, with extra class meetings and remedial topics introduced "just-in-time." Course redesigns may include in-person help clinics, drop-in classes, and remedial uses of on-line homework systems. Modular materials may be offered to remediate particular topics, again "just-in-time," combined with in-person tutoring or coaching.

At the other extreme are incoming students who have already accumulated college credits in mathematics. Some of these prefer to avoid further mathematics; others dig right in. College graduation requirements may obligate these students to take some further mathematics, but we should offer courses—required or not—that challenge and excite these students and encourage them to continue in mathematics, no matter their ultimate majors. Attractive course possibilities may be distinct from or independent of calculus; they might focus on discrete mathematics or on special topics such as number theory, voting theory, and knot theory. Such courses can attract students to mathematics by revealing more of what mathematics really is, without repeating material already seen in high school.

Efforts of other organizations

Both AMATYC and CUPM's subcommittee on Curriculum Renewal Across the First Two Years (CRAFTY) have looked closely at developmental mathematics, college algebra, and general liberal arts courses. In particular, the Curriculum Foundations Project has addressed the needs of beginning students in various other disciplines.

More recently, *Transforming Post-Secondary Education in Mathematics (TPSE Math)*, a new program, sponsored jointly by the Carnegie Corporation of New York and the Alfred P. Sloan Foundation, aims to effect constructive change in mathematics education at community colleges, 4-year colleges and research universities. Among the areas of inquiry for TPSE Math is to determine what areas in mathematics students really need to know. Then the task is to align the curriculum to address these needs. One challenge is to supply a route to the major for students who begin in precalculus or courses designed for non-majors.

Recruitment strategies

Mathematics faculty who want to recruit students can begin by offering courses appropriate to students' needs and mathematical maturity. Promising students in service courses in statistics, applied calculus, or finite mathematics should be individually encouraged, regardless of major, to take the next course or the first course in the major sequence. A necessary next step is to assure that students who begin in service courses or precalculus can indeed complete a major. Minority students and women who start below the standard entrance point (calculus) for majors may be particularly reluctant to continue in mathematics.

Warm and welcoming personal invitations can encourage students beyond what we expect. Coupling initial words of encouragement with departmental tutorial and social support can help students learn to study and to succeed at every level. Even when a curriculum is structured to allow alternative entry points to the major, many students need advisors, coaches, professors, and peers to help clear the way. This structure should be built into the organization of the department by building a corps of faculty who intentionally reach out to students.

Building community

Establishing a supportive and encouraging learning community is crucial to attracting new majors and to nurturing them as mathematics learners. As a first principle, a department must show that it respects and values students, recognizing needs of different student populations—residential, commuter, full-time, part-time, underrepresented minorities, men, women, those from different cultures—and working to ensure that all feel welcome.

Student-led groups (math club, MAA student chapter, Pi Mu Epsilon, Mu Alpha Theta, Kappa Mu Epsilon) allow students who like mathematics to find each other and learn more about math. These groups need active and committed student leaders, meaningful activities and programs, and departmental support.

When mathematics departments cultivate good working relationships with other departments, especially in STEM fields, students follow their example. Student clubs from several departments should be encouraged to jointly invite speakers who can point out how the various disciplines are used in their research or job-related responsibilities. It is equally important to be sure that invited speakers are diverse in gender, culture, race, and undergraduate major. Students respond well to such visitors because they want to see how their education will be integrated into their careers. They need to be able to see the possibilities for themselves in their life after college. They also benefit from seeing that gender and racial stereotypes can be overcome.

Technology can be used in various modes to foster community: email, web pages, Facebook, Twitter, blogs. And we should pay close attention to the department's physical space as it can enhance the sense of community, by providing spaces where students can gather in small or large groups to work (or play, or eat lunch) together; by placing comfortable seating areas near faculty offices; by providing blackboards, journals, a coffee pot, and refrigerator in student areas.

A critical mass of department faculty must be committed to and involved in the process of creating and sustaining community. The department must value faculty time (including in the tenure and promotion process) spent on community-enhancing activities and must commit resources to them. Forming a vibrant student community begins with the department's conscious commitment to this effort.

Orientation

Several schools have found that an orientation program for potential majors that might have been developed in connection with various federally-funded STEM-scholarship and support programs proves interesting and inviting for first-year students (freshmen and transfer students). In addition to introducing students with similar interests to each other and forming a cohort that might also be formally organized into a learning community, a good deal of information can be transmitted. One possibility is to offer a course meeting one or two hours a week. The syllabus for such a course should emphasize degree requirements, possible career paths, suitable double majors or integrated majors, and alumni success stories.

The course should encourage participation in professional organizations (everyone is invited to an MAA Section meeting) or in the campus mathematics club or Pi Mu Epsilon by requiring that students attend a meeting or two and write a short paragraph about the experience. Students should be introduced to possible undergraduate research experiences for mathematics majors and told what they will have to do to prepare for professional examinations (Praxsis, GREs, or actuarial exams, for example) or for graduate programs. Having librarians, financial aid experts (including those who can help students to make reasonable budgets), tutoring coordinators, and peer mentors speak with students can make the department seem accepting of and interested in them. Food and informal conversation among students and faculty encourage community and a feeling of professional identification.

Different paths to, through, and beyond a major

We should pay attention to the provision of diverse paths through the major. This *Curriculum Guide* has presented some new paths for those interested in mathematics. Traditionally the upper-division courses in mathematics, abstract algebra, linear algebra, analysis and complex variables, prepared students for graduate study in mathematics. Designing "options" that allow for a different selection of upper-division courses that lead students to the workforce or graduate programs in the life sciences, engineering, or business can increase interest in the mathematics major.

Students appreciate opportunities to learn off campus. The mathematical and scientific establishment can be harnessed to deepen mathematics majors' experience. Research positions in scientific labs, REU programs and summer internships integrate students into the academic culture and expand their career choices. Communicating mathematics through conferences, tutoring, and school outreach activities give mathematics majors an opportunity for personal and professional growth.

Many beginning college students see a mathematics major as leading to only one profession: high school teaching. While we do indeed need more teachers, college departments should educate students early about the full range of careers available to mathematics majors. The mathematical community should also support high school mathematics teachers in informing their own students of mathematical career opportunities.

Conclusion

Successful programs around the country demonstrate how a few simple strategies can dramatically increase the number of students studying mathematics. While all of the diverse ideas laid out here have worked for some colleges and universities, the common thread is that successful departments are intentional in their efforts to get more students interested in, thriving at, and majoring in mathematics.

Articulation Issues: High School to College Mathematics

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The Mathematical Education of Teachers II (MET II) (CBMS, 2012) chapter on high school teachers opens by noting that the double discontinuity experienced by prospective high school mathematics teachers and described by Felix Klein (1908) still exists today. As stated in MET II, this double discontinuity consists of the jolt experienced by the high school student moving from high school to university mathematics, followed by a second jolt moving from the mathematics major to teaching high school (p. 53). MET II addresses ways of smoothing the second jolt, but both jolts and extensions will be considered here, as they are the essence of the articulation issues between school and college mathematics.

This article is being written as the Common Core State Standards in Mathematics (CCSSM, 2010) are being implemented in almost all states in the U.S. Consequently, there is little evidence beyond speculation as to how CCSSM will impact the transition from school to college mathematics. The impact of the new standards and the associated assessments will not be evident for several years, but inertia in the K-20 education system will likely prevent major changes.

Readiness for College Mathematics

As in most state standards that preceded CCSSM, and in college admissions testing by ACT and the College Board, readiness for college mathematics has been a much discussed and sought after goal. For example, ACT has a benchmark score of 22 on the mathematics test for readiness for college mathematics (approximately 560 on SAT mathematics reasoning). This definition of college readiness is narrow, focusing only on the likelihood of making a grade of C or better in the first degree-credit bearing college mathematics course, often college algebra. CCSSM aims at college and career readiness, built on the principle that all students will meet the standards laid out for high school mathematics as with the NCTM 2000 standards. The assessments associated with CCSSM are being developed by two consortia of states to be implemented in the 2014–2015 school year. The CCSSM comprehensive assessment results, scheduled for the last part of grade 11, should therefore be available for consideration for the entering class of college students for 2016–2017. Some higher education institutions and systems in the states that signed on to CCSSM have agreed to use the CCSSM assessment results as a measure of readiness for college level mathematics, analogous to many institutions currently using the results of the ACT mathematics score or the SAT mathematics reasoning score. The agreements to use the CCSSM assessment scores in this way typically have been made at upper administrative college and university levels and not by mathematics departments. How mathematics faculties will accept and react to CCSSM assessments is largely unknown. In any event, validation of these assessment results as a reliable measure of readiness for college mathematics is likely half a dozen years or more away at this point.

However, the practice of placing entering college students in one of several possible entry-level mathematics courses is not likely to change with the implementation of CCSSM (as outlined above). Whether a student is prepared for calculus, for precalculus, for college algebra, or some other entry-level course likely will not be determined by CCSSM assessments. Therefore, placement programs will still be necessary.

What do we know? We know that for the near future, placement programs will continue to be needed and will probably not change.

What would we like to know? We would like to know how the implementation of CCSSM will affect the mathematical preparation of entering college students and what the CCSSM comprehensive assessments will tell us about these students. In addition, what changes might be needed in current placement tools to better align outcomes with the mathematics students will be expected to learn in high school.

What resources are available? College Board and ACT each offer placement testing systems, Accuplacer and COMPASS, respectively. The MAA placement tests are offered by Maplesoft and consist of the traditional tests in basic algebra, advanced algebra, elementary functions, trigonometry, and calculus readiness. The newest (2010) test is the Calculus Concept Readiness (CCR) instrument, and another new test, Algebra and Precalculus Concept Readiness (APCR), was field tested in 2013–2014. Other placement testing systems are available from publishing companies, and some, e.g. ALEKS, have tutorial systems included.

Including Remediation[1] in College Algebra

One of the continuing articulation issues is that of requiring developmental courses in colleges and universities. For example, in Arkansas, state regulations require that students who have an ACT mathematics score less than 19 must complete a developmental mathematics course prior to enrolling in a degree-credit bearing mathematics course. This results in more than 40% of the entering college students being placed in developmental mathematics courses, namely courses that are prerequisites for college algebra. Since ACT sets as a benchmark an ACT mathematics score of 22, that creates a band of scores 19–22 that ACT does not believe indicate college readiness, yet Arkansas policies say otherwise. In response to this, some institutions (e.g., University of Arkansas, Fayetteville) have created an alternate college algebra course with more class time and more support for students with ACT scores of 19–22. This has proved far more efficient and effective than placing these students in a developmental course and then expecting them to finish college algebra.

Some placement examination systems (e.g., ALEKS) have instituted ways to provide learning tutorials to move the student from one placement level to a higher level. Often the purpose is analogous to the college algebra with support scheme above, namely to move the student from a developmental placement to one that is degree credit bearing.

What do we know? We know that developmental mathematics courses in college are minimally successful in moving traditional age college freshmen from being unsuccessful in mathematics to being successful.

What would we like to know? We would like to know better ways to improve learning of mathematics in high school. Better yet, we would like to know how to better motivate students to learn mathematics, especially algebra, the first time they study it.

Calculus and Precalculus

Attrition from the algebra-precalculus-calculus sequence is known to be significant and affects the number of students in the science, technology, engineering and mathematics (STEM) pipeline. As these courses

[1] We are adhering to the distinctions between remedial and developmental as outlined, for example, by Ross (1970). Remedial instruction provides instruction in prerequisite material that is not a part of the course's objectives. Developmental courses have specific learning objectives that are required of subsequent courses, e.g., college algebra.

overlay the intersection of high school and college mathematics, clarity and consistency in content and cognitive demands are needed for good articulation and realization of the potential for understanding in the next course and beyond. Although the content of precalculus courses may be the same, the cognitive demands of courses can be very different. Recently, the content and cognitive demands of algebra, precalculus, and calculus courses have been studied. As Carlson, Oehrtman, and Engelke (2010) point out, "there is now substantial research on what is involved in learning key ideas of algebra through beginning calculus. However, a cursory examination of the commonly used curricula suggests that this research knowledge has had little influence on precalculus level curricula" (p. 114). (See also, Tallman & Carlson, 2012.)

Currently, an NSF-funded project of the Mathematical Association of America (MAA) is aimed at using the research results pointing to conceptual understanding needed to succeed in algebra, precalculus, and calculus to construct placement tests to measure this essential understanding. The first of these tests, the Calculus Concepts Readiness (CCR), is being used by some institutions to test for calculus readiness. Preliminary results indicate that many beginning calculus students do not have strong understandings of fundamental concepts, the major one being that of a function, especially viewed as a process. Of the twenty-five multiple choice precalculus items on CCR, students in calculus 1 at major universities on average get fewer than half of them correct. Moreover, the results are similar when CCR was administered to a sample of a couple hundred high school mathematics teachers, mostly teachers of algebra and precalculus. A second test, Algebra and Precalculus Concept Readiness (APCR), based on the same research, is nearing completion.

The fact that many students in the calculus 1 courses in college are achieving passing grades without having strong understandings of seeming essential precalculus concepts suggests that the research results may be wrong. However, it more likely points to the lack of cognitive demands of the calculus courses themselves. This latter point is supported by the results of Tallman and Carlson (2012) from examining a sample of 150 final examinations from college and university calculus 1 courses. They determined that about 15% (slightly different for different kinds of institutions) of the examination items required understanding of concepts or applying understanding of concepts. That means that 85% of the items required recall of information or recall and application of a procedure. Interestingly, a similar examination of Advanced Placement (AP) calculus free-response items found that approximately 40% of the items required applying conceptual understanding. The fact that performance on the AP examination has a different scale for determining grades dilutes this comparison and does not necessarily indicate that the AP calculus students exhibit stronger conceptual understanding of calculus than the university calculus students.

The above points strongly to weaknesses in the algebra-precalculus-calculus sequence caused by lack of cognitive demand. Surely, these weaknesses cause difficulties in subsequent STEM courses, indicating a lack of articulation between school and college or within colleges themselves. Another place that this lack of articulation surfaces is within the mathematics major. Many mathematics majors experience a jolt when they move from the more methodological calculus courses into the more abstract courses in algebra and advanced calculus. In fact, many (if not most) mathematics departments have instituted bridge courses (e.g. introduction to proof) to soften this jolt. Computer-based homework systems that provide testing using multiple choice items, often used in courses up through the calculus sequence, can aggravate this jolt, as the cognitive demand of such computer managed courses is often well below that of a junior-level course in abstract algebra or advanced calculus.

What do we know? Strong evidence suggests that the algebra-precalculus-calculus sequence, whether in high school or college, is not meeting its potential for use in subsequent courses or in preparing students, particularly mathematics majors, for smooth transitions to more abstract and advanced study.

What would we like to know? We would like to know how to influence schools and colleges to offer precalculus and calculus courses that are more cognitively demanding.

Differing Systems and Pedagogies

High school mathematics classrooms often differ from college and university classrooms. Most high school mathematics classes are in interactive classrooms with less than 30 students. Many incorporate collaborative learning situations, frequently with inquiry-based instruction. Contrast that with a lecture style university classroom with more than 100 students, sometimes many more than 100. This system of large lecture-style classes, present in many large universities, is not only different from the system in most high schools, but it is also inconsistent with what research in learning theory tells us that is most effective for long-term retention and transfer, which provides another instance where research results are not significantly changing classroom practices.

These differences will potentially increase with the use of online courses and degree programs in colleges and universities. The potential of delivering high-quality instruction by expert teachers to all corners of the world is indeed attractive, but many questions remain about promoting interaction and keeping the cognitive levels of grading high. Some of these questions are raised in the following section on what is known about how people learn best.

Ignoring How People Learn Best for Long-Term Retention and Transfer

The expanded edition of *How People Learn* (Bransford, Cocking, & Brown, 2001) reported research results on learning and how these results can improve teaching and learning. Subsequent to the publication of *How People Learn,* Diane Halpern and Milton Hakel (2003) reported the results of a consensus agreement among 30 experts on the science of cognition in "Applying the Science of Learning to the University and Beyond." They summarized the findings by giving ten basic laboratory-tested principles (listed in brief below) needed for enhancing long-term retention and transfer. In the opening paragraphs Halpern and Hakel (2003) write, "We have found precious little evidence that content experts in the learning sciences actually apply the principles they teach in their own classrooms. Like virtually all college faculty, they teach the way they were taught. But, ironically (and embarrassingly), it would be difficult to design an educational model that is more at odds with the findings of current research about human cognition than the one being used today at most colleges and universities" (pp. 37–38). So, many of us in collegiate mathematics are unaware of or ignoring the research results on what concepts students need to understand to be successful in calculus, and we are seemingly joined in this by our high school colleagues. However, high school classroom practices are much more in tune with the ten Halpern and Hakel (2003) principles than are most college classrooms.

1. The single most important variable in promoting long-term retention and transfer is "practice at retrieval."
2. Varying the conditions under which learning takes place makes learning harder for learners but results in better learning
3. Learning is generally enhanced when learners are required to take information that is presented in one format and "re-represent" it in an alternate format.
4. What and how much is learned in any situation depends heavily on prior knowledge and experience.
5. Learning is influenced by both our students' and our own epistemologies.
6. Experience alone is a poor teacher. Too few examples can situate learning. Many learners don't know the quality of their comprehension and need systematic and corrective feedback.
7. Lectures work well for learning assessed with recognition tests, but work badly for understanding.
8. The act of remembering itself influences what learners will and will not remember in the future. Asking learners to recall particular pieces of information (as on a test) that have been taught often leads to "selective forgetting" of related information that they were not asked to recall.

9. Less is more, especially when we think about long-term retention and transfer. Restricted content is better.
10. What learners do determines what and how much is learned, how well it will be remembered, and the conditions under which it will be recalled.

What do we know? We know what research on learning tells us about teaching for long-term retention and transfer. We also know that most college mathematics faculty members do not apply the principles gleaned from this research in their classrooms.

What do we need to know? We need to know how to teach – both in high school and college – to maximize long-term retention and transfer. There are research opportunities here.

Statistics Articulation

The situation in statistics articulation between high school and college is far less structured than that in mathematics. This is due in large part to the relative newness of statistics to the K–12 curriculum and its growing presence in undergraduate studies. However, with the growth of AP statistics over the past 15 years and the more definitive inclusion of probability and statistics in CCSSM, articulation possibilities are increasing. Currently, many college and university statistics courses cover content that is included in CCSSM, and developmental courses in statistics are rare. So are placement examinations in statistics. However, with a more determined effort via CCSSM to include and assess statistics in grades 6–12, entry-level college statistics courses have a chance to build on previous knowledge. In colleges and universities, statistics teaching is frequently dispersed across several departments—social sciences, agriculture, engineering, business, and mathematics. This most likely means that the effects of increased attention to statistics in K–12 because of CCSSM will be delayed a bit longer as these non-mathematics disciplines are likely to be slower to recognize the changes.

Teachers from High School to College

This is the jolt that is directly addressed by MET II. As noted by Klein (1908), this jolt stems mostly from the lack of any apparent connections between the mathematics the teachers studied as mathematics majors and the mathematics and statistics that they are expected to teach. There is no good reason for this lack of connections. Geometry, history of mathematics, abstract algebra, probability and statistics, and functions and relations form the foundations of school mathematics. Most importantly they give teachers the conceptual frameworks on which to hang their facts. Providing these conceptual frameworks, or versions thereof, to their students can bring coherence to learning, thereby promoting long-term retention and transfer. As concluded in *How People Learn*, "To develop competence in an area of inquiry, students must: (a) have a deep foundation of factual knowledge, (b) understand facts and ideas in the context of a conceptual framework, and (c) organize knowledge in ways that facilitate retrieval and application."

Prospective teachers would be better able to teach if their undergraduate mathematics courses modeled what is known about effective teaching for long-term retention and transfer. Some of the more prominent works in this area include the eight CCSSM standards for mathematical practice, the ten principles from Halpern and Hakel (2003) given above, the five elements[2] of effective thinking by Burger and Starbird (2012), the six core competencies for quantitative reasoning (Boersma, et al.,2011), and the conclusions reached in *How People Learn*. Although the eight CCSSM standards for mathematical practice were derived from the five strands of mathematical proficiency from *Adding It Up* (National Research Council,

[2] See appendix for the lists of the elements of effective thinking, the strands of mathematical proficiency, the core competencies for quantitative reasoning, the eight CCSSM practice standards, and the conclusions from *How People Learn*.

2001) and the five NCTM (2000) process standards, these alternate expressions help in understanding the practices in slightly different ways. For example, the *Adding It Up* strand of productive disposition (habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence in one's own efficacy) is very important to keep in mind whether teaching college students or K-12 students.

What do we know? We know that most college and university teachers teach the way they were taught and in ways that are different from the ways that high school teachers will be expected to teach.

What would we like to know? We need more examples of ways to effectively connect the undergraduate mathematics major courses to school mathematics.

What resources are available? MET II is available online and outlines the CBMS recommendations for the education of teachers. The references given here – *How People Learn, Adding It Up, Five Elements of Effective Thinking* (Burger and Starbird), Halpern and Hackel's *Change* paper, CCSSM's eight standards for mathematical practice, and the six core competencies for quantitative reasoning (Boersma, et al.) are excellent readings for helping prospective teachers develop.

Two books that connect undergraduate mathematics to school mathematics are *Mathematics for High School Teachers: An Advanced Perspective* by Usiskin, Peressini, Marchisotto and Stanley and *An Introduction to Abstract Algebra with Notes to the Future Teacher* by Nicodemi, Sutherland, and Towsley.

Quantitative Reasoning Courses

Quantitative reasoning (QR) or quantitative literacy (QL) courses are increasing as the newest entry in the mathematics courses for general education. Since these courses rely most heavily on proportional reasoning and number sense that is developed mostly in middle school mathematics and not emphasized in high school, there can be an articulation issue when college students encounter these. One of the weaknesses of CCSSM is its attention to the third readiness "C," citizenship readiness, to go along with college and career. High school courses have not emphasized general education issues, being hard pressed to cover all the content in geometry, algebra, statistics, and functions. However, teachers are beginning to use applications to the everyday worlds of their students as motivation for the students to learn mathematics. This has become more possible because of the increased number of quantitative issues in contemporary society and is especially effective in teaching statistics and data analyses.

What do we know? We know that courses in QR and QL are increasingly offered at many colleges and universities, sometimes by units other than mathematics departments.

What would we like to know? We would like to know how to evaluate QR and QL courses that are unusual in mathematics departments because they are not defined by their content as are most mathematics courses.

What resources are available? The National Numeracy Network (NNN) (an interdisciplinary organization) and the QL SIGMAA of MAA can provide information about QR and QL courses. NNN publishes a free access online journal, *Numeracy*, twice annually containing papers on QR and QL education.

College Credit for Courses Taken while in High School

There are two types of courses taken by high school students that may earn them college credit. The first are courses that are validated by examinations: Advanced Placement (AP) courses by the College Board and International Baccalaureate (IB) courses. Students entering a college or university and wanting credit for either AP or IB courses should have their grades in those courses reported to the college or university.

Individual colleges or universities have to determine what credit in what courses to award for what AP or IB grades. The courses of interest to mathematical science departments are AP Calculus AB, AP Calculus BC, AP Statistics, and IB Mathematics.

The second type of courses is dual enrollment courses (also called concurrent enrollment courses). Credit for these courses is arranged by way of agreements between a high school and a college or university. Sometimes these courses are taught on a college campus with a mix of school and college students, and sometimes they are taught in a high school. The teacher may be either a college teacher or a high school teacher, and there may or may not be an examination fashioned by the college. The mathematics courses range from college algebra through the calculus sequence.

Both AP course enrollments and dual enrollment course enrollments have been increasing, and are likely to continue for the near future.

What do we know? There is considerable information about students who take AP Calculus, less about students who take AP Statistics, and IB programs are relatively few compared to AP. David Bressoud has written extensively about the interaction of calculus in high school and calculus in college.

What would we like to know? The literature on the effectiveness of dual enrollment courses paints a mixed picture of their effectiveness. There is a professional organization that promotes concurrent enrollment courses. The central question is this: Do dual enrollment courses increase learning in postsecondary education? If so, how? The same question can be asked about AP or IB courses. Studying college material in high school can hasten finishing college, but does it increase learning?

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Appendix

The six core competencies for quantitative reasoning (Boersma, et al., 2011)

1. *Interpretation*: Ability to glean and explain mathematical information presented in various forms (e.g., equations, graphs, diagrams, tables, words).
2. *Representation*: Ability to convert information from one mathematical form (e.g., equations, graphs, diagrams, tables, words) into another.
3. *Calculation*: Ability to perform arithmetical and mathematical calculations.
4. *Analysis/Synthesis*: Ability to make and draw conclusions based on quantitative analysis.
5. *Assumptions*: Ability to make and evaluate important assumptions in estimation, modeling, and data analysis.
6. *Communication*: Ability to explain thoughts and processes in terms of what evidence is used, how it is organized, presented, and contextualized.

The five strands of mathematical proficiency from Adding It Up (National Research Council, 2001)

1. *Conceptual understanding*: Comprehension of mathematical concepts, operations and relations.
2. *Procedural fluency*: Skill in carrying out procedures flexibly, accurately, efficiently, and appropriately.
3. *Strategic competence*: Ability to formulate, represent, and solve mathematical problems.
4. *Adaptive reasoning*: Capacity for logical thought, reflection, explanation, and justification.
5. *Productive disposition*: Habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence in one's own efficacy.

The mathematical practice standards from CCSSM (2010)

1. *Make sense of problems and persevere in solving them:* Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution.
2. *Reason abstractly and quantitatively*: Mathematically proficient students make sense of quantities and their relationships in problem situations.
3. *Construct viable arguments and critique the reasoning of others:* Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments.
4. *Model with mathematics:* Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace.
5. *Use appropriate tools strategically:* Mathematically proficient students consider the available tools when solving a mathematical problem.
6. *Attend to precision:* Mathematically proficient students try to communicate precisely to others.
7. *Look for and make use of structure:* Mathematically proficient students look closely to discern a pattern or structure.
8. *Look for and express regularity in repeated reasoning:* Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts.

Four of the five elements of effective thinking (Burger and Starbird, 2012).

- Earth — Understand deeply. Don't face complex issues head-on; first understand simple ideas deeply. Clear the clutter and expose what is really important.
- Fire — Ignite insights by making mistakes. Fail to succeed. Intentionally get it wrong to inevitably get it more right. Mistakes are great teachers – they highlight unforeseen opportunities and holes in your thinking.
- Air — Raise questions. Constantly create questions to clarify and extend your understanding. What's the real question? Working on the wrong question can waste a lifetime. Be your own Socrates.
- Water — Follow the flow of ideas. Look back to see where ideas came from and then look ahead to see where the ideas may lead. A new idea is a beginning, not an end.

Conclusions from How People Learn (Bransford, Brown & Cocking, 2001).

1. Students come to the classroom with preconceptions about how the world works. If their initial understanding is not engaged, they may fail to grasp the new concepts and information that they are taught, or they may learn them for purposes of a test but revert to their preconceptions outside the classroom.

   Therefore: Teachers must draw out and work with preexisting understandings that their students bring to them.

2. To develop competence in an area of inquiry, students must: (a) have a deep foundation of factual knowledge, (b) understand facts and ideas in the context of a conceptual framework, and (c) organize knowledge in ways that facilitate retrieval and application.

   Therefore: Teachers must teach some subject matter in depth, providing many examples in which the same concept is at work and providing a firm foundation of factual knowledge. Burger and Starbird (2012) get at this in several ways. While giving advice on how to understand deeply, they say, "Sweat the small stuff." (p. 25). They note that when studying some complex issue, instead of attacking it in its entirety, find one small element of it and solve that part completely.

3. A "metacognitive" approach to instruction can help students learn to take control of their own learning by defining learning goals and monitoring their progress in achieving them. Burger and Starbird's five elements are aimed at students (and others) taking control of their own learning. Although there are anecdotes from their classrooms that illustrate the five elements in action, the real message is to the learner-thinker.

Assessment of the Mathematics Major

Bonnie Gold, *Monmouth University*

Do you want to do everything you reasonably can to ensure that the students you graduate with majors in the mathematical sciences have learned the central concepts and acquired the essential skills you are offering them? Virtually every faculty member will reply, "Of course!" This is the principal purpose of assessment. And yet, assessment is viewed as an unwanted chore in almost all mathematics departments, done grudgingly and postponed for as long as possible. The purpose of this short note is to show how assessment can yield significant program improvements without becoming a major burden.

First, an attitude change is needed. Rather than doing assessment because it is imposed from the outside, do the assessment with one question in mind: how can we improve the experience of our mathematics majors so that they better attain the goals of our program?

Goals and objectives

This leads immediately to the first step in good assessment: articulate the goals of the program! The goals should come from the department's mission statement (if it has one) and be aligned with the institution's mission. Until the department agrees on what the goals are, you cannot coherently assess how well your students are achieving them. You should keep in mind as you do this, that some goals are long-term, and even as students graduate, they may still be in the early stages of attaining them. Also, some goals—say, that students attain an understanding of the deep connections among mathematical subjects – may not be assessable without considerable effort. But usually once you have articulated the overall program goals, you can find at least several more detailed learning objectives (or "learning outcomes"—the terminology varies a bit) that indicate whether students are achieving each major goal of the program. To write a learning objective, finish the sentence "The student can…." The objectives should be measurable—not necessarily with a number, but with a clear concept of what it would mean to attain them and to partially attain them. This stage, of articulating the program's goals and objectives, will take several hours, maybe even a department retreat. And it is extremely important that all full-time faculty involved in the program participate in it. One immediate benefit of doing this is that the department often realizes how much agreement it has about its mission. There may be a few goals that not everyone agrees on, or agrees are high priorities, but generally a department really does have a largely common purpose.

Revise courses

The next step is to determine where in its program the department is giving students the opportunity to work toward achieving its goals for them. Since most complex goals are not attained when students first work on them, the department needs to look at how the program builds toward that goal. Usually a serious analysis of students' paths toward the goals leads the department to realize that there are gaps in the program.

Modifying or adding courses to fill these gaps is the first positive outcome of the assessment process. More generally, when problems are found at any point during the assessment process, that is a result of assessment and it is appropriate to make improvements at that time, rather than waiting until the full assessment cycle has run its course. Once your program is sufficiently coherent, it is time to start involving students in assessment.

Identify artifacts and develop rubrics

This is done by identifying what artifacts should be collected from the students to determine their degree of success in meeting a given objective. At least one such artifact needs to be identified for each objective, though sometimes the same artifact may be used to assess multiple objectives. The artifacts may come from activities that are already part of some courses. (Often not all faculty teaching the courses may be using these activities. However, there has to be agreement that, if an activity is being used for the assessment plan, all instructors will use it.) Or they may be summative activities covering the student's career at the institution, such as comprehensive examinations, exit interviews, or alumni surveys used toward the end of a student's work or after graduation. While eventually the whole department needs to agree on the assessment artifacts to be collected, the identification of appropriate artifacts for individual objectives can be split among small subcommittees charged with bringing proposals to the full department.

At the same time that artifacts are being chosen, drafts of rubrics should be developed to determine how to interpret what is collected. Otherwise, the department ends up with a huge mass of material and no idea how to interpret what it has gathered. The point of developing rubrics is to find a uniform standard that will enable those looking at student work to determine to what extent each student has met the specified learning outcome.

Collect and analyze data

Of necessity, to assess student learning, you need to collect samples from students. Most departments find that dividing the learning objectives into four to six clusters, and examining one cluster each year, makes the work manageable. Students should be involved in the process and be told that the material is being gathered to improve the program for future students. Generally a subcommittee of the department will do the initial analysis of data that has been collected, using the agreed-upon rubrics. They then prepare a report for the department of what has been found. Normally this would include how many students have achieved the given objective, how many have surpassed it, and how many have not achieved it (and perhaps how many have partially achieved it). Ideally this report should be quite specific about the existence of any problems. It should also include a summary of areas in which the department is achieving its goals and areas which require further work or discussion.

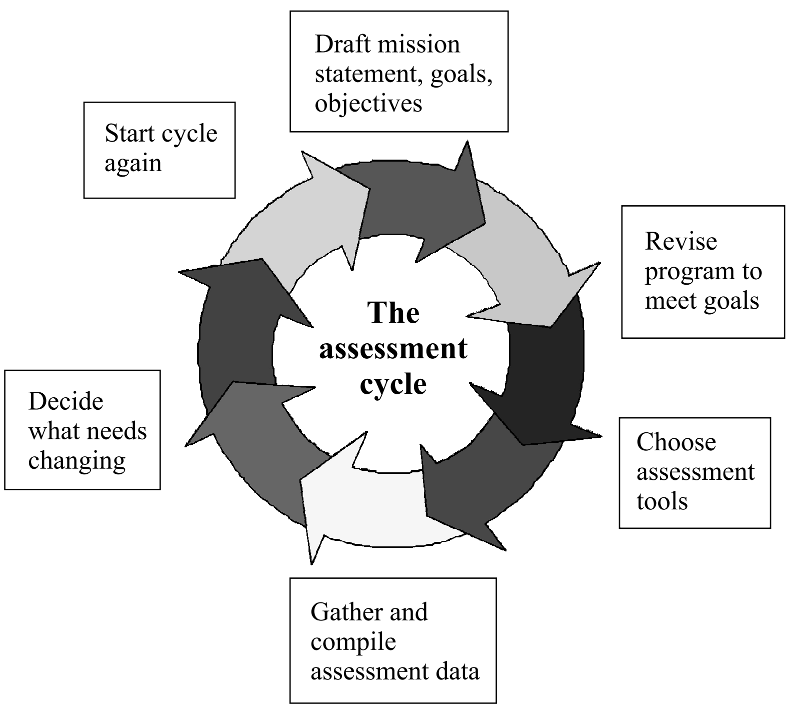
Report results to the department

Once the subcommittee has put together an initial report, the department needs to meet to discuss what is to be done. Sometimes the data that has been collected are ambiguous: it is not clear whether students are acquiring the desired skills and understandings. Or a substantial proportion of students may not have achieved the objectives, but it is not clear why. In this case, additional work may need to be done before the department can decide on a remedy. More data, not in the initial assessment plan, may need to be collected and analyzed. Good assessment should lead to program improvements, but when problems become apparent, care needs to be taken to identify the source.

Identify changes to be made. When an objective is not being attained, there can be many causes. Students may not have had enough opportunities to acquire the desired skills, the skills may not have been taught or not taught well, or the expectations may be set too high for the students in the program. The department thus needs to examine the program more closely to see where the problem is. For example, if the first time students' mathematical writing is examined critically is on the item used for assessment, they are not likely to do well. When this happens, courses earlier in the program where students can begin work on the skill need to be identified. On the other hand, sometimes the level of expected achievement is not reasonable for the students the institution attracts and an adjustment of the department's goals and objectives is called for. Or the types of artifacts being collected are not appropriate for the objectives they are supposed to measure, or are no longer an appropriate activity because the course has changed for other reasons. Often it is a combination of program issues and assessment practice that needs revision. In any case, the purpose of assessment is to improve the program—that is, what students get out of their mathematics major. So the identification of changes to be made—and implementing the changes—is the most important step of the assessment process, often referred to as "closing the loop."

Assessment must be cyclic

Making changes based on assessment data is called "closing the loop" because assessment is not a "once-and-done" activity. Every goal should be assessed at least every 4–6 years. When a department makes changes to improve student learning, it needs to check that those changes have worked. In addition, changes in the outside culture or in the students coming to an institution, mean that what was working five years earlier may not still be working. Perhaps a couple of crucial faculty members have retired or left, and students are not getting the same kinds of experiences through their studies. So it is important that all the issues are re-examined periodically, and that the assessment process also be revised as needed. In addition, later iterations of the cycle provide longitudinal data that allow comparison of student outcomes over time. Of course, if a problem becomes apparent at some point when the particular issue is not scheduled for assessment, it is appropriate to respond to the problem immediately, rather than wait for its turn in the cycle—as long as there is sufficient care taken to determine what the problem is and what possible remedies there might be!



To quote from the original CUPM assessment document [1], the questions to ask include:

First, about the learning strategies:

- Are the current strategies effective?
- What should be added to or subtracted from the strategies?
- What changes in curriculum and instruction are needed?

Second, questions should be raised about the assessment methods:

- Are the assessment methods effectively measuring the important learning of all students?
- Are more or different methods needed?

Finally, before beginning the assessment cycle again, the assessment process itself should be reviewed:

- Are the goals and objectives realistic, focused, and well-formulated?
- Are the results documented so that the valid inferences are clear?
- What changes in record-keeping will enhance the longitudinal aspects of the data?

Resources

The following books are good sources for information about assessing the major.

1. A good introduction to the assessment process is the initial 1995 CUPM report, "Assessment of Student Learning for Improving the Undergraduate Major in Mathematics," reprinted in the first volume mentioned below and available online.
2. B. Gold, W. Marion, and S. Keith, eds., *Assessment Practices in Undergraduate Mathematics*, MAA Notes # 49, Washington, DC: Mathematical Association of America, 1999
3. L. Steen, ed., *Supporting Assessment in Undergraduate Mathematics*, Washington, DC: Mathematical Association of America, 2006

Technology and the Mathematics Curriculum

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Introduction

What role should technology play in teaching and learning undergraduate mathematics? We assume the primary goals for using technology are to deepen student understanding of mathematics and statistics and to increase student motivation and engagement. We believe the informed and intelligent use of technology helps meet these goals. We address teachers who seek to enhance current use of technology and colleagues who do not use technology now. We discuss departments and institutions concerned with the implications of technology and also faculty incentives and rewards for the risks and hard work of creating and implementing technology innovations.

We have a broad and flexible view of what technology means. Wherever feasible we specify technologies by function, referring to computer algebra system (CAS), graphing calculator (or a smart phone with a graphing app), online applet, course management system, spreadsheet, or statistical package, rather than by "brand" names. We include both free software and online utilities and packages sold commercially. For example, a formal assignment could require students to use the software licensed by the school, free online utilities such as WolframAlpha [39], Sage [30], SOCR [35], or freely downloadable software such as R [28] or GeoGebra [11]. Some commercial products are better documented and supported than free ones, but not always. The landscape is changing rapidly. Many technologies work on multiple platforms. Students do (and could be encouraged to) use their smart phones and other hand-held devices to access online resources in addition to computers and calculators. Different platforms will be appropriate in different environments and we do not specify them. Although it is imprecise, we use "mathematics" and "mathematical" to refer to the range of subjects taught by departments of mathematical sciences; in particular, we include statistics. We use the term "teacher" in lieu of "instructor" or "professor."

We discuss five ways technology can be used to enhance teaching and learning:

- **Exploration:** Technology can be used to explore ideas and substantial applications.
- **Computation:** Technology can enable students to work with rich examples, realistic applications, and large data sets.
- **Communication:** Technology can facilitate communication between teacher and student and among students, inside and outside the classroom.

- **Assessment:** Technology can give students more ways to demonstrate knowledge and understanding and improve the effectiveness of both formative and summative assessment.
- **Motivation:** The use of technology can increase students' engagement and motivation.

Using technology for visualization could warrant a separate section, but instead we have referred to visualization throughout this document.

The forthcoming *Pedagogy Guide* [15] of the MAA Committee on the Teaching of Undergraduate Mathematics will provide some evidence on how well certain pedagogical strategies work.

We also discuss

- **Benefits after college:** Students' experience using technology can help them in the workplace and in graduate school.
- **Issues for departments and institutions:** The use of technology raises a host of issues for departments, colleges and universities.

An annotated bibliography, including digital sources and websites, appears at the end of this report.

Key to the effective use of technology is using it intelligently. Is the technology appropriate for the given problem? Are hypotheses satisfied to guarantee that the process will produce an answer? If numerical, is the answer plausible? Of the right order of magnitude? These questions can be subtle, even for experienced teachers, but they should always be raised. Also the effort required for successful use of a specific technology by students at a particular level should be balanced against the benefits.

Exploration

The goal of exploration is for students to engage with examples, deepen understanding, notice patterns, and make conjectures, thus providing a natural setting for recognizing the importance of proof to support correct conjectures. To be effective, exploration must culminate in clear statements of what has been discovered. Teachers must follow up and be sure students understand what the exploration has achieved.

Students in a College Algebra course that emphasizes modeling and numeracy may use a spreadsheet to work examples, build familiarity with ideas, and look for patterns. Any graphing utility can enable students to experiment with the effect of changing parameters of a function, say with the quadratic function $f(x) = a(x - b)^2 + c$. For students in multivariable calculus it might be a first look at the "second derivative test" via the plot of a function $f(x, y)$ and the expression $D(x, y) = f_{xx}(x, y)\, f_{yy}(x, y) - (f_{xy}(x, y))^2$ at a critical point. Technology can allow students low-cost attempts to identify parameter values to achieve a desired result, and they can learn even from failed attempts. A graphing utility can also change a "procedure," like the method of Lagrange multipliers, into something that is geometrically meaningful. Technology can help Linear Algebra students visualize the effect of a linear transformation of the plane and notice the role of the determinant in comparing the area and orientation of a finite region and its image.

Assignments can also help students think critically about what technology tells them. A Calculus assignment with a twist asks students to analyze/critique the following argument. The function $1/(\cos x + 2)$ is a positive continuous function for all x, so its antiderivatives must all be continuous, strictly increasing functions, but the antiderivative given by a computer algebra system (CAS) is periodic of period 2π and hence not strictly increasing. Explain this.

In a Transition to Proofs course, students can use a simple program (or a CAS or an applet) to examine the sequence $x_n = f^n(x_0)$ of iterates of a linear function $f(x) = ax + b$ as they vary the initial point x_0 and the parameters a and b. Students observe patterns, formulate conjectures, and try to prove them. Along the way, students have to develop the precise language of convergence.

In a Real Analysis course, students' exploration of the iteration of a function of the form $g_c(x) = cx(1-x)$ introduces them to discrete dynamical systems and the analysis of fixed points. Specialized programs like GAP [10], Magma [16], and Sage [30] permit explorations in Abstract Algebra.

For Differential Equations, the displacement, $y(t)$, at time t of a physical device satisfies a differential equation, $ay''(t) + by'(t) + cy(t) = f(t)$, where $f(t) = \cos(\omega t)$. Students use technology to explore the effects of changes in parameters, discover the relationship between the frequency ω of the function f and the natural frequency of the solution of the differential equation, and experience beats and resonance (especially if their CAS gives output aurally as well as visually). They gather data through numerical or analytic runs of a CAS to create a frequency response plot showing the maximum displacements of the physical device across all input frequencies ω.

Students can explore the online universe, using search engines or online sites dedicated to mathematics, data or computation to look up a mathematical concept or a person/place of mathematical interest. The MAA provides a site [5] to explore resources for a variety of courses. Another source is the site *Demos with Positive Impact* [6]. The American Mathematical Association of Two-Year Colleges (AMATYC) has a project called *The Right Stuff: Appropriate Mathematics for All Students* [29]. There are many special-purpose applets available, but they can be difficult to find. As one example, Wolfram Research [38] has many demos.

Computation

In courses at all levels, substantial and realistic applications involve "messy" mathematics that makes calculation by hand onerous or infeasible. Using technology opens the door for students to set up solution strategies, justify their analyses, and interpret the results. For example, in Calculus there is no reason to restrict the study of optimization to situations where functions are simple and zeroes of derivatives are easy to find by hand.

Linear Algebra is a good place for using technology to carry out lengthy and/or complex computations, so students can concentrate on the underlying ideas. For example, when using a Leslie matrix to study populations, technology can permit a preliminary "what if" analysis by changing parameters through iterations, before a formal analysis of the model via eigenvalues of the matrix.

In Differential Equations, students can see the form of the solution using a CAS and understand the nature of a solution, say of steady state and transient portions. When analytic solutions are not possible, numerical solutions, even using simple Euler methods in a spreadsheet, can give insight as well as permit a study of the importance of step size in approximating a solution. Moreover, estimating parameters in a differential equation model for which data are available can suggest or confirm both the appropriate modeling mechanism and the form of the solution.

There are many options currently available for accessing and using powerful computer algebra systems. The most widely used packages include Maple [17], Mathematica [19], MATLAB [22], and Sage [30].

Introductory Statistics is another course where the use of technology for computation is essential. In this *2015 CUPM Guide*, the report on statistics has six overall recommendations for all introductory statistics courses, including statistics for mathematics majors. These recommendations include: "use real data" and "use technology for developing concepts and analyzing data." Besides technology for exploration and analysis, students can also use technology for simulation to build understanding of probability and distributions, including such basic concepts as the distribution of sample means and the Central Limit Theorem. Many introductory statistics texts come with online resources. Other statistical packages (some commercial, some free) are independent of a text. The free *OpenIntro* [24] offers text plus supporting technology. The SIGMAA on Statistics Education has compiled a list of technology resources for the teaching of statistics [33].

Communication

We don't minimize the importance of communication that does not use technology. Indeed, technology-aided communication, using social media or structured platforms for the special purpose of course communication, often leads to face-to-face communication.

Communication is an important element of exploration, as students write up their findings. In fact, in an Inquiry-Based Learning class, student writing—facilitated by using a wiki—can produce a text for the class by the end of the term [13].

Alternatives to traditional textbooks in mathematics using technology are emerging. Free online and open source electronic textbooks are available in growing number. Examples include [2] and [36]. The American Institute of Mathematics [1] provides a clearing house for such materials.

There is considerable interest in using flipped classrooms. In the "forward" flip, students watch videos, read, and use technology before class to prepare for practice and exploration in class. In the "reverse" flip, students first explore a new idea in class and then use technology as well as watch or read material after class.

Videos can be used to add a coda to a class to be posted for student use before the next class. Some teachers use existing videos. Others use many levels and kinds of technology to produce their own videos in order to personalize the message. In this regard we note the apps EduCreation [7] and Screencast-o-matic [32].

Many teachers use course management software to monitor student success in online homework. Asking students to solve problems and/or submit questions before class informs the teacher about student successes and struggles and guides planning for the class, e.g., to determine what concepts to emphasize and/or how to group students for collaborative work.

The "ink technology" of a tablet or a so-called smart-pen (such as Livescribe [14]), can be used in addition to an online homework system. Students work on the online homework problems, but rather than writing on paper, they write using ink technology. There is no need to type, nor is the students' writing "interpreted," because the ink technology captures each student's process as the image of what they wrote and how they wrote it in real time. Each student can enter the final answer in the online homework. The teacher can review the tabulated online homework results and also the ink technology work and can choose the work of one or more students to share with the class. In the classroom the teacher can pause the display at any point to open discussion, give the solver feedback on his or her approach, and offer suggestions for next steps as well as engage other students in "what would you do here?" discussions. Ink technology solutions can also be posted anonymously using the course management system, and the teacher can ask students to comment and/or perhaps answer specific questions. The students' comments can be posted, or the discussion can continue in class.

There are many ways to use technology to capture what is done in class so it can be posted for students to review (or to preview). In addition, there might be a network of mobile devices used in the classroom, so the work in class can be preceded or followed by work online. The teacher can also use technology to present material in or out of the classroom, using, e.g., PowerPoint, Beamer, or PREZI [26]. The latter permits the teacher to jump to smaller pieces and can be stored online.

Students and teachers can use electronic discussion boards, wikis (from publishers or via course management systems), and commercial social media for collaboration and communication. An advantage of social media is that teacher-student communication can occur where the student already spends time. If some students find the teacher's presence in their social world intrusive, there are alternative cloud-based tools to reach students, such as Google+.

Students' smart phones can be an important platform for communication. Some course management systems let students choose how they prefer to receive communications: email, text messages, listserves, social media, etc. Some teachers use course reminder features on phones, e.g., text messaging, when they do not have course management systems. In some instances students do almost all their course work on their phones, including dictating writing to a voice-to-text system, proof reading in a word processor, and sending to a wireless printer.

Massive Open Online Courses, MOOCs, are a relatively new and growing presence on the landscape. Individual institutions are making their own decisions about whether their students can earn credit for taking certain MOOCs, possibly with local certification. In this way, existing MOOCs can expand their offerings. Others can use parts of MOOCs to enhance their own courses and widen their own students' experiences, perhaps recommending that their students watch selected lectures or demonstrations. Consider one example [23] in Robert Ghrist's Calculus Single Variable through Coursera.

Some schools are producing courseware to reach beyond their own campuses. This can result in a MOOC reaching a very large audience or a course shared among just a few institutions. Technology that captures the entire classroom (aurally and visually) can be used for "blended" classes that have both in-person and virtual participants. This can permit small institutions to team up and take turns offering a specialized course in real time. At Mount Holyoke College, the Mathematics Leadership Program uses a technology called Zoom [41] that unifies cloud video conferencing, online meeting and cross-platform group chat, to offer a hybrid course for in-service teachers with meetings simultaneously for on-campus participants and those from around the world [20].

Writing mathematics is an important form of communication for all students and at all levels. Technology enables a thorough mix of text and graphics for enhanced communication. Students bound for graduate school will be especially helped by learning to use TEX. See the next section for the use of technology to enable and enhance the assessment of writing assignments.

Assessment

Assessment can indicate whether students know facts, understand ideas, or can do a more complex activity, such as modeling or problem solving. Technology can inform both the teacher and the student of what needs improvement. When students are active in the classroom (often facilitated by technology), the teacher can learn more about what students know and what they need to work on by listening to student conversations, live or captured.

Online homework systems (including WeBWorK [37]) are becoming more widely used. These systems collect extensive data that can help teachers and their departments refine their course offerings. Online homework systems can also be adapted for use in course placement. Some online systems incorporate tutorials that can extend the capacity of on-site tutors.

Discussing problem solving using authentic student work, through video and ink technology, helps both the teacher and the students gain insight into what is understood. This technology helps the teacher plan classroom activities that are appropriate and effective. In another use, students can upload their solutions or essays, and their writing can be sent to other students for comments. The quality of a student's editorial comments contributes to his or her grade on the assignment.

Students posting questions and comments reveal their understanding. In the real or virtual classroom, "clickers" [3] and their smart phone, calculator, and computer analogs can provide useful feedback. For example, texting to answer with a phone can provide a teacher with recordable and instant assessment results. Index cards used as low-tech substitutes for clickers can provide the same immediate feedback, but not the recordability nor the anonymity.

Technology also affects assessment practices by challenging teachers to construct examination questions that cannot be entirely "answered" by technology. We say more about this in the section on department issues.

Student motivation

When students take ownership of their learning, they are motivated to persevere and succeed. Pedagogical strategies that engage students actively in exploring and discovering ideas, solving problems, communicating their ideas to others, and reflecting on their own thinking can foster this ownership.

For many students, seeing the mathematics they are studying applied to real-world situations is an important motivator. Students then see how the mathematics they are learning will re-appear in subsequent courses. Technology can enable students to work with more realistic applications of mathematics and with real data. The scale can vary from a simple activity in class to a substantial student project. Indeed, there is growing interest in using applications, coupled with technology, to introduce and motivate the learning of mathematics. For example, SIMIODE [34] is a site for learning and teaching differential equations using realistic modeling and technology in this way. In addition, when students work with large data sets or complex calculations in their mathematics classes, they can draw on those classes for projects in courses offered in other departments.

When using technology, students seem to be more apt to try things and try again, to be playful and less fearful of error. Students may see by-hand solutions as laborious and error-prone and thus be more cautious working by hand. Perhaps the technology environment is less threatening because it can be more private. The experimentation and risk that accompany game playing have taught many current students to explore more boldly with technology. The immediate feedback offered by technology may make students more adventurous and more persistent. Visual images provided by technology can also be a powerful inducement to persist and search for understanding.

Some teachers have used game environments and rewards to motivate students to do activities that build understanding of mathematical concepts. For example, games in mathematical ecology can help students understand the elements of the mathematical models being used to generate the simulations. In fact, some recent books consider the use of games in learning; see for example [31].

Another use of games for teaching has been devised by George Woodbury [40] who uses a grading system that involves game-design features. Although students earn no points for their online homework scores, the students who “level up” (earn essentially perfect homework scores and a mastery score on the online practice quiz) can earn more points for their in-class exam scores than students who do not. Those who have leveled up but did not score well on the in-class exam also have an opportunity to re-take it.

Benefits after college

Employers want graduates to have experience with technology, be it programming or using software applications. Using an appropriate tool to solve a problem is a universally-valued skill. Effective communication of ideas often requires technology for images, data representations, and notation. Most important, for professional and personal needs, students need the ability to learn to use emerging technologies. We therefore have a responsibility to encourage and enable our students to learn technologies alongside their mathematics.

Graduate schools often expect knowledge of TEX to properly and professionally prepare material for communication, be it homework assignments, technical materials for posting, or articles for publication. Sometimes knowledge of a programming language might be appropriate for preliminary or even deep analyses, exploration and analysis of large data sets, or to modify established routines. Indeed, in a 2001 paper [25] a well-known algebraist, Cheryl Praeger, says:

> It is impossible to over-emphasize the impact of computer systems such as MAGMA and GAP on the professional lives of algebraists … both on their teaching and on their research. All of my research students use these computer systems as essential learning tools to explore new concepts. They demand illustrative examples to examine by computer to aid their understanding.

Department and institutional issues

CUPM has long recommended a department culture of "continuous improvement." Departments and institutions have a responsibility to explore how their programs can or should change in light of technology. The place of technology in the curriculum should always be in service of general cognitive and content-specific learning goals, as outlined elsewhere in this *Guide*. We have highlighted ways this can happen by fostering exploration, computation, communication, and motivation, and attending to assessment.

Departments can work on technology and the curriculum in many ways, e.g., study by department committees, external reviews, seeking and obtaining grant support, or as a consequence of institutional commitments. To have positive outcomes, we argue that this programmatic work requires input from all department constituents and education of all on what technology is available and what the possibilities are in the department's setting. It is not healthy if individuals feel disenfranchised because they lack experience or interest in using technology, and we should not assume that these categories necessarily align with age. We recommend collegial processes and shared learning, including demonstrations by faculty who have experience using technology successfully, but not coercion. It is important to pick easy entry points for novice users: key places where technology saves time, enhancements of things they already do (such as illustrating simple Riemann sums), and simple ways to engage students more actively during class.

Technology often comes into our classrooms through student use of calculators and online CAS, very sophisticated CAS. The implications of this and other technology for what is important and/or necessary for our students to know is a large and sometimes contentious issue. It is a truism that a CAS should not be able to pass a calculus exam. But what does this mean? Is it an argument to forbid the use of technology on exams, or is it an argument to re-think our testing and, more broadly, our syllabi? We argue for the latter. If technology is used on an exam, students should be asked to describe the process and goal of any calculation, interpret its result, and explain its significance. An exam question can give the result of a computation and ask students for explanation and interpretation. Students should also be asked higher-order, conceptual questions that go beyond techniques and cannot be answered merely with technology. Thus technology should enable us all to elevate our "game" and our expectations of our students. At the same time, we have to carefully bring our students up to the level to which we aspire.

To illustrate, consider an example from calculus: integration by parts. Rather than merely a specific technique (which a CAS can carry out), this is a process of transformation (a new integral can be *set up* by hand), and the idea of transformation should be emphasized more than facility with the details of the technique. The current interest in building concept inventories for calculus [8] to test more than computational skills is a move in the direction of raising the bar in student understanding and de-emphasizing the ability to do calculations for which technology is well-suited.

Not everything need happen department-wide. Departments should also foster the work of individual teachers in finding or developing new and effective uses of technology in the classroom. This can occur through encouraging exploration and supporting reasonable risk-taking in the use of technology by individuals, particularly younger faculty. Varied mechanisms exist for providing incentives and rewards for individuals doing substantial work on curriculum and teaching in new ways. Annual evaluations of teaching should certainly recognize work of this kind. Some institutions offer their teachers internal grants for course development. For example, Georgia Southern University has for over 25 years offered supporting summer grants of $3–4000. Much of the work that was supported involved the use of technology, sometimes using software and courseware already available on campus. There is a premium on innovation, and a campus-wide committee judges proposals. Recently, development of online courses has been particularly encouraged. At other schools, institution-wide committees select faculty for teaching awards to recognize their work in the classroom, including technology innovations. The recognition itself sends a signal, to the recipients and to others, that the institution values the work. Such efforts can be featured externally (for alumni, media, and prospective students) and internally through invited presentations for faculty gatherings and retreats.

Reasoned and reasonable experimentation in using technology should be encouraged and supported. Acknowledging the risks in the first use of any innovation is important, and institutional encouragement to learn from mistakes and improve should be in place.

In departments where a large fraction of the teaching is done by part-time adjuncts, incorporating the use of technology poses special challenges. These teachers may need help using the technology themselves. More important, they may need help using the technology to enhance the mathematical ideas, not to replace them.

Finally, we note that the use of technology has implications for institutional infrastructure. While some use of technology imposes new costs, there can also be some savings. For example, some departments and institutions are changing their "computer labs" to make use of student-owned devices.

While we cannot ignore the presence of problems, including some that can only be dealt with at the institutional level (such as difficulties with course management software), we believe that moving forward in using technology in teaching mathematics is essential for our discipline. To ignore technology is to ignore the future. We do this at our professional peril.

Conclusion

There is no conclusion, only on-going conversation and inquiry into the uses of technology in the undergraduate mathematics curriculum. We believe this is an evolutionary as well as revolutionary process, with new species and approaches being introduced (and some dying out). We encourage individual teachers, departments, and institutions to foster and support reasonable experimentation in the uses of technology to teach undergraduate mathematics. We believe wider discussion and sharing of implementations through MAA activities (meetings, speakers, newsletters, etc.), journal publications, [4, 9, 12, 18, 21, 27], campus visits, etc. can help support innovators in their local environments. We see this report as part of that process, and we believe that everyone—teachers and students—can benefit from open, objective, and energized discussions about the uses of technology in teaching undergraduate mathematics.

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www.rossmanchance.com/applets/index.html, onlinestatbook.com/2/index.html,
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For dynamic data exploration and simulation in introductory statistics they list *Tinkerplots* at www.srri.umass.edu/tinkerplots and *Fathom* at concord.org/fathom-dynamic-data-software.

They note that most publishers have technology-packed sites that include eBooks and other resources. They also list statistical packages and calculators for doing statistics, such as: TI8x calculators, Minitab, SPSS, JMP, DataDesk, StatKey, StatCrunch, CrunchIt!, Stata and R.

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Undergraduate Research in Mathematics

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The Overview of Majors in the Mathematical Sciences recommends that every major student should "work, independently or in a small group, on a substantial mathematical project that involves techniques and concepts beyond the typical content of a single course. Students should present their results in written and oral forms." A student can have such an experience by participating in undergraduate research, either by working one-on-one with a faculty mentor or in a research group, perhaps with both faculty and student mentors.

Members of CUPM's Subcommittee on Research by Undergraduates have written "Mathematics Research by Undergraduates: Costs and Benefits to Faculty and the Institution." This document summarizes some of the major points in the subcommittee's report. More extensive resources are contained in the longer version of this report included in the online version of the *2015 CUPM Curriculum Guide*.

Definition of undergraduate research

The subcommittee's report uses the definition of undergraduate research as having the following components:

- The student is engaged in original work in pure or applied mathematics.
- The student understands and works on a problem of current research interest.
- The activity simulates publishable mathematical work even if the outcome is not publishable.
- The topic addressed is outside the standard undergraduate curriculum.

This definition excludes situations in which undergraduate students are working on difficult problems whose solution is not known to the students but have been solved by someone in the mathematical community. While such activities can be very beneficial to students (and meet the CUPM recommendation in the *Overview*), they are called by other names: independent study, capstone experiences, inquiry-based projects, etc.

Benefits

Students benefit from doing undergraduate research in many ways.[1] Undergraduate researchers in a STEM discipline gain knowledge and skills, show increases in academic achievement and educational attainment, and grow and advance professionally and personally. For students from underrepresented groups, a research

[1] M. Osborn, K. K. Karukstis, "The benefits of undergraduate research, scholarship, and creative activity," *Broadening Participation in Undergraduate Research: Fostering Excellence and Enhancing the Impact*, Council on Undergraduate Research, Washington, DC, pp. 41–53, 2009.

experience with an experienced faculty mentor is positively correlated with improvements in grades, retention rates, and motivation to pursue and succeed in graduate school. Specifically, research projects for undergraduates can help them prepare for graduate school in mathematics by preparing them to make the transition from structured course work to open-ended research that requires a longer period of work on a single problem without many inherent clues on how to prove results. Students who do not plan on attending graduate school also can benefit from doing undergraduate research. Such students typically are employed in business and industry or teach mathematics in high schools. These professions often require employees to analyze data, work on unsolved problems, and present mathematical ideas orally or in writing—all of which are skills learned while doing undergraduate research.

Some specific studies about the benefits of undergraduate research are given in the online resources. Also, the Council on Undergraduate Research (CUR) is a good resource on the benefits of undergraduate research.

Types of programs

Undergraduate research can occur in an intense summer program or be spread out through the academic year. Each approach has strengths and weaknesses. While undergraduate research is often done with advanced students, there is a growing trend to work with students earlier. Engaging students in undergraduate research early in their academic careers helps to retain students in the major and provides them a chance to work on research over several semesters or summers.

Originally, Research Experience for Undergraduates (REU), was the name given to NSF-funded undergraduate summer research programs. Now the acronym REU has come to mean any intensive, summer-long research program for undergraduate students. These programs are often collaborative, where students work in research groups with peers, supervised by or in collaboration with a faculty mentor. These programs are often residential: students travel to another campus, living and working there for the duration of the program. REU programs are essentially full-time research jobs, for which students receive a stipend. An REU can be a marvelous experience for a student, providing the opportunity to live and work in a new and different environment while engaged in a focused research problem. An advantage of a summer REU program is that the students work exclusively on the research problems and hence can make significant progress in a short time. The condensed nature of a summer program makes it crucial that the selected research problems provide opportunities for quick and meaningful progress. Group dynamics and interactions also play significant roles, and group social activities are beneficial. Since summer research students often are visitors, identifying strategies for following up with students once they have left the program is essential, especially if the students are to be involved in submission and publication of their results.

Academic year programs are spread out over a semester or longer. The work is done while students are enrolled in classes and are participating in on- and off-campus activities. A successful academic year research program therefore requires careful coordination with student schedules. Scheduling regular times to meet with research students is an important component of this planning. Advantages of an ongoing academic year research program are that it provides an opportunity for students to be involved in a project for an extended period of time and it allows experienced students to act as mentors for students who are new to research. Academic year programs face the challenges of finding students, deciding on mechanisms (if any) to provide academic credit or financial support for student researchers, balancing research against student and faculty obligations during the year, and maintaining momentum on the research throughout the year.

Building a successful program

Faculty interested in beginning to mentor undergraduate research should examine successful programs; some are listed in an online Appendix. Relevant articles can be found in MAA *Focus*, the AMS *Notices*, and the MAA online column *Resources for Undergraduate Research*. The Joint Mathematics Meetings and MathFest often have panels, sessions, and mini-courses related to mentoring undergraduate research. Faculty development programs are available at the Center for Undergraduate Research in Mathematics (CURM), the Council on Undergraduate Research (CUR), the Park City Mathematics Institute (PCMI), and American Institute of Mathematics (AIM).

Here are some ingredients of successful programs:

- finding students;
- choosing good research problems;
- mentoring students;
- helping students communicate their work;
- finding funding.

Finding students

Often it is necessary to actively recruit students to do undergraduate research. Faculty can recruit by talking to students about their own research, inviting students in the classes they teach, and soliciting recommendations from faculty colleagues. If no undergraduate research program exists at a school or in a department, a great way for a faculty member to start doing undergraduate research is to work with students as an undergraduate thesis advisor, perhaps as part of an honors program or a capstone experience. If neither of these options is available and a faculty member has a problem or two that could be worked on by students, she should advertise this. Faculty can discuss undergraduate research in their own courses, in student mathematics clubs, and ask colleagues to do the same.

Finding good problems

A good problem should first of all be of interest to the student and to the faculty mentor. Second, it is helpful if the student needs only a limited background to understand and begin working on the problem. If the topic is in an advanced area of mathematics it should have aspects that the student can grasp quickly. Students tend to be more successful in solving problems that are specific and concrete rather than theoretical and abstract. Problems that lend themselves to investigation through computations, to the creation of specific examples, or to computer-based exploration are often good choices. Such problems allow students to quickly explore examples, to develop an intuitive feel for the problem, and to make and explore conjectures. Doing so gives students some ownership of the problem and permits them to direct some of the work. It is important that mentors not over-direct the research.

Projects are more likely to be successful if a problem has several "layers," with multiple steps or subproblems. Initial steps are often easier to solve and provide background to solving the subsequent more difficult steps. Solving aspects of the initial layers of a problem can help students proceed to more advanced layers.

Good problems may be posed by researchers in industry (e.g., data analytics problems) or by faculty from other disciplines (e.g., biology, physics, chemistry, or psychology).

Mentoring

Mentoring students through undergraduate research is an important component of helping them succeed. The mentoring process includes several elements:

- quickly getting students to solve basic problems, explore, and make conjectures;
- ramping up to harder problems;

- addressing social and emotional needs;
- building students' communication skills.

While it is important to communicate work expectations to students, faculty mentors should be realistic and not set goals too high. As the project proceeds, the faculty mentor and the student can modify their research goals depending on what seems feasible for the student.

Students may need help staying motivated while doing undergraduate research. Most students will at some point get stuck or frustrated. Students new to research might feel that being stuck is a result of their not being good enough mathematically. The faculty mentor should point out that being stuck or frustrated is a common experience for anyone doing research. The faculty mentor can also suggest some strategies that may help:

- taking a break;
- explain the difficulty to someone else, mathematically trained or not;
- reviewing background material;
- seeing whether the problem can be modified;
- checking hypotheses and assumptions;
- working out simple examples;
- persevering.

Mentors should not lose patience. Undergraduate research is primarily a learning experience for the student—not part of the mentor's research program. Mentors should be willing to accept what students can reasonably accomplish.

Communicating results

Communicating mathematics is an important skill. Undergraduate research provides at least two activities that can help students improve: oral presentations and written reports.

Student oral presentations. Most students are inexperienced in giving oral presentations about their research. They should practice giving talks to students and faculty, and should be informed about what will occur leading up to, during, and after their presentations. It is also beneficial to advise students explicitly on how to make oral presentations. Here are some guidelines:

- Know your audience; address your talk to people who will be listening.
- Be able to summarize what was accomplished in one sentence, and repeat this idea several times during your talk.
- Include simple examples to help the audience understand definitions, theorems, etc.
- Explain why the problem and the results are interesting.
- Practice your presentation, at least once with an audience not familiar with the research area.
- Ask for and welcome feedback on your presentation.

As a useful resource, see J. Gallian, "How to Give a Good Talk," *Math Horizons,* April 1998.

Student written reports. Students should be expected to write often. For many students, keeping a research journal works well, helping them focus on what they have learned. Written drafts of the report should be regularly handed in to the faculty mentor, who can offer feedback and help the students spot errors and understand how to write a mathematics research paper. Turning in written drafts helps students produce a suitable written report at the end of the project.

If appropriate, the written report may be submitted to a journal for publication. If the faculty mentor has worked with the students on the research, then it is common for her or him to appear as a co-author. This practice allows the faculty mentor to be involved in the revisions of the written report and in the logistics of publication.

Beyond the standard refereed research journals, there are refereed journals that cater specifically to undergraduate authors. See the online resources for a list of such journals.

Funding

Before seeking funds for undergraduate research, faculty should build a track record of working with students outside the classroom, through student projects and presentations. It is also helpful to gain experience judging student talks. Efforts to learn more about how to succeed at undergraduate research should be documented.

Local funding may be needed. A summer undergraduate research program might be supported by a chair or dean, a start-up budget, work-study funds, alumni donors, or an institutional development office. Consider external funding, too, perhaps through the MAA, the NSF, and the Center for Undergraduate Research in Mathematics (CURM). The Appendix has further information.

Certain MAA programs can fund aspects of undergraduate research, including travel funds for students, Tensor grants, and National Research Experience for Undergraduates Program. More about these programs can be found in the online resources. Pi Mu Epsilon also provides funding for students to present at MathFest.

The National Science Foundation funds many summer REUs in mathematics. These typically run for eight weeks with 8–12 undergraduate students working in groups on research problems under the direction of a faculty mentor. Faculty with an existing NSF research grant can also apply for additional NSF funding in the form of Research Experience for Undergraduates supplements. These supplements support the inclusion of undergraduate students in a faculty member's research. Several other NSF programs (see the online resources) support efforts to increase the number of students completing STEM (science, technology, engineering, and mathematics) degrees; grants in these programs can be used to fund undergraduate research.

The Center for Undergraduate Research in Mathematics, directed by Michael Dorff, awards mini-grants of up to $20,000 to support and fund faculty to do undergraduate research during the academic year with students from their own institutions. Activities include a three-day summer workshop to train faculty in doing successful undergraduate research and a spring research conference for students to present their work. The mini-grants provide a $3,000 stipend for each student, a $6000 faculty stipend to reduce the faculty member's teaching load, and funds to travel to the summer workshop and spring conference.

The future of undergraduate research

Over the past decade undergraduate research activities have increased dramatically at colleges and universities nationwide. This development raises important questions for the future.

- Should undergraduate research in mathematics move away from the traditional model of a faculty member working with students individually to a model similar to the laboratory sciences, in which a faculty member works with groups of students on a single project?
- What are appropriate models for rewarding faculty for doing undergraduate research?
- How can undergraduate research in mathematics be integrated into courses and into the mathematics curriculum?

- How can the effectiveness of undergraduate research be assessed?
- Should a significant research experience be required for an undergraduate mathematics major?

Individuals or institutions will have to answer these questions for themselves; this report's authors take no stand on them.

Open Questions

Important questions arose, and remain unanswered, after several years' work in preparation of this *Guide.* We collect several such questions here. Exploring some of them will require large studies and financial support; we expect CUPM and other committees of MAA and other CBMS organizations to investigate some of them. Other questions may prompt individuals in our community to explore and experiment.

1. Getting started in college mathematics

a. *Broader bridges to further mathematics.* A persistent question for CUPM and its subcommittee, CRAFTY, is what mathematics courses to recommend or design for students entering the university. There is interest in offering courses that present a broad view of the subject and make students aware of the wide variety of disciplines that benefit from mathematical thinking and techniques. These courses should serve as bridges to courses that serve both mathematics majors and others. We hope to collect examples of innovative introductory courses distinct from and independent of college algebra and calculus. Several such courses exist, but they are seldom seen as leading to further mathematics courses, let alone to a major. Many institutions now require First-Year Seminars in various areas; CUPM expects to compile a list of such seminars with mathematical themes. Could these seminars provide a bridge to the major?

b. *Discrete mathematics courses.* There was an effort in the 1980s to promote first-year discrete mathematics courses as alternatives to elementary calculus. Recommendations from this effort remain timely. Should we revive this effort? Can discrete mathematics serve as an entry point to a mathematics major? Together with calculus, discrete mathematics would prepare students for further work in the social and behavioral sciences, computer science, information technology, and business.

c. *Serving underprepared students.* How can we best serve the many students who arrive underprepared for college mathematics? Experiments with "stretch courses" are ongoing. Credit-bearing college courses in calculus were expanded to include developmental topics on an as-needed basis in the 1980's. Newer examples of courses outside the standard pre-calculus track designed to incorporate such developmental mathematics in credit-bearing courses are being designed and tested. Are such efforts successful? Can we scale them up?

d. *Data analysis.* The American Statistical Association recommends that *all* students take a course in data analysis. CUPM recommends such a course for all mathematical sciences majors. If this is the first mathematics-related course that new college students take, what are the implications? Will such courses satisfy university mathematics requirements? What developmental mathematics will students need to succeed? Will calculus (and pre-calculus) courses remain preferred for beginning STEM majors? How well does the AP Statistics examination reflect the content of a college data analysis course? Who will staff such courses? What can we expect from beginning college students who have been prepared in programs based on the Common Core State Standards for Mathematics? The CCSSM recommends added emphasis on probability and statistics; will that change the recommended college-level data analysis course? What professional development will faculty need?

e. *Emphasis on modeling.* There have been repeated calls for a modeling course at the college algebra and pre-calculus levels, and textbooks have appeared. What is meant by modeling? We are struggling to define the term as it applies to various levels of mathematics courses. How common and popular are modeling courses at the entry level? Do they successfully prepare students for a next course in mathematics? What should the next course(s) be for biology, chemistry, and other STEM majors?

f. *Persistence in the pipeline.* How can we best encourage students who succeeded in high school calculus to take college mathematics and consider a mathematics major?

2. Calculus

a. *National studies.* Following on the MAA study of Calculus I, *Characteristics of Successful Programs in College Calculus* , we need to know more about the success and persistence of potential STEM majors across the entire precalculus through calculus sequence as well as adequacy of their preparation for downstream courses, both in mathematics and in other disciplines. The new NSF-funded MAA study, *Progress through Calculus*, will address some of these issues.

b. *Restructuring college calculus.* Several efforts are underway to redesign beginning college calculus courses to interest and challenge students who have met calculus ideas in high school. Can we describe syllabi and goals of these alternative courses and measure their success? How transportable are these courses to other institutions?

c. *Serving biology majors.* Should the large and growing population of biology majors take the same introductory mathematics courses as other STEM majors? Perhaps these students' early experience should include elements of modeling, data analysis, and linear algebra—as well as calculus.

d. *Blended calculus.* What examples exist of successful team-taught interdisciplinary courses, blending calculus with other subjects such as physics or biology?

3. Technology: an evolving challenge

a. *"Smart" technologies*. How does—and will—the ubiquity of computer algebra systems, tablets, smart phones, and sophisticated calculators affect the teaching and learning of mathematical content? What concepts and techniques must students master? What does "mastery" mean, and how should we assess it?

b. *Technology in the classroom.* How does, and how should, technology change the ways mathematical material is presented in class?

c. *Online resources.* How can we use online resources (You Tube lectures, MOOCs, etc.) to make students' experiences in mathematics more interesting and successful? This question may be especially significant for large, first-year courses.

d. *Technology and learning.* How can and should technology change the ways students learn and how they mature mathematically?

4. "High-impact" learning experiences

a. *Capstone courses.* What makes an effective capstone course? CUPM continues to call for examples of senior-level capstone courses. Examples will ideally include course objectives, syllabi, requirements, and evaluation of course effectiveness.

b. *Research and research-like experiences*. Many different strategies exist to offer undergraduates research or research-like experiences. Such offerings are often expensive; we need new models and alternatives that allow departments to offer such experiences at larger scale. What are alternative ways for students to participate? Can team projects with one advisor and several students provide the desired experience? Might industrial or business sponsors rather than full-time faculty direct team projects that explore applied problems ?

c. *Internships.* Internships are popular among students. How can the mathematical community encourage partners in business, industry, and government to sponsor more and mathematically richer internships? How can information be shared with advisors and students?

5. Teaching for cognitive growth

a. *Achieving cognitive goals.* What are our best pedagogical strategies for helping students achieve the cognitive goals set out in the Overview? What evidence supports these "best" strategies? How can we help departments design courses that intentionally build students' mathematical sophistication and maturity throughout their undergraduate programs?

b. *Encouraging innovation.* Faculty—including junior faculty— should be encouraged to experiment with new pedagogy and content, without feeling unduly at risk in the promotion and tenure processes. Faculty should be accountable for their experiments, but they should also be invited to document their work and provide evidence for what works well or poorly. Departments should foster a culture that rewards innovation—and accepts that not all innovations succeed.

c. *Setting standards.* Can the mathematics community work toward agreement on ways to measure pedagogical effectiveness ?

6. Professional rewards and course development

a. *Encouraging course development*. How can the mathematical community support course development? Individuals in many institutions are developing materials that change their own courses. Many good ideas emerge. How can we encourage departments to encourage and reward innovative development of content?

b. *Defining alternative pathways into the major*. Can we define new pathways into the mathematics major that can be widely applied? What are reasonable routes distinct from the traditional entry from calculus?

c. *Scaling up good ideas.* How do we communicate with each other to promote broader adoption of new ideas?

d. *Facilitating restructuring large-enrollment course sequences.* Can the community effectively help departments restructure their large-enrollment course sequences when we have to consider transfer students, other disciplines, the nature of the teaching corps? How do we generate cooperation and enthusiasm for such changes?

e. *Rewarding efforts in reform.* Do departments have internal mechanisms for professional rewards for those who develop, communicate, and inspire curriculum reform?

7. Building interdisciplinary cooperation

Many of our colleagues in other departments see us as isolated loners. Mathematicians should meet regularly with colleagues in other disciplines to advance common interests. What strategies have succeeded? How can institutions avoid the silo effect? How can young faculty be encouraged to initiate and sustain interdisciplinary relationships? How can the mathematical community support these efforts?

8. Professional development

a. *Supporting adjunct faculty*. Reliance on non-tenured, non-tenure track adjunct faculty to teach at the introductory level is common at many institutions, and likely to continue. It can be difficult to change curricula, to design innovative courses, and to monitor success when, as at some institutions, adjunct faculty teach more than 50% of students in their first two years. How can departments support professional development for adjuncts?

b. *Supporting permanent faculty.* Tenured and tenure-track faculty may be set in their ways, eager to pursue their own research (for which they are rewarded), and reluctant to try time-consuming new approaches. What are effective ways to change attitudes and reward systems to encourage professional development, new ideas, and practices?

9. Articulation

How can the professional national organizations and the statewide mathematics associations work with high schools to better communicate what students will know when they reach college? What does it really mean to be "college and career ready"? Does the same standard apply to everyone? How will the Common Core State Standards for Mathematics (CCSSM) change our students' mathematical background? How should placement testing reflect these changes? It seems inane to say that college mathematics departments should make themselves familiar with the CCSSM. But many myths about the Standards persist. Are changes needed in the placement system in colleges, or in the CCSSM itself, to help our students succeed in college-level mathematics? Does the college-level mathematics that we offer have to change?

10. Increasing diversity, broadening interest

a. How can departments increase the diversity of students in mathematics courses?

b. How can we best scale up already successful programs for supporting minorities and women in mathematics?

c. How can we better advocate for the importance of mathematics in all fields and applications? How can we better convince parents, high school students, and legislators that mathematics matters to all of us, and deserves support?

Where do we go from here?

This *Guide* is inevitably incomplete, and its recommendations likely to change over time. The curriculum is always evolving—inevitably, as mathematics grows and its applications develop. Still, we hope that the mathematical community, together with its academic partners, will engage in discussion and build consensus on curriculum as we move forward. The Internet can help us negotiate our rapidly changing environment by supporting the exchange of new ideas and evidenced-based models.

How can the mathematical community identify good ideas that have been successfully implemented? How can we measure success? What mechanism is available for scaling up the curricular and pedagogical innovations that seem to be effective?

CUPM welcomes the challenge of starting virtual communities dedicated to improving, supplementing, and evaluating its recommendations. It is an exciting, if awesome, task. What more can the mathematical community offer?

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